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Cohomology of comodules II

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1. Preliminaries. Let C be a vector space over an arbitrary field F, and let Δ and ε ,be F-linear maps, $\Delta : C \longrightarrow C \otimes C$, $\varepsilon : C \rightarrow F$. We say that the triple (C, Δ, ε) is a coalgebra, if the maps verify the following equations: $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$, and $(id \otimes \varepsilon)\Delta = (\varepsilon \otimes id)\Delta = id$. Here,id stands for the identity map. The map Δ is called the comultiplication of the coalgebra,and the map ε is called the counit. A coalgebra C is said to be coaugmented,if there is an element 1_C in C,such that $\Delta(1_C) = 1_C \otimes 1_C$ and $\varepsilon(1_C) = 1_F$, where 1_F stands for the identity of F. Let M be a vector space over F,a linear map $\chi_M : M \rightarrow M \otimes C$, is called a C-comodule structure on M,if χ_M satisfies the following equations: $(\chi_M \otimes id)\chi_M = (id \otimes \Delta)\chi_M$ and $(id \otimes \varepsilon)\chi_M = id$.

We denote by $\underline{\underline{CM}}(C)$ the abelian category of all C-comodules and all C-comodule maps,these being defined in the obvious way. We denote by $\underline{\underline{M}}(F)$, the category of F-vector spaces.

In [1] , we observed that $\underline{\underline{CM}}(C)$ has enough injectives,(C is an injective object when considered as a C-comodule with structure Δ),and defined a connected sequence of functors,denoted by $H^i(C,-)$,from $\underline{\underline{CM}}(C)$ to $\underline{\underline{M}}(F)$. The functors $H^i(C,-)$ are the derived functors of the functor $H^0(C,M)=M^C$, where for any C-comodule M we define M^C as:$M^C = \{m \in M \,/\, \chi_M(m) = m \otimes 1_C\}$.

For any vector space V,defined over F,V^* denotes its dual space. If C is a coalgebra (coaugmented or not),C^* has a natural structure of associative algebra. If $f,g \in C^*$,we define $fg \in C^*$ as $(fg)(x) = \sum f(x_i)g(y_i)$,where $\Delta(x) = \sum x_i \otimes y_i$. Clearly ε is an identity of the product defined above, thus C^* becomes an associative algebra with identity.

In a similar way,any C-comodule M can be endowed with a C^*-module structure.If $f \in C^*$ and $m \in M$, $f.m = \sum f(c_i)m_i$,where $\chi_M(m) = \sum m_i \otimes c_i$. It follows immediately that $\varepsilon.m = m$.

There is a finiteness condition built up in the very definition of a C-comodule structure,that forces M,when considered as a C^*-module,to be locally finite. Recall that if R is an F-algebra with identity and M an

R-module,we say that M is a locally finite R-module,if for every m in M, the dimension of Rm,when considered as an F-space,is finite.

In the situation of C-comodules, if $m \in M$,and $\chi_M(m) = \sum m_i \otimes c_i$,we have that $C^* m \subset \text{span}_F(m_1,\ldots,m_t)$,i.e.,M is locally finite when considered as a C^*-module.

Conversely,let M be an arbitrary C^*-module,and let $m \in M$. Let $m_1,\ldots,m_t$ be a basis of the subspace $C^* m$. There are elements $\lambda_1,\ldots,\lambda_t \in C^{**}$,such that for every $f \in C^*$, $fm = \sum \lambda_i(f)\, m_i$.

Definition 1. In the notation above,we say that the locally finite C^*-module M is of type C,if for every m,the functionals λ_i defined above, are in fact elements of C. (Here we are identifying C with its image in C^{**}).It is clear that if we take another basis for $C^* m$,the new functionals will still be elements of C.

Thus, in the case that M is a locally finite C^*-module of type C,we can find for every m in M and every basis $m_1,\ldots,m_t$ of $C^* m$,elements c_i in C such that $f.m = \sum f(c_i) m_i$. It is a matter of routine fo check,that in that case the map $\chi_M : M \to M \otimes C$,defined as $\chi_M(m) = \sum m_i \otimes c_i$, is a C-comodule structure on M.

The observations above guarantee that the category $\underline{\underline{CM}}(C)$ is naturally equivalent with the category of locally finite C^*-modules of type C.

Let $\underline{\underline{M}}(C^*)$, $\underline{\underline{M}}_f(C^*)$ and $M_f(C^*,C)$,denote the abelian categories of the C^*-modules,locally finite C^*-modules and locally finite C^*-modules of type C,respectively.It is well known,and easy to prove that all the three categories above,have enough injectives.

If R is an arbitrary augmented algebra with augmentation $a : R \to F$,for any R-module M,we define M^a as $M^a = \{ m \in M \ / \ rm = a(r)m \}$ for every r in R .

In the case in which C is a coaugmented coalgebra,C^* becomes an augmented algebra with augmentation $a(f) = f(1_C)$. In this particular case for every C-comodule M,we have that $M^C = M^a$. Take $m \in M^a$,then for every f in C^*, $fm = f(1_C)m = \sum f(c_i) m_i$. As $m = \sum \varepsilon(c_i) m_i$,we have that $\sum f(c_i) m_i = \sum f(1_C)\, \varepsilon(c_i)\, m_i$ .If we choose the m_i to be linearly independent over F,we deduce that for every f in C^*,$f(c_i) = f(1_C)\, \varepsilon(c_i)$.

Then, we have that $c_i = 1_C\,\varepsilon(c_i)$, i.e., $\chi(m) = \sum m_i \otimes \varepsilon(c_i)1_C = m \otimes 1_C$. Conversely, let $m \in M^C$, as $\chi(m) = m \otimes 1_C$, for any f in C^* we have that $f.m = f(1_C)\, m = a(f)m$, i.e. m belongs to M^a.

Thus, our cohomology theory appears as a locally finite version of Hochschild's cohomology of augmented algebras. We recall the basic definitions.

Let A be an associative F-algebra, with identity denoted as 1, and augmentation a. If M is an arbitrary F-space and N a right A-module, then $Hom_F(N,M)$ has a natural structure of left A-module as follows: $(r.f)(n) = f(nr)$. In particular if N=A with the cannonical right A-module structure, the left A-module $Hom_F(A,M)$, is injective. This is because if $\varphi: X \to Y$ is an injective left A-module map, and t is an arbitrary left A-module map from X to $Hom_F(A,M)$, we can define $\bar{t}$ from Y to $Hom_F(A,M)$ as : $\bar{t}(y)(r) = t(\beta(ry))(1)$, where β is an arbitrary F-linear splitting of φ. It is easy to verify that $\bar{t}$ is a left A-module map and that $\bar{t}\varphi = t$.

Moreover, if M is a left A-module, the map $\xi_M: M \longrightarrow Hom_F(A,M)$, defined as $\xi_M(m)(r) = rm$, is an injective left A-module map.

One defines the Hochschild cohomology functors in the category of left A-modules, as the derived functors of the functor $M \to M^a$, where $M^a = \{ m \in M \ / \ rm = a(r)m \}$ for every r in A .

We say that the (left) A-module M is locally finite, if for every m in M, Am is a finite dimensional vector space. Given an arbitrary A-module M, we define M_f as the maximal locally finite A-submodule of M. It is clear that the correspondence that associates M_f to M, is functorial in the sense that if $f: M \to N$ is a map of A-modules, f maps M_f into N_f.

We denote by $\underline{\underline{M}}(A)$, the abelian category of (left)A-modules and by $\underline{\underline{M}}_f(A)$, the abelian category of locally finite A-modules. If I is an injective object in $\underline{\underline{M}}(A)$, I_f is an injective object in $\underline{\underline{M}}_f(A)$. Let X and Y be objects of $\underline{\underline{M}}_f(A)$, $\varphi: X \to Y$ an injective A-module homomorphism, and

$t: X \to I_f$ an arbitrary A-module homomorfism. As I is injective as A-module there is an A-module map $\bar{t} : Y \to I$, such that $\bar{t}\varphi = i\, t$, where i stands for the inclusion of I_f into I. As $Y=Y_f$, $\bar{t}(Y_f)=\bar{t}(Y) \subset I_f$. Consequently I_f is injective in $M_f(A)$. Thus, we can also derive the functor $M \to M^a$, in the category $\underline{\underline{M}}_f(A)$. We denote by $H^i_f(A,-)$ the derived functors considered above. Clearly for any M in $\underline{\underline{M}}_f(A)$, there are natural homomorfism from $H^i_f(A,M)$ into $H^i(A,M)$.

Suppose that C, isa coaugmented coalgebra, and call $C^*=A$. There are three categories that are relevent in this case. These are: $\underline{\underline{M}}_f(C^*,C)$, $\underline{\underline{M}}_f(C^*)$ and $\underline{\underline{M}}(C^*)$. All three of them have enough injectives, the first one because it is equivalent to $\underline{\underline{CM}}(C)$, and the last two because of what we just observed. Thus, we can derive the functor $M \to M^a$ in the three categories, obtaining three cohomology theories. Thus, if M is a C-comodule, we have for every $i \geq 0$, naturally defined maps from $H^i(C,M)$ into $H^i_f(C^*,M)$, and from $H^i_f(C^*,M)$ into $H^i(C^*,M)$, that for $i = 0$ are isomorfisms.

2.Extension of Comodules. We beguin with some general considerations on comodules.Let C be a Hopf algebra,with multiplication μ unit 1_C,comultiplication Δ,counit ε and antipode η. We call u the map from F to C that sends λ to $\lambda 1_C$,for every $\lambda \in F$. If M and N are C-comodules,we endow $Hom_F(N,M)$ with a C-comodule structure $\chi_{N,M}$ as follows (we call χ_N and χ_M the comodule structures on N and M respectively):

$\chi_{N,M}$: $Hom_F(N,M) \longrightarrow Hom_F(N,M) \otimes C = Hom_F(N,M \otimes C)$ is defined by the formula $\chi_{N,M}(f) = (id \times \mu)(id \otimes \eta \otimes id)(\chi_M \otimes id)(f \times id)\chi_N$. It is an easy computation to verify that $\chi_{N,M}$ is a C-comodule structure for $Hom_F(N,M)$. We want to identify $Hom_F(N,M)^C$. Take f in $Hom_F(N,M)^C$. Then f verifies the following equality: $(id \otimes \mu)(id \otimes \eta \otimes id)(\chi_M \otimes id)(f \otimes id)\chi_N = f \otimes 1_C$. Applying $(id \otimes \mu)(\chi_M \otimes id)$ to the equality above we get:

$(id \otimes \mu)(\chi_M \otimes id)(id \otimes \mu)(id \otimes \eta \otimes id)(\chi_M \otimes id)(f \otimes id)\chi_N = \chi_M f$.

Changing the order of the composition factors in the right hand side we get:

$(id \otimes \mu)(id \otimes id \otimes \mu)(id \otimes id \otimes \eta \otimes id)(\chi_M \otimes id \otimes id)(\chi_M \otimes id)(f \otimes id)\chi_N =$
$= \chi_M f$.

Using the fact that χ_M is a C-comodule strucutre,we deduce that:

$(id \otimes \mu)(id \otimes id \otimes \mu)(id \otimes id \otimes \eta \otimes id)(id \otimes \Delta \otimes id)(\chi_M \otimes id)(f \otimes id)\chi_N =$
$= \chi_M f$.

Then : $(id \otimes \mu)(id \otimes \mu(id \otimes \eta)\Delta \otimes id)(\chi_M \otimes id)(f \otimes id)\chi_N = \chi_M f$.

By the definition of antipode,we have that $\mu(id \otimes \eta)\Delta = u\varepsilon$.

Then $(id \otimes \mu)(id \otimes \mu(id \otimes \eta)\Delta \otimes id)(\chi_M \otimes id) = id \otimes id$. Thus, $(f \otimes id)\chi_N = \chi_M f$. Thus f is a C-comodule map. We have proved that $Hom_F(N,M)^C \subset Hom_C(N,M)$. Conversely,if f is in $Hom_C(N,M)$,we have that $(f \otimes id)\chi_N = \chi_M f$. Thus,

$\chi_{N,M}(f) = (id \otimes \mu)(id \otimes \eta \otimes id)(\chi_M \otimes id)\chi_M f =$
$=(id \otimes \mu)(id \otimes \eta \otimes id)(id \otimes \Delta)\chi_M f = (id \otimes \mu(\eta \otimes id)\Delta)\chi_M f =$
$=(id \otimes u\varepsilon)\chi_M f$.

If $(\chi_M f)(n) = \sum m_i \otimes c_i$, we deduce that $\chi_{N,M}(f)(n) = \sum m_i \otimes \varepsilon(c_i)1_C =$

$= f(n) \otimes 1_C$. Then f belongs to $Hom_F(N,M)^C$.
The construction of a C-comodule structure on $Hom_F(N,M)$ is functorial in both variables in the following sense:(we just explain the situation for the second variable). Let $\alpha: M \to M'$ be a morphism of C-comodules.Then $\alpha^*: Hom_F(N,M) \to Hom_F(n,M')$,given by composition with α ,is a morphism of C-comodules,when we endow the spaces of linear maps,with the structures defined above.

Let C be a Hopf algebra over F. If M and N are C-comodules,we endow $M \otimes N$ with a C-comodule structure as follows.Call χ_M and χ_N the structures on M and N respectively,and call s the switching map that sends $x \otimes y$ into $y \otimes x$. Define, $\chi_M \boxtimes \chi_N = (id \otimes id \otimes \mu)(id \otimes s \otimes id)(\chi_M \otimes \chi_N)$. It is easy to see that $\chi_M \boxtimes \chi_N$ is a C-comodule structure on $M \otimes N$. When endowed with such an structure,the vector space $M \otimes N$ will be denoted by $M \boxtimes N$. In the case that N=C,we have the following result.

Lemma2. If C is a Hopf algebra,then for any C-comodule M,$M \boxtimes C$ and $M \otimes C$, are isomorfic as C-comodules.This last one,endowed with the C-comodule structure $id \otimes \Delta$.

Proof. Consider the map $\psi: M \boxtimes C \to M \otimes C$, $\psi = (id \otimes \mu)(\chi \otimes id)$ where χ stands for the structure map on M. The map ψ is a C-comodule map. This amounts to prove that:

$$(\psi \otimes id)(id \otimes id \otimes \mu)(id \otimes s \otimes id)(\chi \otimes \Delta) = (id \otimes \Delta)\psi.$$

The left hand side of the above equality is :

$$LHS = (id \otimes \mu \otimes id)(\chi \otimes id \otimes id)(id \otimes id \otimes \mu)(id \otimes s \otimes id)(id \otimes id \otimes \Delta)(\chi \otimes id) =$$
$$= (id \otimes \mu \otimes id)(id \otimes id \otimes id \otimes \mu)(id \otimes id \otimes s \otimes id)(id \otimes id \otimes id \otimes \Delta)(\chi \otimes id \otimes id)(\chi \otimes id) =$$
$$= (id \otimes \mu \otimes \mu)(id \otimes id \otimes s \otimes id)(id \otimes \Delta \otimes \Delta)(\chi \otimes id).$$

The right hand side of the above equality is:
$RHS = (id \otimes \Delta)(id \otimes \mu)(\chi \otimes id)$. As $\Delta\mu = (\mu \otimes \mu)(id \otimes s \otimes id)(\Delta \otimes \Delta)$, see [3] ,Chapter 3 ; we deduce that LHS=RHS.
The map $\gamma: M \otimes C \to M \boxtimes C$, defined by $\gamma = (id \otimes \mu)(id \otimes \eta \otimes id)(\chi \otimes id)$, is a two sided inverse of ψ.

We will check that $\gamma\psi = id$.

$\gamma\psi = (id \otimes \mu)(id \otimes \eta \otimes id)(\chi \otimes id)(id \otimes \rho)(\chi \otimes id) =$

$= (id \otimes \mu)(id \otimes id \otimes \mu)(id \otimes \eta \otimes id \otimes id)(id \otimes \Delta \otimes id)(\chi \otimes id) =$

$(id \otimes \mu)(id \otimes \mu(\eta \otimes id)\Delta \otimes id)(\chi \otimes id) = id_{M \otimes C}$.

The verification that $\psi\gamma = id$, is analogous.

Q.E.D.

If X is an arbitrary F-space, any comodule of the form $X \otimes C$, with structure $id \otimes \Delta$, is called a coinduced comodule. In [1], we proved that any induced comodule is injective

Corollary 3. Let C be a Hopf algebra over F, let I and M be C-comodules with I injective. Then $M \boxtimes I$ is an injective C-comodule.

Proof. In [1], we proved that any C-comodule can be embedded in a coinduced one. As I is injective I is a direct summand of a coinduced C-comodule. Then there is a C-comodule J and a vector space X, such that $I \oplus J \cong X \otimes C$. Then: $M \boxtimes (X \otimes C) \cong (M \otimes X) \boxtimes C =$

$= (M \boxtimes I) \oplus (M \boxtimes J)$.

In the expression above the C-comodule structure on $M \otimes X$ is given by $\bar{\chi}: M \otimes X \to M \otimes X \otimes C$, $\bar{\chi} = (id \otimes s)(\chi \otimes id)$.

By Lemma 2 $(M \otimes X) \boxtimes C$ is isomorphic to a coinduced comodule and consequently injective. Then $M \boxtimes I$, is injective.

Q.E.D.

We go back now to the case in which C is an arbitrary coaugmented coalgebra.

Let M and N be C-comodules and let $0 \to M \xrightarrow{\alpha} M_0 \xrightarrow{\alpha_0} M_1 \ldots$, be an injective resolution of M by C-comodules.

Consider the complex:

$\underline{\underline{C}}(N,M) : Hom_C(N,M_0) \xrightarrow{\alpha_0^*} Hom_C(N,M_1) \ldots$, and define $Ext_C^n(N,M)$ to be the n-th homology group of the complex $\underline{\underline{C}}(N,M)$.

In other words, we consider for a fixed C-comodule N, the functor from the category of C-comodules to the category of vector spaces that sends M to $Hom_C(N,M)$. This functor is righ exact and $Ext_C^n(N,M)$ is its n-th derived functor.

In the case in which N is the base field F, with the comodule structure $\chi_F: F \to F \otimes C, \chi_F(1_F) = 1_F \otimes 1_C$, the map from $Hom_C(F,M)$ to M that sends f to $f(1_F)$, establishes an isomorphism from $Hom_C(F,M)$ onto M^C. Then, we have that $Ext^n_C(F,M) = H^n(C,M)$. Thus, the cohomology functors defined in §1, are a particular case of the Ext functors.

In the case in which C is a commutative Hopf algebra, and N a finite dimensional C-module, we can obtain the Ext from the cohomology, in the following fashion.

We endow N^* with the dual C-comodule structure as follows. Let $n_1 \ldots, n_t$, be an F-basis of N, and let $f_1, \ldots, f_t$, be the dual basis of N^*. Then the comodule structure χ_N of N defines elements (c_{ij}), by the formulaë: $\chi_N(n_j) = \sum n_k \otimes c_{jk}$. We define $\chi^*: N^* \to N^* \otimes C$, as $\chi^*(f_i) = \sum f_k \otimes \eta(c_{ki})$. It is clear that χ^*, defines a C-comodule structure on N^*.

In that situation, Lemma 4.1 of [1], guarantees that for any C-comodule M, $(N^* \boxtimes M)^C = Hom_C(N,M)$.

<u>Theorem 4</u>. Let C be a commutative Hopf-algebra, and let N and M be C-comodules, the first one finite-dimensional. Then $H^i(C, N^* \boxtimes M) = Ext^i_C(N,M)$, for every $i \geqslant 0$.

<u>Proof</u>. Let $0 \to M \to M_0 \to M_1 \to \ldots$, be an injective resolution of M by C-comodules, then $0 \to N^* \boxtimes M \to N^* \boxtimes M_0 \to N^* \boxtimes M_1 \ldots$, is an injective resolution of $N^* \boxtimes M$ by C-comodules, (see Corollary 3). If we call $\hat{\underline{C}}$, the complex obtained by taking the C-fixed part of the last resolution and deleting the first term, we have that $H^i(C, N^* \boxtimes M)$ is isomorphic to the homology of $\hat{\underline{C}}$. Now, the complex $\hat{\underline{C}}$, is isomorphic to the complex: $0 \to Hom_C(N,M_0) \to Hom_C(N,M_1) \to \ldots$,, whose homology is, by definition, $Ext^i_C(N,M)$.

Q.E.D.

We also have long exact sequences, one in the first and the other in the second variable of Ext, in the same way than in the case of the ordinary Ext functor in the category of R-modules.

In particular we deduce that a C-comodule J is injective if and only if $\mathrm{Ext}^1_C(N,J) = 0$ for every C-comodule N.

We finish this section showing that $\mathrm{Ext}^1_C(N,M)$ can be identified with the set of equivalence classes of short exact sequences of C-comodules. We skip most of the details because they are similar to the "classical" case.

If $0 \to M \xrightarrow{\gamma} M_0 \xrightarrow{\gamma_1} M_1 \xrightarrow{\gamma_2} M_2 \quad \ldots$, is an injective resolution of M and $\mathrm{Hom}_C(N,M_0) \xrightarrow{\gamma_1^*} \mathrm{Hom}_C(N,M_1) \xrightarrow{\gamma_2^*} \mathrm{Hom}_C(N,M_2)$ is the associated complex, we have that $\mathrm{Ext}^1_C(N,M) = \mathrm{Ker}\gamma_2^* / \mathrm{Im}\gamma_1^*$.

If $0 \; M \xrightarrow{i} E \xrightarrow{j} N \to 0$ is a short exact sequence of C-comodules, as M_0 is injective, the map $\gamma : M \to M_0$, can be extended to a map $\tilde{\gamma} : E \to M_0$, that defines a map ρ from N into $M_0/\gamma(M)$. The map γ_1, factors through $\gamma(M)$, inducing a map $\bar{\gamma}_1 : M_0/\gamma(M) \longrightarrow M_1$. Now, the map $\bar{\gamma}_1 \rho$ is an element of $\mathrm{Ker}\gamma_2^*$, then it defines an element of $\mathrm{Ext}^1_C(N,M)$.

Conversely, an element of $\mathrm{Ker}\gamma_2^*$, is a C-comodule map from N to M_1, such that composed with γ_2 gives the zero map. Then, its image is contained in the $\mathrm{Ker}\gamma_2$, this kernel is isomorphic to $M_0/\mathrm{Im}\gamma$.

Thus, an element of $\mathrm{Ker}\gamma_2^*$ can be thought as an element h of $\mathrm{Hom}_C(N, M_0/\mathrm{Im}\gamma)$. Define $E_h = \{ (n,x) \in N \oplus M_0 \; / \; h(n) = x + \mathrm{Im}\gamma \}$. We endow E_h with the direct sum comodule structure and define i and j as: $i : M \to E_h \quad i(m) = (0, \gamma(m))$, $j : E_h \to N \; j(n,x) = n$. If we start with an element $\gamma_1 \alpha$ from $\mathrm{Im}\gamma_1^* \subset \mathrm{Ker}\gamma_2^*$, the corresponding h from N into $M_0/\mathrm{Im}\gamma$, is just $\pi\alpha : N \to M_0 \to M_0/\mathrm{Im}\gamma$. In this case $E_h = \{(n,x) \; \exists \, m \in M \;$ with $\alpha(n) - x = m\}$.

The map $\theta : E_h \to M \oplus N \quad \theta(n,x) = (m,n)$ establishes an isomorphism of extensions between $0 \to M \to E_h \to N \to 0$ and $0 \to M \to M \oplus N \to N \to 0$. This shows that the correspondence given above, factors through $\mathrm{Im}\gamma_1^*$, and thus it defines a map from $\mathrm{Ext}^1_C(N,M)$ into the isomorphisms classes of extensions.

It is a matter of routine to verify that the above constructions are inverses of each other.

<u>3.Normal maps and spectral sequences</u>.Suppose that C and D are coalgebras over a field,and let $\pi: C \to D$ be an arbitray coalgebra map. We define a functor $\pi_*: \underline{\underline{CM}}(C) \longrightarrow \underline{\underline{CM}}(D)$, as follows. If (M, χ_M) is a C-comodule,we define $\pi_*(M,\chi_M)$ to be $(M,(id \otimes \pi)\chi_M)$.
In particular it follows easily from the definitions that if N and M are C-comodules $\pi_*(Hom_F(N,M),\chi_{NM}) = (Hom_F(N,M),\chi_{\pi_*(N)\pi_*(M)})$.

Suppose now that C is a commutative Hopf algebra defined over a field.The map $\theta: C \to C \otimes C$, defined as $\theta=(\eta \otimes \mu)(\Delta\mu \otimes id)\Delta$, is called the conjugate comodule structure on C.(See[1], for the motivation of that definition). If C and D are a pair of Hopf algebras that are commutative,and π is a surjective bialgebra map from C to D, we say that the map π is normal if $\theta(Ker\pi) \subset Ker\pi \otimes C$.

The following Lemma, is contained,though not in explicit form in[1],we state it without proof because it is an obvious consequence of results proved in the mentioned paper.

<u>Lemma 5</u> .Let C,D and π be as above,and let M be an arbitrary C-comodule.Then $\pi_*(C)^D$ is a subHopf algebra of C.Moreover $\pi_*(M)^D$ is a $\pi_*(C)^D$-comodule in such a way that when considered as a C-comodule via the inclusion $\pi_*(C)^D \to C$, it becomes a sub C-comodule of M. We also have that $(\pi_*(M)^D)^{\pi_*(C)^D} = M^C$, and if $f: M \to M'$ is a morphism of C-comodules f restricted to $\pi_*(M)^D$ is a morphism of $\pi_*(C)^D$-comodules into $\pi_*(M')^D$.

Suppose that N and M are C-comodules.From the considerations above we deduce that $Hom_D(\pi_*(N),\pi_*(M)) = \pi_*(Hom_F(N,M))^D$, can be endowed with a structure of $\pi_*(C)^D$-comodule,in such a way that

$$Hom_D(\pi_*(N),\pi_*(M))^{\pi_*(C)^D} = Hom_C(N,M)$$

Moreover if $\alpha: M \to M'$ is a morphism of C-comodules,the induced map $\alpha^*: Hom_F(N,M) \to Hom_F(N,M')$ is also a morphism of C-comodules with the corresponding structures, then α^*,when restricted to

$Hom_D(\pi_*(N),\pi_*(M))$, is a morphism of $\pi_*(C)^D$-comodules from $Hom_D(\pi_*(N),\pi_*(M))$ into $Hom_D(\pi_*(N),\pi_*(M'))$.

Finally we recall, see [1], that in the situation above, the D-comodule $\pi_*(C)$ is injective. That implies that the functor π_* carries injective objects into injective objects.

If $0\to M\to X_0\to X_1\to\ldots$, is an injective resolution of M by C-comodules,

$0\to\pi_*(M)\to\pi_*(X_0)\to\pi_*(X_1)\to\ldots$, is an injective resolution of $\pi_*(M)$ by D-comodules.

We take now an arbitrary C-comodule N, and construct the complex

$$Hom_D(\pi_*(N),\pi_*(X_0))\to Hom_D(\pi_*(N),\pi_*(X_1))\to\ldots,$$

this is a complex in the category of $\pi_*(C)^D$-comodules, whose homology coincides with $Ext^n_D(\pi_*(N),\pi_*(M))$.

In [1], §3, we observed that if $A_1\to A_2\to A_3\to\ldots$, is a complex in the category of X-comodules where X is an arbitrary coalgebra, its homology groups have a natural structure of X-comodules in such a way that all the morphisms that appear in the long exact sequence of homology, are also morphisms of X-comodules.

In our especial case, $Ext^n_D(N,M)$ inherits a natural structure of $\pi_*(C)^D$- comodule, in such a way that for n=0 it coincides with the natural structure of $\pi_*(C)^D$-comodule on $Hom_D(\pi_*(N),\pi_*(M))$.

We summarize the above considerations in the following Lemma.

<u>Lemma 6</u>. Let C and D be commutative Hopf algebras over a field F, and let π be a surjective normal bialgebra map from C to D. Then for any pair of C-comodules N and M, there is a natural $\pi_*(C)^D$-comodule structure $\tilde{\chi}_n$ on $Ext^n_D(\pi_*(N),\pi_*(M))$ such that:

a) $\tilde{\chi}_0$ is the natural $\pi_*(C)^D$-comodule structure in $\pi_*(Hom_F(N,M))^D$.

b) If $0\to M_1\to M_2\to M_3\to 0$ is an exact sequence of C-comodules, the maps in the long exact sequence:

$\mathrm{Ext}^n_D(\pi_*(N),\pi_*(M_1)) \to \mathrm{Ext}^n_D(\pi_*(N),\pi_*(M_2)) \to \mathrm{Ext}^n_D(\pi_*(N),\pi_*(M_3)) \to$
$\mathrm{Ext}^{n+1}_D(\pi_*(N),\pi_*(M_1)) \to \mathrm{Ext}^{n+1}_D(\pi_*(N),\pi_*(M_2)) \to \dots$,

are $\pi_*(C)^D$-comodule maps.

c) If $0 \to N_1 \to N_2 \to N_3 \to 0$ is an exact sequence of C-comodules, the maps in the long exact sequence:

$\mathrm{Ext}^n_D(\pi_*(N_3),\pi_*(M)) \to \mathrm{Ext}^n_D(\pi_*(N_2),\pi_*(M)) \to \mathrm{Ext}^n_D(\pi_*(N_1),\pi_*(M)) \to$
$\mathrm{Ext}^{n+1}_D(\pi_*(N_3),\pi_*(M)) \to \mathrm{Ext}^{n+1}_D(\pi_*(N_2),\pi_*(M)) \to \dots$,

are $\pi_*(C)^D$-comodule maps.

The basic machinery of Groethendieck's spectral sequence,see 2 , yields the following result:

Theorem 7. Let C and D be commutative Hopf algebras over a field F,and let $\pi: C \to D$ be a surjective bialgebra map from C onto D.Let X be an object in $\underline{\underline{CM}}(\pi_*(C)^D)$ and M and N objects in $\underline{\underline{CM}}(C)$. Call i the inclusion of $\pi_*(C)^D$ into C. There is a third quadrant spectral sequence $\{E_r^{pq}\}$,such that :

$$E_2^{pq} = \mathrm{Ext}^p_{\pi_*(C)^D}(X, \mathrm{Ext}^q_D(\pi_*(N),\pi_*(M))) \underset{p}{\Longrightarrow} \mathrm{Ext}^{p+q}_C(i_*(X) \boxtimes N , M),$$

where we regard $\mathrm{Ext}^q_D(\pi_*(N),\pi_*(M))$ as a $\pi_*(C)^D$-comodule with the structure described above.

Proof.Fix X and N and consider the functors:

$\underline{\underline{F}}: \underline{\underline{CM}}(C) \to \underline{\underline{CM}}(\pi_*(C)^D)$, $\underline{\underline{G}} : \underline{\underline{CM}}(\pi_*(C)^D) \longrightarrow \underline{\underline{M}}(F)$, defined as follows

a) If M is a C-comodule $\underline{\underline{F}}(M) = \mathrm{Hom}_D(\pi_*(N),\pi_*(M))$

b) If Y is a $\pi_*(C)^D$-comodule $\underline{\underline{G}}(Y) = \mathrm{Hom}_{\pi_*(C)^D}(X,Y)$.

Then,the general theory developped in[2], yields our result once we prove:

1) I injective in $\underline{\underline{CM}}(C)$ implies $\underline{\underline{F}}(I)$ is $\underline{\underline{G}}$-acyclic.

2) The derived functors of $\underline{\underline{F}}$,are $\mathrm{Ext}^q_D(\pi_*(N),-)\,\pi_*$.

3) $\mathrm{Hom}_{\pi_*(C)^D}(X,\mathrm{Hom}_D(\pi_*(N),\pi_*(M)) = \mathrm{Hom}_C(i_*(X) \boxtimes N , M)$.

1) As $\mathrm{Hom}_D(\pi_*(N),\pi_*(I)) = \pi_*(\mathrm{Hom}_F(N,I))^D$,we prove the acyclicity in

two steps. First we prove that if I is an injective C-comodule and N an arbitrary one, then $Hom_F(N,I)$ is injective as a C-comodule. Then we apply Theorem 5.1 of [1], (That says that if L is an injective C-comodule, then $\pi_*(L)^D$ is injective as a $\pi_*(C)^D$-comodule) to deduce our result. If N is a finite dimensional vector space that is also a C-comodule, and N^* denotes its dual vector space with the natural C-comodule structure defined above, we know that $Hom_F(N,I)$ is isomorphic to $N^* \boxtimes I$, and consequently it is injective, (see Corollary 3). As any C-comodule is the limit of finite dimensional C-comodules, it follows that the restriction on the dimension of N is superfluous.

2) Let $0 \to M \to X_0 \to X_1 \to \ldots$, be an injective resolution of M by C-comodules. Then $0 \to \pi_*(M) \to \pi_*(X_0) \to \pi_*(X_1) \to \ldots$, is an injective resolution of $\pi_*(M)$ by D-comodules, that -as we observed before- are injective. Using this resolution to compute $Ext^q_D(\pi_*(N), -)$, our conclusion follows immediately.

3) It follows in a straightforward way from the definitions.

Q.E.D.

If we take X=F with the trivial $\pi_*(C)^D$-comodule structure, we get an spectral sequence whose E_2 and E_∞ terms are:

$$H^P(\pi_*(C)^D, Ext^q_D(\pi_*(N), \pi_*(M))) \underset{p}{\Longrightarrow} Ext^{p+q}_C(N,M).$$

If we consider N=F with the trivial C-comodule structure, we get Hochschild-Serre spectral sequence (see [1]).

4. Dimension of Coalgebras. Here we assume that C is an arbitrary coaugmented coalgebra over a field F. The following theorem can be proved in the same way than the corresponding theorem for modules over a ring.

Theorem 8. For n an arbitrary positive integer, the following conditions on a C-comodule N are equivalent.

i) For all C-comodules M, $\mathrm{Ext}^{n+1}_C(M,N) = 0$

ii) For every exact sequence of C-comodules of the form $0 \to N \to X_0 \to X_1 \to \dots \to X_n \to 0$, with all the X_i, for $0 \leq i < n$, injectives, the comodule X_n is also injective.

iii) N has an injective resolution of lenght n.

Defn. 9. We say that the cohomological dimension of a C-comodule N is n, if $\mathrm{Ext}^{n+1}_C(M,N)=0$ for every C-comodule M, and there is a C-comodule M_0, such that $\mathrm{Ext}^n_C(M_0,N) \neq 0$. The cohomological dimension of N is denoted as chd(N).

Defn. 10. We define the global dimension of an coaugmented coalgebra, and denote it by gd(C), as gd(C)= sup $\{$chd(N) / N is a C-comodule$\}$. If gd(C)=0, then chd(N)=0 for every C-comodule N. Then for every pair of C-comodules M and N, $\mathrm{Ext}^1_C(N,M)=0$. That implies that every C-comodule is injective. The argument above is reversible. Thus, we have proved that gd(C)=0 if and only if every C-comodule is injective. Obviously, this is equivalent to the assertion that every locally finite C^*-module of type C is completely reducible. Now, Sweedler proves, see [3] Lemma 14.01, that this last condition is equivalent with the assertion that C is the sum of simple coalgebras. Such a coalgebra is called semisimple. Thus, we have proved the following result.

Theorem 11. A coalgebra C is cosemisimple if and only if gd(C)=0.

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