Feasibility set evaluation of multiple NMPC formulations

Authors: Guilherme Souza[1*]; Bruno Faccini Santoro[2]; Darci Odloak[1]

[1] *Laboratório de Simulação e Controle de Processos, Departamento de Engenharia Química, Escola Politécnica da Universidade de São Paulo*

[2] *Op2B – Optimization to Business, São Paulo, Brazil*

Corresponding Author: [*]guilherme_souza@usp.br

Abstract

Four formulations were compared feasibility-wise with simulations of varying initial points and horizon lengths: no stabilizing constrained, terminal equality constrained, terminal inequality constrained and terminal contractive constrained formulations. As expected, horizon length increases cause an important computational effort requirement increase but for most of them the performance gain was sufficiently small to discourage large controller horizons in general.

Keywords

Model predictive control, nonlinear systems, KKT conditions, feasibility, closed-loop stability.

1. Introduction

Model predictive control (MPC) is an advanced control strategy that consists of computing future inputs via an optimization problem solved at every sampling time. Said optimization consists of an objective function, which is often composed of a stage cost and a terminal cost. With this function, different objectives such as control move attenuation, setpoint tracking and a multitude of economy-based targets can be attained. Its predictions are calculated through mathematical models used to describe the system to be controlled, which can be linear or nonlinear.

Linear MPC was, and still is, the preferred choice for industrial implementation, due to lower computational effort required when compared to nonlinear MPC. Unconstrained linear MPC can be represented by a linear programming (LP) problem. With addition of state and input constraints, it is then represented by a nonlinear programming (NLP) problem (Rawlings, 2000). Early implementations of linear MPC managed to circumvent this inconvenience (Morshedi, Cutler and Skrovanek, 1985; Garcia and Morshedi, 1986). Nonlinear MPC had no such recourse, and remained in theoretical studies of recursive feasibility and guaranteed stability until sufficient advances in optimization algorithms and computational power availability.

Both properties are sought after to this day in the literature, as well as an increase in nonlinear MPC, with operational safety and product quality in mind, as well as the constant increase in computational power, optimization algorithms and nonlinear model identification techniques available to model and solve rigorous NLP problems.

Recursive feasibility is a desired property since it guarantees solutions to the optimization problem as long as a first solution is found. However, this is true as long as it is deployed inside its feasibility region. Aside from sensitivity-based adaptive control, there is little interest in feasibility assessment outside of theoretical proof of recursive feasibility. At the time of writing, to the best of our knowledge, no works have compared their proposed formulations with traditional or recent formulations in this sense. This leaves the choice of formulation up to trial and error before deploying a selected formulation.

The aim of this work is to compare four NMPC formulations with guaranteed stability and recursive feasibility with respect to their feasibility region, as well as assess the effects of enlarging said feasibility region on closed-loop performance.

2. Methodology

In this section, four controller formulations will be exposed and their simulation process will be explained. Then, the optimality condition used to determine the feasibility region will be exposed.

2.1. Controller formulations

The first controller deployed has no stabilizing properties, and is shown here for brevity as other controllers will be shown.

2.1.1. Nonlinear model predictive controller with no stabilizing constraint (NSC)

$$\min_{u_k, x_{sp}} J(x) = \sum_{j=0}^{N-1} \| x(k+j|k) - x_{sp} \|_Q^2 \tag{1}$$

$$\text{s.t. } x(k|k) = x_0, \tag{2}$$

$$u(k-1|k) = u_0, \tag{3}$$

$$x(k+j+1|k) = f(x(k+j|k)u(k+j|k)), \qquad j = 0, \dots, N-1 \tag{4}$$

$$x(k+j|k) \in X \subset \mathbb{R}^{n_x}, \qquad j = 0, \dots, N, \tag{5}$$

$$x(k+j+1|k) \in U \subset \mathbb{R}^{n_u}, \qquad j = 0, \dots, N-1. \tag{6}$$

$$x_{sp} \in X_z \subset \mathbb{R}^{n_x} \tag{7}$$

The state prediction is built via a nonlinear model, as in (4), with states and inputs constrained, as per (5) and (6), and setpoint is a decision variable constrained to the desired output zone X_z, represented by (7).

2.1.2. Nonlinear model predictive controller with terminal equality constraint (TEC)

With properties granted by dead-beat conditions (Keerthi and Gilbert, 1998; Sencio et al., 2020), terminal inequality constraints (Rajhans et al., 2019), or contractive constraints (Sencio et al., 2020), these closed-loop stable and recursively feasible were deployed. First, the dead-beat constrained formulation, which adds the following constraints to the optimization problem

$$x(k+N|k) - x_{sp} = 0 \tag{8}$$

$$x_{sp} = f(x_{sp}, u(k+N-1|k)). \tag{9}$$

These constraints enforce the controlled system to reach a steady state, and due to constraint (7), it has to be inside the control zone. Dead-beat constraints were the first approach to closed-loop stability of NMPCs (Keerthi and Gilbert, 1998), and this formulation was further adapted to work along with zone control (Sencio et al., 2020). There are no changes to the objective function, unlike the next controllers.

2.1.3. Quasi-infinite horizon nonlinear model predictive controller

The terminal inequality constrained (TIC) and the terminal contractive constrained (TCC) formulations propose additional constraints as well as a terminal penalty function. First, the quasi-infinite horizon formulation, which enforces the controlled system to reach a region where it is closed-loop stable when under action of a LQR (Rajhans et al. 2019).

$$\left\| \begin{bmatrix} x(k+N|k) \\ u(k+N-1|k) \end{bmatrix} - \begin{bmatrix} x_{sp} \\ u_{sp} \end{bmatrix} \right\|_P^2 \leq \gamma, \tag{10}$$

$$J(x) = J(x) + \left\| \begin{bmatrix} x(k+N|k) \\ u(k+N-1|k) \end{bmatrix} - \begin{bmatrix} x_{sp} \\ u_{sp} \end{bmatrix} \right\|_P^2. \tag{11}$$

In this formulation, the practitioner has to compute the reference output and input values and then the terminal region is characterized around it. The characterization step was done according to the procedure described by Rajhans et al. (2019).

2.1.4. Nonlinear model predictive controller with terminal contractive constraint (TCC)

The terminal contractive constraint enforces the contraction of an artificial variable which is commonly referred to as a slack for a terminal dead-beat constraint. The constraints added to the optimization problem and objective function are:

$$\delta_k = x(k+N|k) - x_{sp}, \tag{12}$$

$$\|\delta_k\|_S^2 - \alpha\|\delta_{k-1}\|_S^2 \leq 0, \tag{13}$$

$$J(x) = J(x) + \|\alpha - \alpha_{min}\|_W^2 \tag{14}$$

2.2. Case study

A CSTR reactor to be controlled by these controllers was first studied by Hicks and Ray, with the following ODE system describing its dynamics:

$$\frac{dx}{dt} = \begin{bmatrix} \dfrac{1-x_c}{\theta} - k_0 x_c \exp\left(\dfrac{-E_A}{x_t}\right) \\ \dfrac{x_{t,f}-x_t}{\theta} + k_0 x_c \exp\left(\dfrac{-E_A}{x_t}\right) - \alpha_T u(x_t - x_{cw}) \end{bmatrix}, \tag{15}$$

Where known parameters are θ = 10, k_0 = 300, E_A = 25.2, $x_{t,f}$ = 3, α_T = 1.95 × 10^{-4}, and x_{cw} = 2.9. A control zone considered for the feasibility studies are intervals for the dimensionless concentration and temperature, z_c and z_t of [0.20.3] and [3.63.7], respectively.

The controllers shown in Subsection 2.1 were simulated with horizon lengths of 3, 5, 10, 15 and 25 sampling intervals, with starting input of u_0 = 300. Controller tuning consisted of $Q = I_{nx}$, $R = I_{nu}$, $S = I_{nx}$, W = 100, α_{min} = 0.5.

2.3. Optimality characterization

The nonlinear programming problems shown in the previous section are subject to the Karush-Kuhn-Tucker (KKT) conditions for optimal solution existence. For the feasibility study, we focus on the stationarity condition based on the following Lagrange function associated to the optimization problem:

$$\begin{aligned}\mathcal{L} = {} & F(x(k+N|k)) + \lambda_0(p)^T(x_0 - p) \\ & + \sum_{j=0}^{N-1}\left[\|x(k+j|k) - x_{sp}\|_Q^2 + \lambda_{j+1}(p)^T(x(k+j+1|k) - \right. \\ & \left. f(x(k+j|k), u(k+j|k))\right],\end{aligned} \tag{16}$$

Abbreviating the arguments for predicted state and input:

$$\begin{aligned}\mathcal{L} = {} & F(x_N) + \lambda_0(p)^T(x_0 - p) \\ & + \sum_{j=0}^{N-1}\left[\|x_j - x_{sp}\|_Q^2 + \lambda_{j+1}(p)^T(x_{j+1} - f(x_j, u_j)\right],\end{aligned} \tag{17}$$

and computing the stationarity condition from the Lagrange function:

$$\nabla_{\lambda_0}\mathcal{L} = x_0 - p, \tag{18}$$

$$\nabla_{x_j}\mathcal{L} = \nabla_{x_j}\|x_j - x_{sp}\|_Q^2 - \nabla_{x_j} f(x_j, u_j)\lambda_{j+1} + \lambda_j, \qquad j = 0, \ldots, N-1 \tag{19}$$

$$\nabla_{u_j}\mathcal{L} = -\nabla_{u_j} f(x_j, u_j)\lambda_{j+1}, \qquad j = 0, \ldots, N-1 \tag{20}$$

$$\nabla_{\lambda_{j+1}}\mathcal{L} = x_{j+1} - f(x_j, u_j), \qquad j = 0, \ldots, N-1 \tag{21}$$

$$\nabla_{x_N}\mathcal{L} = \nabla_{x_N} F(x_N) + \lambda_N. \tag{22}$$

With $F(x_N)$ as a terminal penalty function based on the final predicted state, the system (18)-(22) admits a nontrivial solution for predicted states, computed inputs and the Lagrange multipliers, when the optimization problem has an optimal solution. The stationarity condition has been used for computing an explicit control law along with the next control move to be injected to the system (Zavala and Biegler, 2009). In this work, the idea is to use the stationarity condition parametrized by the initial point *p* to search for feasible points, instead of solving the optimization problem itself at every point. As was done by Zavala and Biegler, state and input bounds were asssumed to be satisfied, hence they were not included in the stationarity condition.

3. Results

The stationarity condition was obtained via symbolic math by CasADi v.3.6.4 (Andersson et al., 2019). The resulting nonlinear system was solved via IPOpt (Wächter and Biegler, 2005), with the feasibility domains plotted via MPT3 v. 3.2.1 tools. Required LQR was computed via MATLAB 9.13.0.2502115 R2022b (The Mathworks Inc., 2022), where all simulations were executed. The computer used has *bench* function output [0.32220.22490.23750.24720.25240.1813].

Use of the stationarity condition alone has lead to trivial solutions, where the controller computes null control moves along with Lagrange multipliers which satisfy the stationarity condition. The primal and dual feasibility as well as complementarity conditions are to be solved with the stationarity condition in order to obtain control moves which minimize the objective function.

Every formulation feasibility was evaluated over the interval [0.21] for dimensionless concentration and [13.75] for dimensionless temperature. Discretization of the viable space has length of 0.05 for dimensionless concentration and 0.1 for dimensionless temperature. The only formulation that showed changes in the feasibility set as the horizon length was varied (from 3 to 25) was the dead-beat constrained NMPC. The other formulations were feasible for the entire interval and every horizon length. The change in feasibility set can be seen in Figure 1.

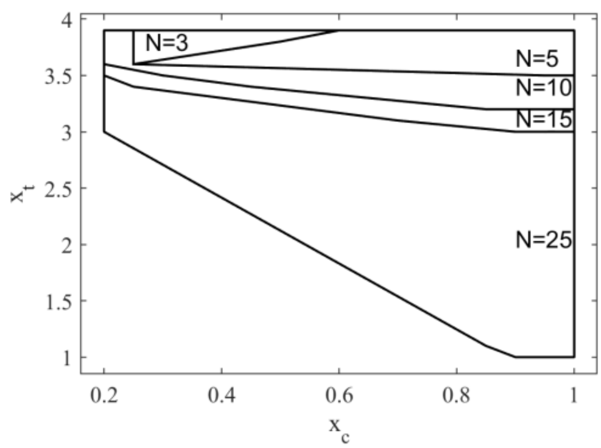

Figure 1: Feasibility set for dead-beat constrained NMPC

As expected, the feasibility set enlarges with horizon length increase. This shows dead-beat constraints are undesirable in this sense, because of their restrictiveness of initial points. Increases in horizon length lead to larger optimization problems, that may lead to delays in delivering control moves in time for the next sampling interval. This is investigated as well and the results are presented next.

Each formulation was solved over the entire viable set, with the discretization step just shown. The computer times and the first predicted state were collected. The square root of the Euclidean distance between the first prediction and current point in the discretized viable space was computed. Then, the mean value of this metric was taken over the entire viable space was computed and gathered in Table 1.

Table 1: Mean state change promoted by formulations, observed over discretized viable space

N	**5**	**10**	**15**	**25**
NSC	0.2978	0.2978	0.2978	0.2982
TEC	0.1936	0.1647	0.1444	0.2039
TIC	0.2987	0.2983	0.2981	0.2991
TCC	0.2987	0.2978	0.2978	0.2991

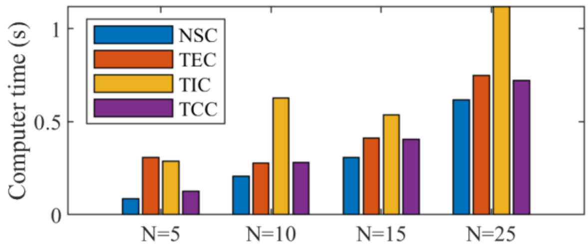

Figure 2: Mean computer times observed in feasibility set computation

From Table 1, the terminal equality constrained formulation shows its conservativeness regarding changing state. As it forces the closed-loop to reach steady- state at the end of the prediction horizon, this results in poor performance when navigating the state space. All other controllers show similar results, gaining little from increasing the controller horizon.

As for computer times seen in Figure 2, every formulation shows a dramatic increase in mean computer times as horizon length increases, as expected. However, the poor gains observed in Table 1 is evidence against horizon length increases.

4. Conclusions

Feasibility studies performed in this work show that a dead-beat constraint causes poor performance when compared to other formulations. Although this formulation shows a performance gain when its horizon length is increased, it is still unable to compare to other formulations in recent literature. It is also evident that more recent developed formulations do not benefit greatly, performance-wise, from controller horizon length increments.

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