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Maps of  $\mathbb{R}^2$

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# On Locally Diffeomorphic Polynomial Maps of $\mathbb{R}^2$

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## Resumo

Se consideram aplicações polinomiais não singulares  $F = (P, Q)$  do plano real nele próprio, que satisfazem a seguinte condição  $J$  de regularidade no infinito: “Não existe nenhuma seqüência  $S$  de pares de números complexos tal que a seqüência  $T$  do plano real, formada pelas partes reais das coordenadas dos elementos da seqüência  $S$  tende para infinito e a sequência  $F(T)$  tende para algum valor  $a$  do plano real”. Se prova que  $F$  é um difeomorfismo global do plano real, sempre que ele satisfizer a Condição  $J$  e, além disso, sempre que a restrição de  $F$  a todo nível real  $\{P = c\}$  for uma aplicação própria, para valores  $|c|$  suficientemente grandes.

# On Locally Diffeomorphic Polynomial Maps of $\mathbb{R}^2$

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## Abstract

We consider nonsingular polynomial maps  $F = (P, Q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  under the following regularity condition at infinity ( $J_\infty$ ): **There does not exist a sequence  $\{(p_k, q_k)\} \subset \mathbb{C}^2$  of complex singular points of  $F$  such that the imaginary part  $(\Im(p_k), \Im(q_k))$  tends to  $(0, 0)$ , the real part  $(\Re(p_k), \Re(q_k))$  tends to  $\infty$  and  $F(\Re(p_k), \Re(q_k)) \rightarrow a \in \mathbb{R}^2$ .** It is shown that  $F$  is a global diffeomorphism of  $\mathbb{R}^2$  if it satisfies Condition ( $J_\infty$ ) and, in addition, if the restriction of  $F$  to every real level  $P^{-1}(c)$  is proper for values  $|c|$  large enough.

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## 1. Introduction.

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This paper is addressed to the question of whether a polynomial map  $F = (P, Q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  which is a local diffeomorphism is also a global diffeomorphism. Pinchuck's example [P] showing the existence of polynomial local diffeomorphisms which are not global diffeomorphism proves the necessity of some extra condition. In this paper, we propose Condition  $(J_\infty)$  below as one we believe to be necessary and sufficient to ensure that a polynomial map, which is a local diffeomorphism, be a global diffeomorphism. In this way, we provide a positive answer in the most simple (but not trivial) topological situation. Everything indicates that the very important problem of characterizing polynomial maps of the real plane that are global diffeomorphisms is very difficult; this explains why we will need to use a lot of real geometry. Condition  $(J_\infty)$  ensures the nice feature that the complex singularities, of the complexification of a polynomial map defined in the real plane, do not have direct influence in the behavior at infinity of the considered map. Certainly, polynomial maps satisfying the well known constant Jacobian Keller condition satisfy the  $(J_\infty)$  condition which we define now.

For a complex point  $p = (a + ib, c + id) \in \mathbb{C}^2$ ,  $a, b, c, d \in \mathbb{R}$ , we denote  $\Im(p) = (b, d)$  and  $\Re(p) := (a, c)$ , the imaginary part and the real part of  $p$ . The condition  $(J_\infty)$  can be formulated as:

$(J_\infty)$  There does not exist a sequence  $\{p_k\} \subset \mathbb{C}^2$  of singular points of the complexification  $\tilde{F} : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  of  $F$  such that  $\Im(p_k) \rightarrow (0, 0)$ ,  $\Re(p_k) \rightarrow \infty$  and  $F(\Re(p_k)) \rightarrow a \in \mathbb{R}^2$ .

**Theorem 1.1.** *Suppose  $F = (P, Q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a nonsingular polynomial map satisfying Condition  $(J_\infty)$ . Then,  $F$  is a global diffeomorphism of  $\mathbb{R}^2$  provided that, for all  $|c| > 0$  large enough, either*

- (i)  $P^{-1}(c)$  is connected or
- (ii)  $F|_{P^{-1}(c)}$  is proper.

This theorem improves the main result of [CG], where the non-zero constant Jacobian case was considered. Its proof, presented in §3, is based on the examination, in §2, of the behavior at infinity of nonsingular polynomial maps of  $\mathbb{R}^2$  satisfying  $(J_\infty)$ . Theorem 1.1 is not valid for analytic maps of  $\mathbb{R}^2$ . Indeed, in [CG], it is constructed a non-zero constant Jacobian analytic map  $F = (P, Q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  which is non-injective, non-surjective, at most 2-to-1 and such that for all  $|c|$  large enough,  $P^{-1}(c)$  is connected and  $F|_{P^{-1}(c)}$  is proper. This example will be presented in §4. In that section we will also see that Pinchuck's example [P] does not satisfy Condition  $(J_\infty)$  and can be modified so that the resulting map

$F = (P, Q)$  is a non-injective nonsingular polynomial map which, when restrictive to every level  $P^{-1}(c), Q^{-1}(c)$ , is proper for  $c \ll 0$ .

Before continuing, we wish to mention some results related to ours. Fernandes, Gutierrez and Rabanal [FGR] (see also [CGL]) obtained that if  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a differentiable map (not necessarily  $C^1$ ) and if, for some  $\epsilon > 0$ ,  $\text{Spec}(f) \cap [0, \epsilon) = \emptyset$ , then  $f$  is injective; here  $\text{Spec}(f)$  denotes the set of (complex) eigenvalues of the derivative  $Df(x)$  when  $x$  varies in  $\mathbb{R}^2$ . Campbell [Ca1] classified the  $C^1$  maps  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$  whose eigenvalues are both 1; all such maps are diffeomorphisms having explicit inverse.

## 2. Condition $(J_\infty)$ .

In the following, the Euclidean space  $\mathbb{R}^2$  will be viewed as a subset of  $\mathbb{C}^2$ . Let  $F = (P, Q): \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a polynomial map which is dominant (i.e.  $F(\mathbb{R}^2)$  is an open set). Recall that the *non-proper value set*  $A_F$  of  $F$  is the set of all values  $a \in \mathbb{R}^2$  which have no neighborhood with compact inverse image under  $F$ . It is well-known that the non-proper value set  $A_F$ , if non-empty, is composed by the images of a finite number of non-constant polynomial maps from  $\mathbb{R}$  into  $\mathbb{R}^2$  (See for example [J]). When  $F$  is locally homeomorphic, the map  $F: \mathbb{R}^2 \setminus F^{-1}(A_F) \rightarrow \mathbb{R}^2 \setminus A_F$  is an unbranched covering and  $A_F$  is just the discontinuity set of the integer-valued function  $\#F^{-1}(a)$  defined on  $\mathbb{R}^2$ . When  $A_F = \emptyset$ ,  $F$  is a homeomorphism of  $\mathbb{R}^2$ . We start by describing both the behavior of  $F$  at infinity and the set  $A_F$  by means of the Newton-Puiseux expansion. Let  $(u_0, v_0) \in A_F$  be a smooth point of  $A_F$ . Take a line segment  $L$  that intersects transversally  $A_F$  at the point  $(u_0, v_0)$ . Then, the inverse image  $F^{-1}(L)$  have some branches at infinity along which  $F$  tends to the value  $(u_0, v_0)$ . Assume that  $\gamma \subset F^{-1}(L)$  be such a branch. In suitable linear coordinates of  $\mathbb{R}^2$  the branch  $\gamma$  can be given by  $(\mathbb{R}, +\infty) \ni x \mapsto (x, \gamma(x)) \in \gamma$  with a Newton-Puiseux expansion at infinity

$$\gamma(x) = \sum_{k=0}^{\infty} c_k x^{1-\frac{k}{m}},$$

where  $c_k \in \mathbb{R}$ ,  $\gcd(\{k : c_k \neq 0\} \cup \{m\}) = 1$  and the series  $\sum_k c_k t^k$  is absolutely convergent in the complex disk  $|t| < \epsilon$ . Then, following [C] we can find a unique finite fractional series  $\varphi(x, \xi)$  with parameter  $\xi$  such that  $\gamma(x) = \varphi(x, \xi_0) + \text{lower terms in } x$  and

$$F(x, \varphi(x, \xi)) = (P_\varphi(x, \xi), Q_\varphi(x, \xi)) = (p_\varphi(\xi), q_\varphi(\xi)) + \text{lower terms in } x,$$

with  $p_\varphi, q_\varphi \in \mathbb{R}[\xi]$  and  $\max(\deg p_\varphi, \deg q_\varphi) > 0$ . Such a fractional power series  $\varphi(x, \xi)$  is called *real dicritical series* of  $F$ . Let us represent  $\varphi(x, \xi)$  in the normal form

$$\varphi(x, \xi) = \sum_{j=0}^{n_\varphi-1} a_j x^{1-\frac{j}{m_\varphi}} + \xi x^{1-\frac{n_\varphi}{m_\varphi}}, \quad \gcd(\{j : a_j \neq 0\} \cup \{m_\varphi, n_\varphi\}) = 1$$

and define the maps  $\Phi : F_\varphi : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}^2$  by  $\Phi(t, \xi) := (t^{-m_\varphi}, \varphi(t^{-m_\varphi}, \xi))$  and  $F_\varphi := F \circ \Phi$ . One may easily check the following properties:

i)  $\Phi$  is an analytical homeomorphism from  $\mathbb{R}^+ \times \mathbb{R}$  onto its image  $U_\varphi := \Phi(\mathbb{R}^+ \times \mathbb{R})$  which is a neighborhood of the considered branch  $\gamma$ .

ii)  $F_\varphi$  is a polynomial map in  $(t, \xi)$  and the image of the polynomial map  $f_\varphi(\xi) = (p_\varphi(\xi), q_\varphi(\xi))$  contains the considered value  $(u_0, v_0)$  and is an irreducible component of  $A_F$ .

In other words, the map  $\Phi$  gives local analytic coordinates  $(t, \xi)$  in the neighborhood  $U_\varphi$  of the branch  $\gamma$  and the polynomial map  $F_\varphi$  gives a representation of  $F$  in these coordinates.

As noted above, the set  $A_F$ , if non-empty, is a union of the images of a finite number of non-constant polynomial maps from  $\mathbb{R}$  into  $\mathbb{R}^2$ . In this way one can construct a finite family  $\Lambda$  of dicritical series  $\varphi$  such that  $A_F = \cup_{\varphi \in \Lambda} f_\varphi(\mathbb{R})$  and the corresponding family of open sets  $\{\Phi(\mathbb{R}^+ \times \mathbb{R})\}$  covers all curve branches at infinity along which  $F$  tends to a value in  $A_F$ . This structure can be used to examine the behavior at infinity of  $F$ . In the following we will use this structure to study the geometry of a nonsingular polynomial map  $F$  satisfying Condition  $(J_\infty)$ .

Recall that the *exceptional value set*  $E_h$  of a polynomial  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  is the smallest subset of  $\mathbb{R}$  such that the restriction  $h : \mathbb{R}^2 \setminus h^{-1}(E_h) \rightarrow \mathbb{R} \setminus E_h$  is a locally trivial fibration. When  $E_h = \emptyset$ , the map  $h$  gives a trivial fibration with fiber homeomorphic to  $\mathbb{R}$ . By a *local irreducible complex branch* of a real curve  $V \subset \mathbb{R}^2$  we mean a complexification of a local irreducible real branch of  $V$ . With this meaning, a local irreducible complex branch of  $A_F$  can be seen as the image  $f_\varphi(D(\xi_0, \epsilon))$  of a small disk  $D(\xi_0, \epsilon) := \{\xi \in \mathbb{C} : |\xi - \xi_0| < \epsilon\}$ , where  $\xi_0 \in \mathbb{R}$  and  $\varphi$  is a dicritical series of  $F$ .

**Theorem 2.1.** *Suppose  $F = (P, Q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a nonsingular polynomial map satisfying Condition  $(J_\infty)$ . If the complex line  $L := \{u = c\}$  intersects transversally to all local irreducible complex branches of  $A_F$  located at  $A_F \cap L$ , then  $c \notin E_P$ .*

To prove this theorem we need the following elementary fact

**Lemma 2.2** *Let  $f(t, \xi) = (u, v)$  be a holomorphic map from a neighborhood  $U \subset \mathbb{C}^2$  of  $(0, 0)$  into  $\mathbb{C}^2$ ,  $f(t, \xi) = (p(\xi), q(\xi)) + \text{higher terms in } t$ ,  $f(0, 0) = (0, 0)$ . Assume that the line  $u = 0$  intersects transversally the local branch  $\Gamma := f(\{0\} \times D)$ ,  $D := \{\xi \in \mathbb{C} : |\xi| < \delta\}$  for enough small  $\delta > 0$ , and that  $\{(t, \xi) \in U : \det Df(t, \xi) = 0\} \subset \{(t, \xi) \in U : t = 0\}$ . Then,  $\dot{p}(0) \neq 0$ .*

*Proof.* Since the line  $u = 0$  intersects transversally  $\Gamma$  at  $(0, 0)$ , we can assume that  $\Gamma$  is a smooth branch of a curve parameterized by  $v = h(u)$ , where  $h$  is an analytic diffeomorphism, with  $h(0) = 0$ , defined in a small neighborhood of 0. Define the new coordinates  $(\bar{u}, \bar{v}) = (u, v - h(u))$  in a neighborhood of  $(0, 0)$ . Let  $\bar{f}$  be the representation of  $f$  in these coordinates. Then,

$$\bar{f}(t, \xi) = (p(\xi), 0) + \text{higher terms in } t,$$

$\bar{f}(0, 0) = (0, 0)$ ,  $\bar{f}(\{t = 0\}) \subset \Gamma = \{\bar{v} = 0\}$  and  $\det D\bar{f}(t, \xi) \neq 0$  for  $t \neq 0$ . By examining Newton diagrams of  $\bar{f}_1$ ,  $\bar{f}_2$  and  $\det D\bar{f}$  we can verify that

$$\bar{f}(t, \xi) = (\xi u_1(t, \xi) + t u_2(t, \xi), t^k v_1(t, \xi)),$$

where  $u_1$ ,  $u_2$  and  $v_1$  are holomorphic functions defined in  $U$ ,  $u_1(0, 0) \neq 0$  and  $v_1(0, 0) \neq 0$  (See, for example [O, Lemma 4.1]). It follows that  $\dot{p}(0) = u_1(0, 0) \neq 0$ . ■

*Proof of Theorem 2.1.* Let  $c \in \mathbb{R}$ . Assume that the complex line  $L := \{u = c\}$  intersects transversally to all local irreducible complex branches of  $A_F$  located at  $A_F \cap L$ .

We will see that it is enough to construct a smooth vector field  $V$  on the neighborhood  $P^{-1}(\Delta)$  for an enough small open interval  $\Delta \ni c$  with two properties:  $\langle \text{Grad}P(z), V(z) \rangle = 1$  and the solutions of the differential equation  $\dot{z} = V(z)$  do not tend to infinity. In fact, if  $\Psi(z, t)$  is the local flow induced by such a vector field  $V$ , then,  $\Psi : P^{-1}(c) \times \Delta \rightarrow P^{-1}(\Delta)$  is well defined and it is a diffeomorphism satisfying  $P(\Psi(z, t)) = c + t$ . Hence,  $P : P^{-1}(\Delta) \rightarrow \Delta$  is a trivial fibration and, by definition,  $c \notin E_P$ .

To construct the vector field  $V$  as above, first we consider a given branch at infinity  $\gamma$  of the real curve  $P = c$ . Let  $\varphi$  be the dicritical series with coordinates  $(t, \xi)$  associated to  $\gamma$ . Assume that in these coordinates  $\gamma$  locates at the point  $(0, \xi_0)$ . Taking the derivative of  $F_\varphi$  we have

$$\det DF_\varphi(t, \xi) = -m_\varphi \det DF(\Phi(t, \xi)) t^{n_\varphi - 2m_\varphi - 1}$$

The condition  $(J_\infty)$  ensures that  $\det DF_\varphi(t, \xi) \neq 0$ , for  $|t| \neq 0$  small enough; in fact, otherwise, there would exist a holomorphic map  $\bar{\xi} : \{t \in \mathbb{C} : |t| < \epsilon\} \rightarrow \mathbb{C}$  with  $\bar{\xi}(0) = \xi_0$  such that  $\det DF(\Phi(t, \bar{\xi}(t))) \equiv 0$ . Therefore,  $\Re(\Phi(t, \bar{\xi}(t))) = \Phi(t, \xi_0) + \text{higher terms in } t$  for  $t \in \mathbb{R}$  and  $F(\Re(\Phi(t, \bar{\xi}(t))))$  tends to  $(p_\varphi(\xi_0), q_\varphi(\xi_0)) \in \mathbb{R}^2$  as  $t$  tends to the zero. Hence, we can apply Lemma 2.1 to determine the positive numbers  $\alpha$  and  $\beta$  and the neighborhood  $W_\gamma = \{(t, \xi) \in \mathbb{R}^2 : 0 < t < \alpha, |\xi - \xi_0| < \beta\}$  of  $\gamma$  such that

$$\dot{p}_\varphi(\xi) \neq 0 \text{ for } |\xi - \xi_0| < \beta$$

$$\frac{\partial}{\partial \xi} P_\varphi(t, \xi) \neq 0 \text{ for } (t, \xi) \in W_\gamma.$$

The property we need here is that in  $W_\gamma$  the motions in the direction  $V_\gamma := (\frac{\partial}{\partial \xi} P_\varphi, 0)$  can not tend to the line  $t = 0$  and  $\langle \text{Grad}P(z), V_\gamma(z) \rangle \neq 0$  for  $z \in W_\gamma$ .

Now, making the above process for all branches at infinity  $\gamma$  of the real curve  $P = c$  we obtain a finite number of neighborhoods  $V_\gamma$  of  $\gamma$  and the vector fields  $W_\gamma$  defined on  $V_\gamma$ . Then, we can choose a real number  $R > 0$ , large enough and a small interval  $\Delta \ni c$  so that the family  $\{W_\gamma\}_\gamma$  together with the open ball  $B_R$  of radius  $R$  centered at  $(0, 0)$  is an open covering of  $P^{-1}(\Delta)$ . Let  $V_R(z) := \text{Grad}P(z)$  defined on  $B_R$ . Note that  $V_R(z) \neq 0$ , since  $F$  has not singularity. Then, by using a smooth partition of unity we can construct, from the fields  $V_\gamma$  and  $V_R(z)$ , a smooth vector field  $V(z)$  defined on  $P^{-1}(\Delta)$  with the desired properties. ■

The following is an immediate consequence of Theorem 2.1, which will be used to prove Theorem 1.1.

**Corollary 2.3.** *Suppose  $F = (P, Q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a nonsingular polynomial map satisfying Condition  $(J_\infty)$ . Then, the non-proper value set  $A_F$  can not be the finite union of lines and semi-lines.*

*Proof.* Assume by contradiction that  $A_F$  is the union of some lines and semi-lines. We can choose a direction  $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$  so that the lines  $au + bv = c$ ,  $c \in \mathbb{R}$ , intersect transversally all complexifications of the lines and semi-lines in  $A_F$ . Then, applying Theorem 2.1 to the pair  $(aP + bQ, Q)$  we get that the exceptional value set  $E_{aP+bQ}$  is empty and that the map  $aP + bQ : \mathbb{R}^2 \rightarrow \mathbb{R}$  gives a trivial fibration with fiber diffeomorphic to  $\mathbb{R}$ . As  $F$  has not singularity,  $F$  is monotone along fibers of  $aP + bQ$ . It

follows that  $F$  is injective. Hence,  $F$  is a diffeomorphism of  $\mathbb{R}^2$  and  $A_F = \emptyset$ , since every injective polynomial map of  $\mathbb{R}^2$  must be bijective [N,Ku]. This contradiction proves the corollary. ■

**Remark. 2.4.** In fact, in the mentioned representation  $A_F = \cup_{\varphi \in \Lambda} f_{\varphi}(\mathbb{R})$ , each component  $f_{\varphi}(\mathbb{R})$  can not be a semi-line. This can be easily proved by applying Lemma 2.2 to the map  $F_{\varphi}$ .

### 3. Proof of Theorem 1.1

This proof will follow from Corollary 2.3 and the following lemmas.

**Lemma 3.1.** ( Lemma 2.2 in [CG]) *Let  $F = (P, Q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a polynomial map. Assume that for every  $|c|$  large enough, the restriction of  $F$  to the level  $P^{-1}(c)$  of  $P$  is proper. Then, the non-proper value set  $A_F$  of  $F$ , if not empty, must be composed by some lines and semi-lines parallel to the vertical axis.*

*Proof.* Assume that for  $|c| > R > 0$  the restriction of  $F$ , to every level  $P^{-1}(c)$  of  $P$  is proper. From the definitions above, we can easily see that if  $L \subset \mathbb{R}^2$  is a line and the restriction of  $F$  on  $F^{-1}(L)$  is proper, then  $L \cap A_F = \emptyset$ . This implies that the non-proper value set  $A_F$  of  $F$  must be contained in the set  $\{(c, d) \in \mathbb{R}^2 : |c| \leq R\}$ . On the other hand, by Proposition 2.1,  $A_F$  is composed by the images of some non-constant polynomial maps  $(p, q) : \mathbb{R} \rightarrow \mathbb{R}^2$ . Thus, the first component of such polynomial maps must be constant, and hence,  $A_F$  must consists of some lines and semi-lines parallel to the vertical axis. ■

**Lemma 3.2.** *Let  $F = (P, Q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a nonsingular polynomial map. Let  $R$  be a positive number. If to every  $|c| > R$  the levels  $P^{-1}(c)$  is connected, then the restriction of  $F$  to every such level  $P^{-1}(c)$  of  $P$  is proper.*

*Proof.* Let  $F = (P, Q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a nonsingular polynomial map. Assume that for every  $|c| > R > 0$  the level  $P^{-1}(c)$  is connected. Let  $W := \{(c, d) \in \mathbb{R}^2 : |c| > R\}$ . Since  $Q$  is monotone along each connected component of a level of  $P$ ,  $F$  takes injectively  $F^{-1}(W)$  onto  $W$ . It follows from the definition that  $W \cap A_F = \emptyset$ . Therefore, for  $|c| > R$  the restriction of  $F$  to each level  $P^{-1}(c)$  of  $P$  is proper. ■

*Proof of Theorem 1.1.* Combining the lemmas 3.1 and 3. 2 and Corollary 2.3. ■.

#### 4. Discussions and Examples.

The following discussion and examples will clarify the statements and the corresponding assumptions of the preceding results.

##### (4.1) The condition $(J_\infty)$

This condition is a variant of the Jacobian condition,  $\det DF \equiv c \neq 0$ , for the real case, and ensures that the complex singularities of  $F$  do not directly influence the behavior of  $F$  at infinity. As a real analogue of the Jacobian Conjecture, it is natural to ask whether a non singular polynomial map of  $\mathbb{R}^2$  is a global diffeomorphism if it satisfies condition  $(J_\infty)$ .

##### (4.2) Condition (i) in Theorem 1.1.

It is clear that the restriction of  $F$  to the inverse image  $F^{-1}(\{(a, b) : \|a\| \gg 0\})$  is injective. For a locally diffeomorphic map  $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  let us denote by  $n(h)$  the minimal integer number  $k$  such that there exists an open disk  $D \subset \mathbb{R}^2$  with  $\#h^{-1}(p) \equiv k$  for all  $p \in D$ . Condition (i) implies that  $n(F) = 1$ . In the Pinchuk's example  $n(F) = 2$  (see [Ca2, Ca3]). We think that it should be true that a nonsingular polynomial  $F$  must be a global diffeomorphism if  $n(F) = 1$ .

##### (4.3) Pinchuk map and Condition $(J_\infty)$

In 1994 Pinchuk in [P] had found a non-singular polynomial map of  $\mathbb{R}^2$  that is not injective. Here, we will point that Pinchuk's map does not satisfy Condition  $(J_\infty)$ .

Pinchuk's map  $F = (P, Q)$  can be constructed in the following way ( see in [E]). Let  $g := xy - 1$ ;  $h := g(xg + 1)$  and  $f := (h + 1)/x(xg + 1)^2$ . then,

$$P := f + h$$

and

$$Q = -g^2 - 6gh(h + 1) - 170fh - 91h^2 - 195fh^2 - 69h^3 - 75h^3f - (75/4)h^4.$$

For such pair  $(P, Q)$  we have

$$\det(P, Q) = g^2 + (g + f(13 + 15h))^2 + f^2$$

which is always positive on  $\mathbb{R}^2$ .

To verify Condition  $(J_\infty)$  for  $(P, Q)$  we choose the function

$$\varphi(x, \xi) := x^{-1} + \xi x^{-3/2}.$$

Then, computing due to Maple V we can get

$$\begin{aligned} P(t^{-2}, \varphi(t^{-2}, \xi)) &= (\xi^4 + 2\xi^2) + t(3\xi^3 + 3\xi) + t^2(3\xi^2 + 1) + t^3\xi \\ Q(t^{-2}, \varphi(t^{-2}, \xi)) &= (-75\xi^{10} - \frac{1155}{4}\xi^8 - 434\xi^6 - 261\xi^4) + t(-698\xi^3 - 1673\xi^5 - \\ &1425\xi^7 - 450\xi^9) + t^2(-608\xi^2 - 2404\xi^4 - 5625/2\xi^6 - 1125\xi^8) + t^3(-170\xi - \\ &1535\xi^3 - 2775\xi^5 - 1500\xi^7) + t^4(-365\xi^2 - \frac{5475}{4}\xi^4 - 1125\xi^6) + t^5(-270\xi^3 - \\ &450\xi^5) - 75t^6\xi^4. \end{aligned}$$

Then, one can see that the exceptional complex curve of  $F_C$  has a component parameterized by  $\mathcal{C} \ni \xi \mapsto (\xi^4 + 2\xi^2, -75\xi^{10} - \frac{1155}{4}\xi^8 - 434\xi^6 - 261\xi^4)$ .

Now, considering the function  $\det DF$  along parameters  $\Phi(t, \xi) := (t^{-2}, \varphi(t^{-2}, \xi))$ . We have

$$\det D\Phi(t, \xi) = -2t^{-6}$$

and

$$\begin{aligned} &\det DF(\Phi(t, \xi)) \det D\Phi(t, \xi) \\ &= 706t^3\xi + 170\xi^4 + 706t\xi^3 + 1074t^2\xi^2 + 3706t\xi^5 + 7468t^2\xi^4 + 7468t^3\xi^3 + 730\xi^6 + \\ &1175\xi^8 + 840\xi^{10} + 225\xi^{12} + 170t^4 + 3706t^4\xi^2 + 7080\xi^7t + 17745\xi^6t^2 + 23680t^3\xi^5 + \\ &17745t^4\xi^4 + 7080t^5\xi^3 + 5880\xi^9t + 17640\xi^8t^2 + 29400\xi^7t^3 + 29400t^4\xi^6 + 17640t^5\xi^5 + \\ &5880t^6\xi^4 + 1800\xi^{11}t + 6300\xi^{10}t^2 + 12600\xi^9t^3 + 15750\xi^8t^4 + 12600t^5\xi^7 + 6300t^6\xi^6 + \\ &1800t^7\xi^5 + 1175t^6\xi^2 + 840t^7\xi^3 + 730t^5\xi + 225t^8\xi^4 \end{aligned}$$

This is a polynomial in  $\mathbb{R}[\xi][t]$  of the free coefficient

$$\xi^4(170 + 730\xi^2 + 1175\xi^4 + 840\xi^6 + 225\xi^8),$$

which vanishes at  $\xi = 0$  and, maybe, some other real values  $a_i$ . This follows that  $\det DF(\Phi(t, \xi)) = 0$  along some branches  $t \mapsto \xi_i(t) = -a_i t +$  terms of  $\deg t > 1$ . Thus,  $\det D(P, Q) = 0$  along some complex branches parameterized by  $t \mapsto (t^{-2}, t^2 + \xi_i(t)t^3)$ , where  $t$  is enough closed to the zero. Along such branches the map  $(P, Q)$  tend to a real value as  $t$  tends to the zero. Hence, we can take  $t$  belong to  $(0, \epsilon)$  to get the sequence for which Condition  $J_\infty$  does not hold.

**(4.4) Condition (ii) of Theorem 1.1.**

Related to this condition, we may apply the following proposition to the non-injective non-singular polynomial Pinchuk map [P].

**Proposition 4.1.** Suppose  $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$  is a non-injective nonsingular polynomial map with the closure of  $F(\mathbb{R}^2)$  equal to  $\mathbb{R}^2$ . Then there is a polynomial diffeomorphism  $\varphi$  of  $\mathbb{R}^2$  such that the restriction of  $\varphi \circ F = (P, Q)$  to every level  $P^{-1}(c), Q^{-1}(c)$  is proper for  $c \ll 0$ .

**Proof:** Since  $F$  is not injective the non-proper value set  $A_F$  is given by

$$A_F = \bigcup_{i=1}^n \{f_i(t) : t \in \mathbb{R}\}$$

for some polynomial maps  $f_i(t) = (p_i(t), q_i(t))$ . By using a linear change of coordinates, if necessary, we can assume that  $\deg p_i \geq 1$  and  $\deg q_i \geq 1$ . Choose a positive integer  $k$  such that  $2k > \deg p_i$ ,  $4k > \deg q_i$ , and define

$$\varphi(u, v) := (u + v^{2k}, v + (u + v^{2k})^2),$$

which is an automorphism of  $\mathbb{R}^2$ . Then, one can see that the branches at infinity of the set  $A_{\varphi \circ F}$  are contained in the positive cone of  $\mathbb{R}^2$ . From this it is easy to obtain the conclusion.

#### (4.5) Non-zero constant jacobian analytic maps

There exists a non-zero constant Jacobian analytic map  $F = (P, Q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  which is non-injective, non-surjective, at most 2-to-1 and such that for all  $|c|$  large enough,  $P^{-1}(c)$  has exactly one connected component and also the restriction of  $F$  to each level  $P^{-1}(c)$  is proper. It will be seen that this maps is a sort of “algebraic map”.

Let us proceed to construct the map  $F$ . First consider the map  $F_1 = (P_1, Q_1) : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}^2$  given by  $P_1(x, y) = x(y^2 - 1)$  and  $Q_1(x, y) = xy(y^2 - 4)$ . then:

- (1)  $\det(DF_1(x, y)) = x(y^4 + y^2 + 4) > 0$  everywhere;
- (2) if  $c < 0$  then  $P_1^{-1}(c)$  is the connected set which is the graph of the map  $y \rightarrow x = c/(y^2 - 1)$  defined in  $(-1, 1)$ ;
- (3) if  $c > 0$  then  $P_1^{-1}(c)$  has two connected components which are the graph of the maps  $x \rightarrow y = \sqrt{(c+x)/x}$  and  $x \rightarrow y = -\sqrt{(c+x)/x}$  defined in  $(0, \infty)$ ;

- (4)  $P_1^{-1}(0)$  has two connected components:  $\{y = 1\}$  and  $\{y = -1\}$ ;
- (5)  $F_1$  is not injective because  $F_1(1, 2) = F_1(1, -2) = (3, 0)$ ;
- (6)  $F_1$  is not surjective because  $(0, 0) \notin F_1((0, \infty) \times \mathbb{R})$ ;
- (7) for all  $c \in \mathbb{R} \setminus \{0\}$ ,  $F_1$  restricted to each level  $P_1^{-1}(c)$  is a proper map.

Now consider the analytic diffeomorphism  $H_1: (0, \infty) \times \mathbb{R} \rightarrow (0, \infty) \times \mathbb{R}$  given by  $H_1(x, y) = (\sqrt{2x}, h(y))$ , where the diffeomorphism  $h: \mathbb{R} \rightarrow \mathbb{R}$  is the solution of the differential equation:

$$h' = \frac{1}{h^4 + h^2 + 4}, \quad h(0) = 0.$$

We may see that  $h(y)$  satisfies the algebraic equation  $(h(y))^5/5 + (h(y))^3/3 + 4(h(y)) = y$ . Let  $F_2 = F_1 \circ H_1$ . We may check that  $\det(DF_2) \equiv 1$ .

As  $H_1$  takes vertical lines onto vertical lines, there is a diffeomorphism  $f: (0, \infty) \rightarrow \mathbb{R}$  such that  $H_1(\{(x, f(x)) : x \in (0, \infty)\})$  is the connected component  $\{(x, -\sqrt{(5+x)/x}) : x \in (0, \infty)\}$  of  $P^{-1}(5)$ . Define the area preserving analytic diffeomorphism  $H_2: (0, \infty) \times \mathbb{R} \rightarrow (0, \infty) \times \mathbb{R}$  by  $H_2(x, y) = (x, y + f(x))$ . Observe that  $H_2$  takes the positive first quadrant  $\{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$  onto the set  $\{(x, y) \in \mathbb{R}^2 : x > 0, y > f(x)\}$  which in turn is taken by  $H_1$  onto  $\{(x, y) \in \mathbb{R}^2 : x > 0, y > -\sqrt{(5+x)/x}\}$ .

We conclude that  $F_3 = (P_3, Q_3) = F_1 \circ H_1 \circ H_2$ , restricted to the first positive quadrant, has the following properties:

- (1)  $\det(DF_3(x, y)) = 1$  everywhere;
- (2) if  $c < 0$  then  $P_3^{-1}(c)$  is a connected set;
- (3) if  $c \in (0, 5)$  then  $P_1^{-1}(c)$  has two connected components and if  $c \geq 5$ ,  $P_3^{-1}(c)$  is a connected set;
- (4)  $P_1^{-1}(0)$  has two connected components;
- (5)  $F_3$  is non-injective and non-surjective;
- (6) for all  $c \in \mathbb{R} \setminus \{1\}$ ,  $F_3$  restricted to each level  $P_3^{-1}(c)$  is a proper map.

Let  $H_3: \mathbb{R}^2 \rightarrow (0, \infty) \times \mathbb{R}$  and  $H_4: \mathbb{R}^2 \rightarrow \mathbb{R} \times (0, \infty)$  be the following area preserving diffeomorphisms:

$$H_3(x, y) = \left( x + \sqrt{x^2 + 4}, y \left( \frac{\sqrt{x^2 + 4}}{x + \sqrt{x^2 + 4}} \right) \right),$$

$$H_4(x, y) = \left( x \left( \frac{\sqrt{y^2 + 4}}{y + \sqrt{y^2 + 4}} \right), y + \sqrt{y^2 + 4} \right).$$

Observe that the function  $k(x) = x + \sqrt{x^2 + 4}$  satisfies the algebraic equation  $(k(x))^2 - xk(x) = 4$ . It can be seen that  $H_3 \circ H_4$  takes  $\mathbb{R}^2$  onto the first positive quadrant and that  $F = (P, Q) = F_3 \circ H_3 \circ H_4$  is the example as required at the beginning of this section.

Now we summarize the properties of the map  $F = (P, Q)$  just constructed:

- (1)  $\det(DF(x, y)) = 1$  everywhere;
- (1) if  $c \in (-\infty, 0) \cup [5, \infty)$ , then  $P^{-1}(c)$  is a connected set; if  $c$  is in the interval  $[0, 5)$  then  $P^{-1}(c)$  has two connected components; in this way the foliation induced by the Hamiltonian vector field  $X_P$  has exactly one Reeb component:  $\{P \leq 0\}$ ;
- (2) if  $c \in \mathbb{R} \setminus \{0\}$ , then  $Q^{-1}(c)$  has two connected components and  $Q^{-1}(0)$  has three connected components; in this way the foliation induced by the Hamiltonian vector field  $X_Q$  has exactly two adjacent Reeb components whose union is the set  $\{Q \leq 0\}$ ;
- (3)  $F$  is non-injective and non-surjective; more precisely, given the point  $(3, 0)$ ,  $F^{-1}(3, 0)$  consists of two points and  $(0, 0) \notin F(\mathbb{R}^2)$ ;
- (5) for all  $c \in \mathbb{R} \setminus \{0\}$ ,  $F$  restricted to each level  $P^{-1}(c)$  is a proper map.

Figure 1 shows the foliations induced by the Hamiltonian vector fields  $X_P$  and  $X_Q$ .

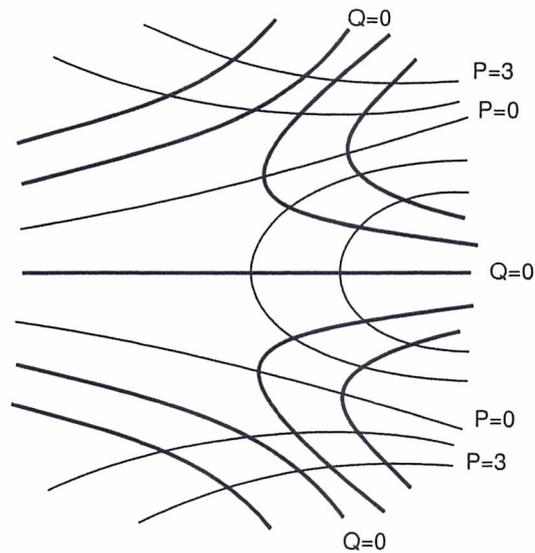


Figure 1

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