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STABILITY OF EQUILIBRIUM OF CONSERVATIVE SYSTEMS WITH TWO DEGREES OF FREEDOM

Garcia, Manuel V.P. & Tal, Fábio A.

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UNIVERSIDADE DE SÃO PAULO
INSTITUTO DE MATEMÁTICA E ESTATÍSTICA
SÃO PAULO — BRASIL

Stability of Equilibrium of Conservative Systems with two Degrees of Freedom

Garcia, Manuel V. P.
Depto. Mat. Aplicada
IME - USP

Tal, Fábio A.
Pós Graduação
MAP - IME - USP*

Abstract

This article intends to study the Liapounof's stability of an equilibrium of conservative Lagrangian systems with two degrees of freedom.

We consider $\Omega \subset \mathbf{R}^2$ an open neighborhood of the origin and the Lagrangian $\mathcal{L} = T - \pi$, where $\pi: \Omega \longrightarrow \mathbf{R}$ of class $\mathcal{C}^2$ is the potential energy with a critical point at the origin and $\mathrm{T}: \Omega \times \mathbf{R}^2 \longrightarrow \mathbf{R}$ is the kinetic energy, of class $\mathcal{C}^2$.

We assume that π has jet of order k at the origin, and this jet shows that the potential energy does not have a minimum in 0. With these hypotheses we prove that (0;0) is an unstable equilibrium according to Liapounof for the Lagrange equations of $\mathcal{L}$. We achieve this by proving that there is an asymptotic trajectory to the origin.

1 Introduction

In this work we consider the Liapounof's stability of conservative Lagrangian systems, for Lagrangians $\mathcal{L}(q;\dot{q}) = \mathrm{T}(q;\dot{q}) - \pi(q)$, where π is the potential energy and T the kinetic energy.

The Lagrange's equations for a system with these features are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} + \frac{\partial \mathcal{L}}{\partial q} = 0. \tag{1}$$

It is a known fact that the equilibria of (1) are the points $(q_0;0)$ in which q_0 is a critical point of π.

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We shall then consider a critical point q_0 of the potential energy π and study the stability according to Liapounof of (1).

One of the major results in this area is the Lagrange -Dirichlet theorem, which states that *if q_0 is a local strict minimum point for the potential energy π, then $(q_0;0)$ is a stable equilibrium according to Liapounof of (1).*

Since Dirichlet proved this result in 1846, many renowned mathematicians have dedicated themselves to the problem known in the literature as the *inversion of the Lagrange - Dirichlet Theorem* .

In short, one considers a conservative Lagrangian system with an equilibrium point in $(q_0;0)$ where q_0 is not a local strict minimum of π, and one tries to study the stability according to Liapounof of $(q_0;0)$.

For a short period of time, many tried to prove that if q_0 is not a local strict minimum of π then $(q_0;0)$ is an unstable equilibrium for (1). Painlévè's example, presented in 1904, showed this to be false, even in the case of 1 degree of freedom.

In order to do so, he considered a system with$(q;\dot{q}) \in \mathbf{R}^2$, kinetic energy $T(\dot{q}) = \frac{\dot{q}^2}{2}$ and potential energy $\pi(q) = e^{-\frac{1}{q^2}} \sin \frac{1}{q}$, if $q \neq 0$, and $\pi(0) = 0$. Painlévè proved, in [3], that (0;0) was a Liapounof's stable equilibrium for (1), while π has neither a minimum nor a maximum in 0.

Although this example shows that a complete reciprocal to the Dirichlet -Lagrange theorem is false, the problem of finding sufficient conditions on $\mathcal{L}$ to ensure the instability of an equilibrium of such systems is still known as the *inversion of the Lagrange - Dirichlet Theorem*, and has received the attention of Liapounof, Tchetaev, Lefchetz, La Salle, E. Hanh e L. Salvadori, among others.

Liapounof proved in 1897 that, if q_0 is not a local minimum of π and *this fact is shown by the second order derivatives of π*, then $(q_0;0)$ is an unstable equilibrium of (1) (see [4]).

In the terminology used in this note this result may be enunciated as *if the jet of order 2 of π in q_0 shows that π does not have a minimum at q_0, then $(q_0;0)$ is an unstable equilibrium.*

Liapounof conjectured that *if the jet of order k of π at q_0 shows that π does not have a minimum at this point, then $(q_0;0)$ is an unstable equilibrium.*

In 1989 and 1991 A. Maffei, V. Moauro and P. Negrine obtained extremely interesting results in this direction (see [5] and [6]).

The central result of these works considers the case in which $q_0 = 0$ and it supposes that, after an eventual change of coordinates,

$$\pi(q_1;\dots;q_n) = \|(q_1;\dots;q_r)\|^2 + \pi_k(q_{r+1};\dots;q_n) + R(q_{r+1};\dots;q_n),$$

where π_k is an homogeneous polynomial of degree $k \geq 2$ which doesn't have a minimum at the origin and R satisfies $\lim_{q \to 0} \frac{R(q)}{\|q\|^k} = 0$. With these hypotheses the authors prove that (0;0) is an unstable equilibrium for (1)

Therefore, if the k jet of π at q_0 is homogeneous and shows that π doesn't have a minimum at this point, then (q_0;0) is an unstable equilibrium for (1)[1].

Another important result was shown by Palamodov who proved that if π is an analytical function without a minimum at q_0 then the equilibrium is unstable (see [8]).

In this work we consider systems with two degrees of freedom and we prove Liapounof's conjecture, with no additional hypothesis, that is, we show that, in the 2 degrees of freedom context, *if the k-jet of π at q_0 shows that π doesn't have a minimum in this point, then* (q_0;0) *is an unstable equilibrium for (1)*. We achieve this by constructing a "cone" with vertex at the origin of the phase space of the Lagrange equations and an auxiliary function which assures that there is a trajectory asymptotic to the origin.

Since Barone, Gorni and Zampieri proved in [2] that for every analytical function f without a minimum at q_0 there is a integer k such that $j^k f$ shows that f doesn't have a minimum at this point, our result extends the Palamodov's result for systems with two degrees of freedom. Moreover we provide a positive answer for a conjecture posed by V. V. Kozlov in [3] concerning the existence of asymptotic trajectories to (q_0;0) for the analytic case, which was not proved by Palamodov in [8].

This article comprehends this introduction and 4 other sections. In section 2 we present the context in which we are going to work in a rigorous way. In Section 3 we demonstrate a technical lemma extremely important to prove our instability result, which is done in section 4. Our work finishes with an appendix where the results in k-decidability needed for the text are exposed. A reader familiarized with the work done by Barone-Netto in this area may skip the reading of a good part of this appendix, and stick to the demonstration of the Fact 1, shown here for the first time.

2 The Problem

Let us consider a conservative Lagrangian system with 2 degrees freedom, with potential energy π defined in an open neighborhood of the origin Ω, and kinetic energy T defined on $\mathbf{R}^2 \times \Omega$.

[1] In these works π is supposed to be of class C^{k+3}, later on S. Tagliaferro improved the result for the homogeneous case for a broader class of potential energies (see [7]).

We admit π of class $C^2, \pi(0) = \|\pi(0)\| = 0$ and $\pi = \mathrm{P} + \mathrm{R}$, where P is a polynomial of degree less than or equal to k and $\lim_{x \to 0} \frac{\pi(x) - \mathrm{P}(x)}{\|x\|^k} = 0$, that is, P is the k-jet of π at the origin, moreover, we assume that there is $j^{k-1}\nabla\pi$. We further suppose that P shows that π doesn't have a minimum at the origin (see the definition below) and $j^{k-1}\pi$ does not show that π does not have a minimum at the origin.

We say that $j^k f$ shows that f does not have a local minimum at 0 if, for every function $g: \Omega \longrightarrow \mathrm{R}$ such that $j^k g = j^k f$, 0 is not a local minimum of g.

The kinetic energy is a defined positive quadratic form in the velocities, and it's supposed to be of class C^2, that is,

$$\mathrm{T} = \frac{1}{2} \langle\, \mathrm{B}(q)\dot{q} \mid \dot{q} \,\rangle \tag{2}$$

where

$$\mathrm{B}(q) = \begin{bmatrix} \mathrm{F}(q) & \mathrm{G}(q) \\ \mathrm{G}(q) & \mathrm{H}(q) \end{bmatrix}$$

is defined positive for every $q \in \Omega$ and F, G and H are C^2 functions defined in Ω. There is no loss of generality in supposing that $\mathrm{B}(0) = \mathrm{I}$.

With these hypotheses[2] $(0;0)$ is an equilibrium for the Lagrange equations of the system with Lagrangian $\mathcal{L} = \mathrm{T} - \pi$. We intend to prove that the hypotheses made above assure the instability according to Liapounof of this equilibrium.

The following k-decidability result is demonstrated in the appendix and plays a major role in this work.

Fact 1 *If the k-jet of π at the origin is the first jet of π that shows that this function does not have minimum at* 0, *then there are, after an eventual rotation of* R^2, *reals* $\lambda > 0$, $\alpha > 0$ *and an algebraic curve* $\Gamma(x) = (x;\gamma(x))$, *where* $\gamma: [0;\varrho[\longrightarrow \mathrm{R}$, *with* $\Gamma(0) = 0$ *and whose versor in* 0+ *is* (1;0), *such that*

$$\min_{-\lambda x < y < \lambda x} \mathrm{P}(x;y) = \mathrm{P}(x;\gamma(x)) = -\alpha x^\beta + o(x^\beta),\ \forall x \in [0;\varrho[, \tag{3}$$

with $\beta \leq k$.

Note that, since Γ is algebraic in the case of this proposition, we can assume with no loss in generality that

[2] Actually we could suppose that π and T are functions of class C^1 and some theorem of unicity and continuous dependence is true for equations (1). The reader will have no problem verifying that the proofs remain valid in this context.

$$\gamma(x) = \sum_{j=1}^{+\infty} b_j x^{\beta_j} \tag{4}$$

with $b_j \in \mathbf{R}$ and (β_j) is a sequence of strictly increasing rationals with $\beta_1 > 1$ (see the Walker's text [2] on algebraic curves for an elegant and complete presentation of the theorem of Puiseaux that shows this result). Further more, except in the case where Γ is the semi-axe of the abscissa (situation in which $\gamma \equiv 0$), we have that $b_1 \neq 0$.

It's an immediate consequence of these observations that, for a possibly smaller ϱ, there are positive constants c_1, c_2 and c_3 such that, for $0 < x < \varrho$

$$\begin{aligned} |\gamma(x)| &< c_1 x^{\beta_1} \\ |\gamma'(x)| &< c_2 x^{\beta_1 - 1} \\ |\gamma''(x)| &< c_3 x^{\beta_1 - 2}. \end{aligned} \tag{5}$$

We point out that, since $\beta_1 > 1$ we have $\beta_1 - 2 > -1$.

These proprieties and estimates on Γ will be used in the forthcoming sections.

3 A Fundamental Lemma

In this section a fundamental result is proved which allows us to establish the existence of a trajectory asymptotic to the origin in the hypotheses mentioned in the previous section.

We recall that $\pi: \Omega \longrightarrow \mathbf{R}$ is a C^2 function, $\pi(0) = \|\nabla\pi(0)\| = 0$, and π has k-jet at the origin and this jet shows that π does not have a minimum in 0.

Let $\mathrm{P} = j^k\pi$ and consider the curve $\Gamma(x) = (x; \gamma(x))$ as in the previous section, satisfying the fact 1. Since Γ obeys (3), we shall call it *curve of vertical minima* of P.

We begin by carrying out a change of coordinates of class C^∞ in the vertical strip $\mathcal{F} = \{(x;y): 0 < x < \varrho\}$ that admits an extension to $\mathcal{F} \cup \{O\}$, which is an homeomorphism. The purpose of this transformation is to move the curve Γ onto the segment $(x;0), 0 < x < \sigma$.

We consider $\varphi(x;y) = (x;w)$, $0 < x < \varrho, y \in \mathbf{R}$, where $w = y - \gamma(x)$.

The fact that this transformation is C^∞ can be seen from (4) and the observations made at the bottom of section 1. Furthermore, it is immediate that defining $\overline{\varphi}: \mathcal{F} \cup \{O\} \longrightarrow \mathbf{R}^2$ by

$$\overline{\varphi}(x;y) = \begin{cases} \varphi(x;y) & \text{se } x > 0 \\ (0;0) & \text{se } (x;y) = (0;0) \end{cases} \tag{6}$$

we have an homeomorfism.

Now, we express P in the $(x;w)$ coordinates. Since P is a polynomial of degree less then or equal to k, we have $\mathrm{P}(x;y) = \sum_{(i;j)\in I} a_{ij}x^i y^j$, therefore, from the definition on φ and (4) it follows by direct substitution that

$$\mathrm{P}(x;w) = \sum_{(i,j)\in I}\sum_{k=0}^{j} a_{ij}x^i \left(\sum_{\ell=1}^{\infty} b_\ell x^{\beta_\ell}\right)^{k-j} w^k \tag{7}$$

We note that φ preserves vertical lines $x = \xi$ and, in each of these lines, it is a translation $(w = y + \gamma(\xi))$. Again, since φ takes Γ onto the x-axis, it follows from (3) that

(i) In the coordinates $(x;w)$, $\mathrm{P}(x;0) = -\alpha x^\beta$.

(ii) For fixed $\xi \in (0,\varrho)$, the function $\ell(w) = \mathrm{P}(\xi;w)$, $w \in \mathbf{R}$ has a local minimum point at 0.

From (ii) it follows that $\frac{\partial \mathrm{P}}{\partial w}(x;0) = 0$, $0 < x < \varrho$ and thus, in the expression(7), there are no linear terms in w.

We distinguish in (7) the terms with degree less then or equal to β, and observe that the exponents of w are all integers greater then or equal to 2, and that the exponents of x are rationals greater then or equal to one, and this yields:

$$\mathrm{P}(x;w) = -\alpha x^\beta + w^2 \sum_{i\in\mathbf{Q},\ j\in\mathbf{N}}^{i+j\leq\beta} \tilde{a}_{ij}x^i w^{j-2} + o(\|(x;w)\|^\beta),$$

so, by making $P_2(x,w) = w^2 \sum_{i\in\mathbf{Q},\ j\in\mathbf{N}}^{i+j\leq\beta} \tilde{a}_{ij}x^i w^{j-2}$, we have

$$\mathrm{P}(x;w) = -\alpha x^\beta + P_2(x,w) + o(\|(x;w)\|^\beta). \tag{8}$$

Direct calculation shows that, taking $\tilde{q} = (x,w)$ we obtain the kinetic energy in the new coordinates $\tilde{T} = \frac{1}{2}\langle\, \tilde{B}(\tilde{q})\dot{\tilde{q}} \mid \dot{\tilde{q}}\,\rangle$,

where

$$\tilde{B} = \begin{bmatrix} \mathrm{F} + 2\mathrm{G} + \mathrm{H}\gamma'^2 & \mathrm{G} + \mathrm{H}\gamma' \\ \mathrm{G} + \mathrm{H}\gamma' & \mathrm{H} \end{bmatrix}. \tag{9}$$

By observing that $\tilde{B}(0,0) = I$ and using the estimates (5), we have

Fact 2 *The matrix* $\widetilde{B}$ *may be written* $\widetilde{B} = I + \widetilde{h}$, *where* $\|\widetilde{h}\|$ *is* $o(\|\widetilde{q}\|^{\delta_1})$ *and* $\|\widetilde{h}'\|$ *is* $o(\|\widetilde{q}\|^{\delta_2})$, *with* $\delta_1 = \min\{\beta_1 - 1, 1\}$ *and* $\delta_2 = \min\{\beta_1 - 2, 0\} = \delta_1 - 1$.

Furthermore $\widetilde{B}$ *is invertible and we have* $\widetilde{B}^{-1} = I + d$, *where* $\|d\|$ *is* $o(\|\widetilde{q}\|^{\delta_1})$ *and* $\|d'\|$ *is* $o(\|\widetilde{q}\|^{\delta_2})$.

With these ingredients, we may write Lagrange equations in the coordinates $(x, w, \dot{x}, \dot{w})$.

Fact 3 *The normal form of the Lagrange equations for the considered system, in the variables* $(x, w, \dot{x}, \dot{w})$, *are*

$$\ddot{\widetilde{q}} = \widetilde{B}^{-1}(-\nabla\widetilde{\pi} + O(\|\widetilde{q}\|^{\delta_2}\|\dot{\widetilde{q}}\|^2). \qquad (*)$$

From here it follows, as a particular case, the equation

$$\ddot{w} = -f(x, w)\frac{\partial\widetilde{\pi}}{\partial w} - g(x, w)\frac{\partial\widetilde{\pi}}{\partial x} + O(\|\widetilde{q}\|^{\delta_2}\|\dot{\widetilde{q}}\|^2), \qquad (10)$$

where $f(0;0) = 1$ and $g(0, 0) = 0$.

In order to build the cone C and the auxiliary function that will help us show the existence of the asymptotic trajectory, we will study in some detail the behavior of P_2 in the curves $w = ax^c$, $x > 0$, for some values of $a \in \mathbf{R}$ and $c \geq 1$.

We assume that $P_2 \not\equiv 0$, and we take $a \neq 0$ and $c \geq 1$; in the curve $w = ax^c$, $x > 0$ the monomial $a_{sr}x^s w^{r+2}$ has order $s + c(r + 2)$.

Since for all $(s;r) \in \widetilde{I}$ we have $s + r + 2 \leq \beta$ we can choose $\bar{c} \geq 1$ such that $\min\left\{s + \bar{c}(r + 2) : (s;r) \in \widetilde{I}\right\} = \beta$.

Clearly, there is only one $\bar{c}$ in these conditions, and we will call J the set $\{(s;r) : s + \bar{c}(r + 2) = \beta\}$. Therefore $J \neq \emptyset$ and, at the curve $w = ax^c$, $x > 0$, we have

$$P_2(x;w) = P_2(x;ax^{\bar{c}}) = \sum_{(s;r)\in J} a_{sr}a^{r+2}x^{\beta} + o(x^{\beta}). \qquad (11)$$

Now, consider the one variable, real polynomial $\widetilde{P}(a) = \sum_{(s;r)\in J} a_{sr}a^{r+2}$. It follows from (11) that $P(x;ax^{\bar{c}}) = \widetilde{P}(a)x^{\beta} + o(x^{\beta})$.

Take $r_0 = \min\{r \in \mathbf{N} : \exists s \in \mathbf{Q}, (s;r) \in J\}$ and note that there is only one s_0 such that $(s_0;r_0) \in J$.

In these settings, we have

Fact 4 *The polynomial* $\widetilde{P}$ *satisfies* $\widetilde{P}(0) = 0$ *and* $\widetilde{P}(a) \geq 0$ *for all* $a \in \mathbf{R}$.

Proof: Since $r_0 \in \mathbf{N}$,we have $r_0+2 \geq 2$, and so, from the way we chose r_0 it follows that $\widetilde{P}$ has a root with multiplicity greater then or equal to 2 in $a=0$.

Furthermore, we note that $P(x;\overline{a}x^{\overline{c}}) - P(x;0) = \widetilde{P}(\overline{a})x^{\beta} + o(x^{\beta})$, thus, if by contradiction there was $\overline{a} \in \mathbf{R}$ such that $\widetilde{P}(\overline{a}) < 0$ then $\overline{a} \neq 0$ and we would have

$$\lim_{x \downarrow 0} \frac{P(x;\overline{a}x^{\overline{c}}) - P(x;0)}{x^{\beta}} = \widetilde{P}(\overline{a}) < 0.$$

So, for sufficiently small $x > 0$ we would have $P(x;\overline{a}x^{\overline{c}}) < P(x;0)$ which contradicts the fact that γ is the vertical minima curve for P, thus establishing the result. ■

Let us recall now the definition of r_0 and let $G(a) = \sum_{\substack{(s;r)\in J \\ r>r_0}} a_{sr}a^{r-(r_0+1)}$ so that (note that $r-(r_0+1) = r+2-(r_0+3)$):

$$\widetilde{P}(a) = a_{s_0r_0}a^{r_0+2} + a^{r_0+3}G(a). \tag{12}$$

In the future, the following inequality will be useful

$$|a^{r_0+3}G(a)| \leq |a^{r_0+3}| \sum_{\substack{(s;r)\in J \\ r>r_0}} |a_{sr}a^{r-(r_0+1)}| \tag{13}$$

Now, let $a > 0$ and consider the cone $C_a = \left\{(x;w) \colon x > 0, -ax^{\overline{c}} \leq w \leq ax^{\overline{c}}\right\}$. We will study the behavior of P_2 and of $x\frac{\partial P_2}{\partial x}$ in C_a.

To do so, we consider $\overline{G}(a) = \sum_{\substack{(s;r)\in J \\ r>r_0}} |a_{sr}a^{r-(r_0+1)}|$ and note that:

Fact 5 *With the previous notations, we have*

(i) For $(x;w) \in C_a$ we have $|P_2(x;w)| \leq (a_{s_0r_0} + a\overline{G}(a))a^{r_0+2}x^{\beta} + o(x^{\beta})$.

(ii) If $\delta(a) = (a_{s_0r_0} - a\overline{G}(a))a^{r_0+2}$, $a \geq 0$ then $P_2(x;ax^{\overline{c}}) \geq \delta(a)x^{\beta} + o(x^{\beta})$.

Proof: To prove (i) note that, if $(x;w) \in C_a$, then $(x;w) = (x;bx^{\overline{c}})$, where $-a \leq b \leq a$ and, remarking that $\overline{G}$ is crescent, it follows

$$\begin{aligned} |P_2(x;w)| = {} & a_{s_0r_0}b^{r_0+2}x^{\beta} + |b^{r_0+3}G(b)|x^{\beta} + o(x^{\beta}) \leq \\ \leq {} & (a_{s_0r_0} + a\overline{G}(a))a^{r_0+2}x^{\beta} + o(x^{\beta}). \end{aligned}$$

As to the second inequality, we recall that, from (11) and (12),

$$P_2(x;ax^{\overline{c}}) = \widetilde{P}(a)x^{\beta} + o(x^{\beta}) = (a_{s_0r_0}a^{r_0+2} + a^{r_0+3}G(a))x^{\beta} + o(x^{\beta}),$$

thus, since $a > 0$, it follows directly from the definition of $\overline{G}$ and from(13), that (ii) is true. ■

Fact 6 *There are* $a_0 > 0$ *and* $\varepsilon > 0$ *such that, if* $0 < a < a_0$ *and* $(x;w) \in C_a$, *with* $x < \varepsilon$ *then*

$$|x\frac{\partial P_2}{\partial x}(x;w)| < (k-1)\delta(a)x^\beta + o(x^\beta).$$

Proof: We begin by noticing that $x\frac{\partial P_2}{\partial x}(x;w) = w^2 \sum_{(s;r)\in\tilde{I}} sa_{sr}x^s w^r$, thus, in C_a we have, taking as before $(x;w) = (x;bx^{\overline{c}})$,

$$\begin{aligned} |x\frac{\partial P_2}{\partial x}(x;w)| = & \ |(s_0 a_{s_0 r_0} b^{r_0+2} + b^{r_0+3} \sum_{\substack{(s;r)\in J\\ r>r_0}} sa_{sr} b^{r-(r_0+1)})|x^\beta + o(x^\beta) \leq \\ \leq & \ |(a_{s_0 r_0} + a \sum_{\substack{(s;r)\in J\\ r>r_0}} \frac{s}{s_0} a_{sr} a^{r-(r_0+1)})|s_0 a^{r_0+2} x^\beta + o(x^\beta) \quad (14) \end{aligned}$$

From the manner in which we defined J and the par $(s_0;r_0)$ it follows that $(s;r)$ is in $J \setminus \{(s_0;r_0)\}$ then $s < s_0 \leq k-2$, therefore, from (14) we have $(x;w) \in C_a$,

$$\begin{aligned} |x\frac{\partial P_2}{\partial x}(x;w)| \leq & \ (k-2)(a_{s_0 r_0} + a\overline{G}(a))a^{r_0+2}x^\beta + o(x^\beta) = \\ = & \ (k-2)(\delta(a) + 2a^{r_0+3}\overline{G}a)x^\beta + o(x^\beta). \quad (15) \end{aligned}$$

Noting that $\delta(a)$ is $O(a^{r_0+2})$ and $a^{r_0+3}\overline{G}(a)$ is $o(a^{r_0+2})$, then, given $\varepsilon > 0$ there is $a_0 = a_0(\varepsilon) > 0$ such that, if $0 \leq a < a_0$, we have $2a^{r_0+3}\overline{G}(a) \leq \varepsilon\delta(a)$.

Now, choosing $\varepsilon > 0$ such that $(k-2)(\varepsilon+1) < k-1$ (for instance $\varepsilon = \frac{1}{4}$), and noting that $a < a_0$, it follows from (15), that in C_a,

$$|x\frac{\partial P_2}{\partial x}(x;w)| \leq [(k-2)(\varepsilon+1)\delta(a)]x^\beta + o(x^\beta) < (k-1)\delta(a)x^\beta + o(x^\beta),$$

which is the desired inequality. ∎

Finally, we are able to prove the fundamental result of this section, a key piece to proving the instability of the origin.

Lemma 1 *There are* $0 < \delta < \frac{1}{4}$ *and* $\sigma > 0$ *such that, in the connected component* C_1 *of*

$$\Omega_1 = \left\{(x;w) \in \mathbf{R}^2 : 0 < x < \sigma,\ P_2(x;w) < \delta x^\beta\right\}$$

that contains$(x;0)$, $0 < x < \sigma$, *we have*

$$x\frac{\partial P_2}{\partial x}(x;w) < (k-1)\delta x^\beta.$$

Proof: We will keep here the notation used in the section.

Note that $\lim_{a\downarrow 0} \frac{\delta(a)}{a^{r_0+2}} = a_{s_0r_0} > 0$, so, by taking an eventually smaller a_0 in the previous expression, we have $\delta(a) > 0$, for $0 < a \leq a_0$.

Also, if $a > 0$, $\frac{\delta(a)}{a} = (a_{s_0r_0} - a\overline{G}(a))a^{r_0+1}$ and so, since $r_0 + 1 \geq 1$ we can choose a_0 sufficiently small so that the function δ is crescent in $[0;a_0]$.

Now, from the fact5 follows that $P_2(x; \pm\, a_0x^{\overline{c}}) \geq \delta(a_0)x^\beta + o(x^\beta)$, and by taking $\delta = \frac{\delta(a_0)}{2}$ we have that there is $\sigma > 0$ such that, if $0 < x \leq \sigma$,

$$P_2(x; \pm\, a_0x^{\overline{c}}) \geq \delta x^\beta. \tag{16}$$

This shows that if $\overline{C}$ is the connected component of

$$\Omega_1 = \left\{(x;w) \in \mathbf{R}^2 : 0 < x \leq \sigma, P_2(x;w) < \delta x^\beta\right\}$$

that contains the segment $(x;0)$, $0 < x < \sigma$, then $\overline{C} \subset C_{a_0}$.

On the other hand, since $[0;a_0]$, $\delta(a)$ is crescent, the fact 6 implies that, after a possible reduction of σ we have, for $(x;w) \in C_{a_0}$,

$$|x\frac{\partial P_2}{\partial x}(x;w)| \leq \frac{\delta a_0}{2}(k-1)x^\beta = \delta(k-1)x^\beta. \tag{17}$$

Since $\overline{C} \subset C_{a_0}$, (17) ends the proof. ■

4 The Instability Theorem

In this section we show that, for the conditions in which we are working, there is an asymptotic trajectory to the origin that solves equations (1).

To achieve this we will construct, in the phase space $(x, w, \dot{x}, \dot{w})$ a cone C and an auxiliary function V that will be useful to establish the existence of an asymptotic trajectory in C.

Since the function φ, used in assigning the new variables $(x;w)$, was defined in $\mathcal{F}$ which is not an open neighborhood of the origin, it is worth pointing out the observation made in the previous section. We defined φ in the strip $\mathcal{F} = \{(x;y) \in \mathbf{R}^2 : 0 < x < \varrho\}$ and we noted that its extension $\overline{\varphi}$ to $\mathcal{F} \cup \{(0;0)\}$, given by(6) is an homeomorfism.

Therefore, since $\varphi(0;0) = (0;0)$, if we prove that there is a solution $\psi(t) = (x(t);w(t))$ of equations (*) defined in $(-\infty;0]$ such that $\lim_{t\to-\infty} \psi(t) = (0;0)$, we will have shown the existence of an asymptotic trajectory to the origin of the equations (1).

We define now the aforementioned auxiliary function. Let $V\colon \mathcal{F}\times\mathbf{R}^2 \longrightarrow \mathbf{R}$ be given by

$$V(x;w;\dot{x};\dot{w}) = \frac{\dot{w}^2}{2f(x;w)} + P_2(x;w). \tag{18}$$

A simple calculation and (10) gives us,

$$\begin{aligned}\dot{V} &= (P_2)_x\dot{x} + (P_2)_w\dot{w} + \frac{1}{f}\dot{w}\ddot{w} - \frac{1}{f^2}f'\dot{w}^2 \\ &= (P_2)_x\dot{x} + (P_2)_w\dot{w} - \frac{1}{f^2}f'\dot{w}^2 + \\ &+ \frac{1}{f}\left[-f\pi_w - g\pi_x + o(\|(x;w)\|^{\delta_2}\|(\dot{x};\dot{w})\|^2)\right]\dot{w}\end{aligned} \tag{19}$$

Thus, if we consider $\tilde{V} = \frac{V}{x^\beta}$, it follows that $\dot{\tilde{V}} = \frac{\dot{V}}{x^\beta} - \frac{\beta\dot{x}V}{x^{\beta+1}}$. Therefore, by using (19), we get

$$x^{\beta+1}\dot{\tilde{V}} = x\dot{V} - \beta V\dot{x} = (x(P_2)_x - \beta V)\dot{x} + S(x;w;\dot{x};\dot{w}), \tag{20}$$

where

$$S(x;w;\dot{x};\dot{w}) = \tfrac{x\dot{w}}{f}\left[-f\pi_w - g\pi_x + o(\|(x;w)\|^{\delta_2}\|(\dot{x};\dot{w})\|^2)\right].$$

These calculations will be fundamental.

We will work in the cone C given by:

$$C = \left\{(x;w;\dot{x};\dot{w}) \in \mathcal{F}\times\mathbf{R}^2\colon \begin{array}{c} E(x;w;\dot{x};\dot{w}) = 0,\ 0 < x < \sigma_1 < \sigma, \\ (x;w)\in C_1,\ V(x;w;\dot{x};\dot{w}) < \delta x^\beta \end{array}\right\} \tag{21}$$

where δ, σ and C_1 are given in lemma 1 and σ_1 is a number yet to be determined.

We prove now an important inequality, expressed in the following result.

Lemma 2 *There is $\varepsilon_0 > 0$ such that, if $0 < \varepsilon < \varepsilon_0$ there exists $\varrho > 0$ such that if $\|(x;w)\| \leq \varrho$ and $(x;w;\dot{x};\dot{w}) \in C$ then*

$$\frac{1}{(1+\varepsilon)}x^\beta \leq \dot{x}^2 \leq \frac{4}{1-\varepsilon}x^\beta \tag{22}$$

Proof: Let $\varepsilon > 0$. Since $B(0) = Id$, there is a neighborhood of the origin in which

$$(1-\varepsilon)\frac{\dot{x}^2+\dot{w}^2}{2} \leq T(x;w;\dot{x};\dot{w}) \leq (1+\varepsilon)\frac{\dot{x}^2+\dot{w}^2}{2}. \tag{23}$$

Since the energy in C is null, we have that $(1+\varepsilon)\frac{\dot{x}^2+\dot{w}^2}{2}+\pi(x;w)>0$. This and $(*)$ show that, for $(x;w;\dot{x};\dot{w})\in C$

$$(1+\varepsilon)\frac{\dot{x}^2}{2}>-\left[-x^\beta+P_2(x;w)+o(\|(x;w)\|^\beta)+(1+\varepsilon)\frac{\dot{w}^2}{2}\right]. \qquad (24)$$

Now, we analyze V and note that, in C, $V(x;w;\dot{x};\dot{w})=\frac{\dot{w}^2}{2f}+P_2(x;w)<\delta x^\beta$. Hence, using the estimate $(**)$ for f, we can choose $\varepsilon_1>0$ and $\widetilde{\sigma}$ so that, if $0<x<\widetilde{\sigma}$, the inequality $\frac{\dot{w}^2}{2}<f(\delta x^\beta-P_2(x;w))<(1+\varepsilon_1)(\delta x^\beta-P_2(x;w))$ holds. Let us choose $\varepsilon_1>0$ so that $\varepsilon+\varepsilon_1+\varepsilon\varepsilon_1<2\varepsilon$.

If we multiply the last inequality by $(1+\varepsilon)$ and replace it in (24), we get

$$\begin{aligned}\tfrac{1+\varepsilon}{2}\dot{x}^2 &> x^\beta-P_2(x;w)+(1+\varepsilon)(1+\varepsilon_1)(P_2(x;w)-\delta x^\beta)+o(\|(x;w)\|^\beta)=\\ &= x^\beta-\delta(1+\varepsilon)(1+\varepsilon_1)x^\beta+(\varepsilon+\varepsilon_1+\varepsilon\varepsilon_1)P_2(x;w)+o(\|(x;w)\|^\beta)=\\ &= (1-\delta)x^\beta-(\delta x^\beta-P_2(x;w))(\varepsilon+\varepsilon_1+\varepsilon\varepsilon_1)+o(\|(x;w)\|^\beta). \qquad (25)\end{aligned}$$

Since in C we have $\delta x^\beta>P_2(x;w)$, if we recall the manner in which we chose ε_1 and use that $1-\delta>\frac{3}{4}$ and (25), we may conclude that

$$\frac{1+\varepsilon}{2}\dot{x}^2>\frac{3}{4}x^\beta+2\varepsilon(P_2(x;w)-\delta x^\beta)+o(\|(x;w)\|^\beta). \qquad (26)$$

Now, we recall that there is $\sigma>0$ such that $P_2(x;w)>0$ if $0<x<\sigma$, thus, it follows from (26) that

$$\tfrac{1+\varepsilon}{2}\dot{x}^2>(\tfrac{3}{4}-2\delta\varepsilon)x^\beta+o(\|(x;w)\|^\beta).$$

We can take $\varepsilon_0>0$ such that $\frac{3}{4}-2\delta\varepsilon_0>\frac{1}{2}$. Clearly, for every $\varepsilon\le\varepsilon_0$, we may choose $\varrho_1>0$ (with $\varrho_1\le\sigma$) such that, for all the points in C where $\|(x;w)\|\le\varrho_1$ the left inequality at (22) is satisfied.

As to the other inequality, it follows from (23) that, restricted to C, $\frac{1-\varepsilon}{2}\dot{x}^2+\frac{1-\varepsilon}{2}\dot{w}^2+\pi(x;w)<0$, therefore

$$\frac{1-\varepsilon}{2}\dot{x}^2<-[\pi(x;w)+\frac{1-\varepsilon}{2}\dot{w}^2]\le-\pi(x;w). \qquad (27)$$

Since $P_2(x;w)\ge 0$, $(*)$ implies that, in C, $\pi(x;w)\ge -x^\beta+o(\|(x;w)\|^\beta)$.

Therefore, there is $\varrho_2>0$ such that, if $\|(x;w)\|\le\varrho_2$ and $(x;w;\dot{x};\dot{w})\in C$, we have $-\pi(x;w)\le 2x^\beta$.This, alongside (27), establishes the right inequality at (22).

So, it suffices to take $\varrho=\min\{\varrho_1,\varrho_2\}$ and the result follows. ■

A last lemma is necessary to prove our instability result.

Lemma 3 *There is σ_1, $0 < \sigma_1 < \sigma$ such that, in $\partial C \setminus (0;0) \cap \{(x;w) : |w| \leq \lambda x\}$ the function $\dot{\tilde{V}}$ does not vanish.*

Proof: It suffices to show that $x^{\beta+1}\dot{\tilde{V}}$ does not vanish because $x > 0$ in $\partial C \setminus (0;0) \cap \{(x;w) : |w| \leq \lambda x\}$.

By formula (20) $x^{\beta+1}\dot{\tilde{V}} = x\dot{V} - \beta \dot{x} V$.

Let us analyze the parcel $x\dot{V}$; we know, by (19), that

$$x\dot{V} = x[(P_2)_x \dot{x} + (P_2)_w \dot{w} - \frac{1}{f^2}\dot{f}\dot{w}^2 - \tilde{\pi}_w \dot{w} + R(x;w;\dot{x};\dot{w})],$$

where $R(x;w;\dot{x};\dot{w}) = \frac{1}{f}[-g\tilde{\pi}_x + o(\|(x;w)\|^{\delta_2}\|(\dot{x};\dot{w})\|^2)]\dot{w}$.

Now, we note that:

(i) Since, restricted to C, the energy is null, it follows from lemma 2 and from the definition of C_1 that $\dot{w}$ is $O(x^{\frac{\beta}{2}})$ at the border of C;

(ii) From (2) we have that $\delta_2 > -1$, and this implies that $\frac{x}{f}o(\|(x;w)\|^{\delta_2}\|(\dot{x};\dot{w})\|^2)$ is $o(x^\beta)$ in ∂C;

(iii) Given that $g(0;0) = 0$, from $P = j^k\tilde{\pi}$ and (8) it follows that, in ∂C, $xg\tilde{\pi}_x$ is $o(x^\beta)$.

(iv) From(ii) and (iii) we have, at the border of C, $xR = \dot{w}R_1$, where R_1 is $o(x^\beta)$, thus, from (i) follows that xR is $o(x^{\frac{3\beta}{2}})$ in ∂C;

(v) Since $\dot{f} = (f_x)\dot{x} + (f_w)\dot{w}$, lemma 2 and (i) imply that $\dot{f}\dot{w}^2$ is $O(x^{\frac{3\beta}{2}})$ in ∂C, and so the parcel $\frac{x}{f^2}\dot{f}\dot{w}^2$ is $o(x^{\frac{3\beta}{2}})$, because $f(0;0) = 1$.

This leave us with the analysis of $x[(P_2)_x\dot{x} + (P_2)_w\dot{w} - \tilde{\pi}_w\dot{w}] - \beta\dot{x}V$.

Once again, we use $P = j^k\tilde{\pi}$ and (8) to show that, at the border of C, we have $\tilde{\pi}_w = (P_2)_w + o(x^{\beta-1})$ and so, due to (i),

$$x[(P_2)_x\dot{x} + (P_2)_w\dot{w} - \tilde{\pi}_w\dot{w}] - \beta\dot{x}V = (x(P_2)_x - \beta V)\dot{x} + o(x^{\frac{3\beta}{2}}). \qquad (28)$$

Note that:

(vi) In ∂C, $V = \delta x^\beta$ and $(x;w) \in \overline{\Omega}_1$, so it follows from lemma 1 that, $x(P_2)_x < (k-1)\delta x^\beta$;

(vii) Since $\beta > k-1$ we have that, in ∂C, $x(P_2)_x - \beta V < 0$ and $|x(P_2)_x - \beta V| \geq \delta x^\beta$.

Therefore, lemma 2 and (vii) imply that if $(x;w;\dot{x};\dot{w}) \in \partial C \setminus (0;0)$ we have

$$|(x(P_2)_x - \beta V)\dot{x}| \geq \delta x^{\beta}|\dot{x}| \geq \frac{\delta}{(1+\varepsilon)} x^{\frac{3\beta}{2}}. \tag{29}$$

Clearly, from (iv), (v), (28) and (29) the result follows. ■

Now, we can state the main result of this paper.

Theorem 1 *Let π be a C^2 function such that $\pi(0) = \|\nabla\pi(0)\| = 0$ and $\nabla\pi$ has jet of order $k-1$ at 0 and suppose that $j^k\pi$ shows that the origin is not a local minimum of π. Then there is a trajectory $\phi(t)$ asymptotic to the origin $(0;0)$, solution of (3), such that for sufficiently large $t > t_0$, $\phi(t) \in C \setminus (0;0) \cap \{(x;w;\dot{x};\dot{w}) : |w| \leq 2x\}$.*

Proof: For the simplicity of notation, we will call Ω_1 the set

$$\Omega_1 = C \setminus (0;0) \cap \{(x;w;\dot{x};\dot{w}) : |w| < 2x \text{ e } \dot{x} < 0\}.$$

Let us recall that, from lemma (2), if we choose a sufficiently small σ_1, then for $0 < x < \sigma_1$, $\dot{x} < -\frac{1}{2}x^{\frac{\beta}{2}}$.

Also, it follows from the definition of C that, if we have $|x|$ sufficiently small and if $(x;w;\dot{x};\dot{w}) \in \Omega_1$, then

$$|\dot{w}| < \frac{4}{3}\delta x^{\frac{\beta}{2}} < 2|\dot{x}|. \tag{30}$$

These two statements show that, for any solution $\overline{\phi}(t) = (\overline{x}(t);\overline{w}(t))$ of (3) , if $\overline{\phi}(t_1) \in \overline{\Omega}_1$ and $|\overline{w}(t_1| = 2\overline{x}(t_1)$ for some instant t_1, then there is ε_1 such that for every t in $[t_1, t_1 + \varepsilon_1)$, we have $\overline{\phi}(t) \notin \overline{\Omega}_1$, and for every t in $[t_1 - \varepsilon_1, t_1)$, we have $|\overline{w}(t| < 2\overline{x}(t)$.

On the other hand, a direct consequence of lemma (3) is that, if $\overline{\phi}(t_2) \in \overline{\Omega}_1$ and $V \circ \overline{\phi}(t_2) = \delta\overline{x}(t_2)^{\beta}$ for some instant t_2, then again we may find ε_2 such that for every t in $[t_2, t_2 + \varepsilon_2)$, we have $V \circ \overline{\phi}(t_2) > \delta\overline{x}(t_2)^{\beta}$, and for every t in $[t_2 - \varepsilon_2, t_2)$, we have $V \circ \overline{\phi}(t_2) < \delta\overline{x}(t_2)^{\beta}$.

Therefore, there is a constant $\sigma_2 < \sigma_1$ such that every solution $\overline{\phi}$ with $0 < \overline{x} < \sigma_2$ that at some instant $\overline{t}$ is in the border of Ω_1 must have been in the relative interior of Ω_1 if for some time interval before $\overline{t}$. Also, this solution is going to be out of $\overline{\Omega}_1$ for a time interval posterior of $\overline{t}$.

Now, we take a sequence $p_k = (x_k;w_k;\dot{x}_k;\dot{w}_k) \in \partial\Omega_1$, with $0 < x_k < \sigma_2$ and such that $lim_{k\to\infty}p_k = (0;0)$, and we consider the solutions ϕ_k of (3) such that $\phi_k(0) = p_k$.

Clearly, for some negative time these solutions have been in Ω_1. Also, they cannot have entered Ω_1 at a point with $x < \sigma_2$. Finally, since in Ω_1 $\dot{x} < -\frac{1}{2}x^{\frac{\beta}{2}}$, there are sequences $t_k < 0$ and $q_k = (\sigma_2;\tilde{w}_k;\tilde{\dot{x}}_k;\tilde{\dot{w}}_k) \in \Omega_1$, such that $\phi_k(t_k) = q_k$ and $\phi_k(t) \in \Omega_1$, for every $t \in (t_k, 0)$.

Of course, we can find a sub-sequence q_{n_k} that converges to a point $\overline{q}$. We will assume that $q_{n_k} = q_k$.

We claim that the solution $\phi(t)$ of (3) starting at $\overline{q}$ is asymptotic to (0;0).

For, if we assume by contradiction that was not the case, then there would be a time t_0 such that $\phi(t_0) \notin \overline{\Omega}_1$.

Let $a = min_{t\in[0,t_0]}\|\phi(t)\|$. Since q_k converges to $\overline{q}$, the continuous dependence gives us that there is k_0 such that, for $k > k_0$

$$min_{t\in[0,t_0]}\|\phi(t) - \phi_k(t_k + t)\| < \frac{a}{2}.$$

This implies that, for $k > k_0$, $t_0 < -t_k$. On the other hand, since $\overline{\Omega}_1$ is closed, the continuous dependence also assures the existence of k_1 such that, for $k > k_1$,

$$\phi_k(t_k + t_0) \notin \overline{\Omega}_1.$$

But, if $k > max\{k_0,k_1\}$, this implies that for some $t \in (t_k,t_k + t_0)$ $\phi_k(t) \notin \Omega_1$, which is absurd, since $t_k + t_0 < 0$, thus concluding the proof. ■

As a final note we would like to remark that if $j^k\pi$ does not show that π does not have a local minimum at the origin, then there is a function $q\colon\Omega \longrightarrow \mathbf{R}$ with $j^kq = 0$ and such that $\pi_1 = j^k\pi + q$ has a local strict minimum at 0. Thus, by the Lagrange-Dirichlet theorem the equilibrium (0;0) is a stable point for the equations of motion of Lagrangian $\mathcal{L} = T - \pi_1$.

Therefore, in the class of functions of $\mathbf{R}^2$ of class C^2 with jet of order k at 0, we have given in theorem 1 a complete characterization of jets that ensure the instability of the equilibrium.

APPENDIX

k-Decidability and Vertical Minima Curve

In this section we show that if $f\colon\Omega \longrightarrow \mathbf{R}$ is a function defined in the open neighborhood of the origin Ω of $\mathbf{R}^2$, $f(0) = 0$ then:

1. If f is C^1 and ∇f has punctual jet of order $k-1$ at the origin then f has punctual jet of order k at the origin and $\nabla j^kf = j^{k-1}(\nabla f)$.

2. If f has punctual jet of order k at the origin such that $j^k f$ shows that f does not have a minimum at 0 and $j^{k-1}f$ does not show that f does not have a minimum, then there is a vertical minima curve for f with the proprieties described in the text.

The first result is a simple calculus' result.

Lemma 4 *Let $U = \mathring{U} \subseteq \mathbb{R}^n$ be a neighborhood of O and take $f\colon U \longrightarrow \mathbb{R}$ of class C^1 such that ∇f has punctual jet of order $k-1$ at O. Then f has punctual jet of order k at O and $j^{k-1}\nabla f = \nabla j^k f$.*

Proof: We define the polynomial

$$P(q) = \int_0^1 \langle j^{k-1}\nabla f(tq)|q\rangle dt.$$

We claim that $P = (j^k f)|_U$.

Of course P has degree less then or equal to k and, if $q \in U \setminus \{O\}$,

$$\begin{aligned}\frac{|f(q)-P(q)|}{\|q\|^k} &= \left|\int_0^1 \frac{\langle \nabla f(tq) - j^{k-1}\nabla f(tq)|q\rangle}{\|q\|^k} dt\right| \leq \\ &\leq \int_0^1 t^{k-1}\frac{\|\nabla f(tq) - j^{k-1}\nabla f(tq)\|}{\|tq\|^{k-1}} dt.\end{aligned}$$

From the definition of $k-1$ jet, and from the last expression, it follows that $\lim\limits_{q\to O} \frac{|f(q)-P(q)|}{\|q\|^k} = 0$, and this proves that $P = j^k f$.

We take $R = f - P$, write $q = \sum\limits_{j=1}^{n} q_j e_j$ and define, for $1 \leq j \leq n$, the functions G_j and $\hat{R}_j$ by:

$$G_j = j^{k-1}\frac{\partial f}{\partial q_j} \text{ e } \frac{\partial f}{\partial q_j} = G_j + \hat{R}_j.$$

In order to finish our proof, it suffices to show that $G_j = \frac{\partial P}{\partial q_j}$.

Note that $\|q - q_j e_j + t e_j\| \leq \|q\|$, for $t \in [0;q_j]$. Therefore, since $j^{k-1}\hat{R}_j = 0$, we have that

$$j^k \int_0^{q_j} \hat{R}_j(q - q_j e_j + t e_j) dt = 0. \qquad (*)$$

Thus,

$$f(q) = f(q - q_je_j) + \int_0^{q_j} \frac{\partial f}{\partial q_j}(q - q_je_j + te_j)dt =$$
$$= (P+R)(q - q_je_j) + \int_0^{q_j} (G_j + \hat{R}_j)(q - q_je_j + te_j)dt.$$

Now, by using (*) and calculating the k jet of both members of the last expression, we get the result. ■

Now, we will prove the result about vertical minimum curves.

For the sake of simplicity we make, also in this appendix, the notation $P = j^k f$.

In [1] Barone proved that in these conditions there exists an algebraic curve $\widetilde{\Gamma}: [0;\varepsilon] \longrightarrow \Omega$

such that $\widetilde{\Gamma}(0) = (0;0)$, $\|\widetilde{\Gamma}(t)\| \neq 0$, if $t > 0$ (and therefore transversal to the circumferences centered at the origin) such that

$$P(\widetilde{\Gamma}(t)) = \min \left\{ P(x) : \|x\| = \|\widetilde{\Gamma}(t)\| \right\}, \text{ if } t > 0, \text{ and}$$

$$\frac{P(\widetilde{\Gamma}(t))}{\|\widetilde{\Gamma}(t)\|} = at^\beta + t.h.o., \text{ with } \beta \leq k \text{ and } a \neq 0.$$

We will show that there is a vertical minima curve with the desired proprieties tangent to $\widetilde{\Gamma}$ at the origin.

Let s be the order of the first non null jet of f. It is easy to see that $s \leq k$, $j^s f$ is a polynomial homogeneous of degree s and one of the following situations must happen:

1. $j^s f$ shows that f does not have a minimum at the origin.

2. $j^s f$ has a strictly weak minimum at 0.

In the first case, we have $s = k$ and the result we seek follows trivially. It is easy to see that the mentioned curve $\widetilde{\Gamma}$ is a semi straight line that satisfies the desired conditions.

We focus now on the situation in which $s < k$ and $j^s f$ is an homogeneous polynomial of degree s with a strictly weak minimum at the origin.

We recall that $j^s f^{-1}\{0\}$ is a finite union of lines all of which pass by the origin.

Lemma 5 *Let $\Lambda: [0;\varepsilon] \longrightarrow \mathbb{R}^2$ be an algebraic curve with $\Lambda(0) = 0$. If Λ is tangent at the origin to a semi straight line in which $j^s f$ is not null, then the order of P at Λ is s and $f \circ \Lambda$ has a strict local minimum at 0.*

Proof: Let ℓ be the semi-line to which Λ is tangent at the origin. We perform a rotation in order to transform ℓ in the x axis.

After this rotation it becomes clear that $j^s f(x;0) = ax^s$, with $a > 0$.

If we use now the canonical parameterization of Λ, the result follows immediately. ■

A direct consequence of this lemma is that the curve $\widetilde{\Gamma}$ above mentioned is tangent at the origin to one of the semi straight lines in which $j^s f$ is null. Let ℓ_0 be this semi line.

We are now able to finish the proof of the existence of the vertical minima curve in this case.

We consider the rotation that takes ℓ_0 onto the positive x axis, and we work in this new set of coordinates.

Since $j^s f$ is a non null homogeneous polynomial, with a strictly weak minimum at the origin, and since it is null on the positive x axis, we can choose $\lambda > 0$ such that in the cone $C_\lambda = \{(x;y) \in \mathbf{R}^2 : x \geq 0,\ -\lambda x \leq y \leq \lambda x\}$ the only points in which $j^s f$ is null are those of the semi-line $(x;0), x \geq 0$. From this and fact 5, it follows that both have a minimum of order s at the origin.

On the order hand, since $\widetilde{\Gamma}$ is tangent at the origin to the semi axis x, there is $\varepsilon_0 > 0$ such that $\widetilde{\Gamma}(t) \in \mathring{C}_\lambda$ if $0 < t < \varepsilon_0$.

Since $\widetilde{\Gamma}$ is algebraic and transversal to the circumferences centered at the origin, this curve is also transversal to the lines $x = x_0$, for sufficiently small values of x_0.

If we write $\widetilde{\Gamma}(t) = (x(t);y(t))$, we will see that, in $[0;\varepsilon_0[$, the function $x(t)$ is strictly increasing and by taking $\varrho = x(\varepsilon_0)$ we have that, for every ξ, $0 < \xi < \varrho$ there is a single $t_\xi \in (0;\varepsilon_0)$ such that $x_1(t_\xi) = \xi$ and there is $y_{x^i} \in [-\lambda\xi;\lambda\xi]$ such that

$$\min\left\{j^k f(\xi;y) : -\lambda\xi \leq y \leq \lambda\xi\right\} = f(\xi;y_\xi) \leq j^k f(\widetilde{\Gamma})(t_\xi) < 0. \qquad (31)$$

We recall that $j^k f(x;-\lambda x)$ and $j^k f(x;\lambda x)$ have a minimum at the origin, and so $y_\xi \in (-\lambda\xi;\lambda\xi)$.

This shows that $\frac{\partial j^k f}{\partial y}(\xi;y_\xi) = 0$ and therefore

$$A = \left\{(x;y) \in \mathbf{R}^2 : \frac{\partial j^k f}{\partial y}(x;y) = 0\right\}$$

is an algebraic set of dimension 1, of whom the origin is not an isolated point.

Thus, there is a neighborhood Δ of the origin, such that $A \setminus \{O\} \cap \Delta$ is a finite reunion of algebraic curves, all of which are adherent to the origin.

Further more, from (31) we have that there exists an algebraic curve Γ defined in $[0;\delta]$, such that $\Gamma(0) = 0$, $\Gamma(t) \in \mathring{C}_\lambda \cap A$, if $t > 0$.

Let$\Gamma_j : [0;\delta_j] \longrightarrow \mathbf{R}^2$, $1 \leq j \leq p$ be the algebraic curves that satisfy the conditions described in the last paragraph and let us consider them with their canonical parameterization. We take $\delta_0 = \min \delta_j : 1 \leq j \leq p$.

Then, for $1 \leq j \leq p$, $\Gamma_j(t) = (t;h_j(t))$, $0 \leq t \leq \delta_j$, where h_j is a function with a power series representation with fractionary exponent.

So, by making $f_j = f \circ \Gamma_j$, there is $j_0 \in \{1, \ldots, p\}$,

such that $f_{j_0}(x) \leq f_j(x)$, for all j and $0 \leq x \leq \delta_0$.

It is clear that Γ_{j_0} is the desired curve.

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