Nº 56

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Maio 1983

CONSTRUCTION OF OPEN SETS OF FREE k-TUPLES OF MATRICES

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0. Introduction

A k-tuple of members of a group is free if it freely generates a free group. In an earlier paper [4] the authors have shown:

Theorem 0: Let F be a locally compact field with a non trivial absolute value. Then the set of free k-tuples of $GL(n,F)$ contains a nonempty open set of $GL(n,F)^k$, where the latter has the topology induced from that of F.

In the present work we refine the proof technique which led to the theorem above, in order to describe an explicit construction for open set of free k-tuples.

Let F be a locally compact field with a nontrivial absolute value $x \mapsto |x|$. Upon identification of a vector space V with F^n, we endow it with the norm $||(v_1,\dots,v_n)|| = \max\{|v_1|,\dots,|v_n|\}$. Thus we identify $M_n(F) = F^{n^2}$, and we have that $GL(n,F)$ is an open and dense subset of $M_n(F)$. Let $D_n(F)$ denote the set of diagonal matrices in $M_n(F)$; thus $(x_1,\dots,x_n) \to \mathrm{diag}(x_1,\dots,x_n)$ is an isometry of V onto $D_n(F)$.

Let $k \in \mathbb{N}^*$ and define the map

$$\phi_k : GL(n,F)^k \times D_n(F)^k \to M_n(F)^k \qquad \text{by}$$

$$\phi_k\Big((A^{(1)},\ldots,A^{(k)}),(D^{(1)},\ldots,D^{(k)})\Big)=\Big(A^{(1)}D^{(1)}(A^{(1)})^{-1},\ldots,A^{(k)}D^{(k)}(A^{(k)})^{-1}\Big).$$

Theorem 1: Let $D_n'(F)$ be the set of diagonal matrices whose diagonal entries are all distinct and non zero. Then the restriction of ϕ_k to $GL(n,F)^k \times D_n'(F)^k$ is an open map.

An m-tuple of members of V is said to be in general position provided each n of its members comprise a basis of the space.

Theorem 2: Let $W=(W^{(1)},\ldots W^{(k)})\subseteq GL(n,F)^k$ be such that the kn-tuple of columns of W is in general position. Then there exist a neighbourhood $\mathcal{A}$ of W in GL(n,F) and a subset $\mathcal{D}$ of $D_n(F)^k$ with nonempty interior, such that $\phi_k(\mathcal{A}\times\mathcal{D})$ consist only of free k-tuples.

As in [4], the discovery of Theorem 2 as well as many of the ideas in its proof are inspired on Tits [6].

Clearly, Theorems 1 and 2 together imply Theorem 0.

The paper is organized in the following sections:

(1) Proof of Theorem 1. That is an exercise in Analysis; we also describe subsets of $GL(n,F)^k \times D'_n(F)^k$ whereupon the the map ϕ_k is locally a homeomorphism.

(2) Proof of Theorem 2, by a quite explicit construction. The numbering of sections (1) and (2) is arbitrary - they may be read the other way around.

(3) Examples of open sets of free pairs in $M_2(\mathbb{R})$ and $M_2(\mathbb{Q}_p)$ are constructed, following the proof of Theorem 2.

(4) A possible alternative setting for Theorem 0 is that given by the Zariski topology in $GL(n,F)^k$. We prove this idea too ambitious, showing that $GL(n,F)^k$ has no Zariski open set of free k-tuples whenever F has infinitely many roots of

unity.

In what follows, whenever the field F is understood implicitly, it will be left out of the notation. That is, we write GL_n for $GL(n,F)$, M_n for $M_n(F)$, D_n for $D_n(F)$, and so on.

1. Open maps into $GL(n,F)^k$

In this section we may drop the hypothesis that F is locally compact. It is enough for it to be complete.

Proof of Theorem 1: It is enough to show that $\phi = \phi_1$ is open, as $\phi_k = (\phi_1)^k$. In order to prove that, we compute the differential $d\phi$ and show that it is surjective at each point of $GL_n \times D'_n$. It will follow then that ϕ is open: when $F = \mathbb{R}$ or $\mathbb{C}$, that is a standard consequence of the "inverse mapping theorem" (see e.g. [1]; in case F is non-archimedean, see Schikhof [5], and the remark following this proof.

Using well known differentiation rules, we obtain:

$$(1) \quad d\phi(A,D)(H,K) = HDA^{-1} - ADA^{-1}HA^{-1} + AKA^{-1}.$$

In order to show that $d\phi(A,D)$ is surjective we consider instead the linear map $T: M_n \times D_n \to M_n$ defined by

$$(2) \quad T(H,K) = A^{-1}(d\phi(A,D)(H,K))A = A^{-1}HD - DA^{-1}H + K.$$

It suffices to show that T is surjective.

Let E_{ij} be the matrix which has a 1 at position ij, and is zero elsewhere. It is enough to show that all E_{ij} are in the image of T. Let $D \in D'_n$, $D = \mathrm{diag}(x_1, \ldots, x_n)$. Direct computations using (2) gives:

$$T((x_j - x_i)^{-1} AE_{ij}, 0) = E_{ij}, \text{ if } i \neq j \text{ and}$$

$$T(0, E_{ii}) = E_{ii}.$$

Remark: It should be noted that the usual formula for the derivative of the map $A \to A^{-1}$ holds also when F is non--archimedean, thus validating formula (1). The proof of Theorem 2.3 of [5] seems to have a mistake. However, some traditional proofs of the inverse mapping theorem (e.g. Lima [1]) also hold in the non-archimedean case, at least for mappings $F^n \to F^m$ all whose components are rational functions.

The mapping ϕ is far from being 1-1, even locally. For actual construction of open sets via ϕ, it is a waste to have excess parameters, so it would be nice to have a subset of $GL_n \times D'_n$ depending on n^2 parameters, over which ϕ is still open and locally 1-1. Such a subset can be easily produced after we characterize the fibers of ϕ.

Lemma 1.1: Let (A,D) and $(\bar{A},\bar{D})$ be elements of $GL_n \times D'_n$. Then $\phi(A,D) = \phi(\bar{A},\bar{D})$ if and only if there exist a nonsingular diagonal matrix K and a permutation matrix P, such that $\bar{A} = AKP$ and $\bar{D} = P^{-1}DP$. Furthermore, there exists a neighbourhood of (A,D) on which any two pairs which are on the same fiber of ϕ have equal second component.

Proof: Clearly $\phi(AKP, P^{-1}DP) = \phi(A,D)$ if K is nonsingular diagonal and P is a permutation matrix. For the converse, suppose that $\phi(A,D)=\phi(\bar{A},\bar{D})=B$. It follows that the diagonal entries of D and $\bar{D}$ are the eigenvalues of B; thus the diagonal of $\bar{D}$ is a permutation of that of D, hence, for a uniquely determined permutation matrix P we have $\bar{D} = P^{-1}DP$. One concludes that $\phi(\bar{A}P^{-1}, D) = \phi(A,D)$; for $i=1,\ldots,n$, the i-th column of A, and that of $\bar{A}P^{-1}$ is an eigenvector of B, corresponding to the same eigenvalue. As each eigenspace of of B is 1-dimensional, each column of $\bar{A}P^{-1}$ is a scalar multiple of the corresponding column of A, whence $\bar{A}P^{-1} = AK$, for some diagonal matrix K.

Consider the simultaneous actions on $GL_n \times D'_n$ of $D^o = D_n \cap GL_n$ and S_n, the group of permutation matrices, given by $(A,D).B = (AB, B^{-1}DB)$. The action of S_n is properly discontinous. If $P \in S_n$ and $K \in D^o_n$, then $P^{-1}KP \in D^o_n$; hence the formula $(A,D).K.P = (A,D).P.(P^{-1}KP)$ shows that the action of S_n preserves the orbits of D^o_n. Hence there is an induced action of S_n on the orbit space of D^o_n, which is also properly discontinous. One concludes that each $(A,D) \in GL_n \times D'_n$ has a neighborhood which is a union of D^o_n-orbits and does not intersect any of its images under $S_n - \{I_n\}$. From this, the second assertion of Lemma 1.1 follows.

Theorem 1.2: Let GL^1_n be the subset of matrices in GL_n whose diagonal consists to ones only. Then the restriction of ϕ to $GL^1_n \times D'_n$ is an open map, and locally a homeomorphism.

Proof: Let GL'_n be the subset of matrices in GL_n with no zeros in the main diagonal, and let $\xi: GL'_n \to GL^1_n$ be division of each row by the corresponding diagonal element. Since ϕ is constant on each fiber of ξ, with $\iota: GL^1_n \to GL'_n$ denoting the inclusion we have the following commutative diagram of maps:

$$\begin{array}{ccccc}
GL^1_n \times D'_n & \xrightarrow{\iota \times 1} & GL'_n \times D'_n & \xrightarrow{\phi} & M_n \\
\uparrow 1 & \swarrow \xi \times 1 & & & \\
GL^1_n \times D'_n & & & &
\end{array}$$

Since ϕ is open and ξ is continuous it follows that $\phi\circ(\iota\times 1)$ is also open.

From the second assertion of Lemma 1.1, each $(A,D) \in GL_n^1 \times D_n'$ has a neighborhood where ϕ is 1-1, since $AK \in GL_n^1$ with $A \in GL_n^1$ and $K \in D_n^o$ implies that $K = I_n$. Therefore, ϕ maps this neighborhood homeomorphically onto its image.
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Finally, we present an obvious consequence of Theorem 1.2, which is the form to be used for constructing examples of free k-tuples.

Corollary 1.3: Let $P=(P^{(1)},\ldots,P^{(k)})$ be a k-tuple of permutation matrices; denote by GL_n^P the set of k-tuples A of GL_n^k such that for each $i=1,\ldots,k$, $A^{(i)}$ has a 1 at each position where $P^{(i)}$ has a 1. Then, the restriction of ϕ_k to $GL_n^P \times (D_n')^k$ is an open map, and is locally a homeomorphism.

Of further interest would be the construction of open sets of projective transformations. These are easily obtained by projecting open sets of GL_n into PGL_n. We state withouth proof an analogue of Theorem 1.2 and leave to the reader the statement of the corresponding analogue of Corollary 1.3.

Proposition 1.4: Denote by $D_n'' = \{D \in D_n' \mid D_{11} = 1\}$. Then with $\pi: GL_n \to PGL_n$ the canonical projection, the restriction of $\pi\circ\phi$ to $GL_n^1 \times D_n''$ is an open map into PGL_n, and locally a homeomorphism.

2. Construction of free k-tuples

Our proof of Theorem 2 is a modification of the proof of Proposition 3.12 of Tits [6], motivated by the idea of making it computational enough, so as to allow one to construct explicit examples. Thus, we have sacrificed some generality in order to simplify calculations. That also explains why the setting is of actual matrices, instead of the more elegant environment provided by projective transformations. In addition, we feel that the main results should still be true over any field with a nontrivial absolute value, regardless of whether its completion is locally compact.

As in [4], our main tool for proving a k-tuple to be free is the following criterion, due to Macbeath [3] Lyndon and Ullman [2] and Tits [6] .

Lemma 2.1: Let G be a group acting an a set P on the left and let $g = (g_1,\dots,g_k)$ be a k-tuple of members of G. Suppose that there exists a k-tuple $(P_1,\dots,P_k)$ of subsets of P and $q \in P-(P_1 \cup\dots\cup P_k)$ such that for all i,j, $1 \le i,j \le k$, $i\neq j$ and for all $n \in \mathbb{Z}^*$, $g_i^n(P_j \cup \{p\}) \subseteq P_i$. Then g is free.

We need some preparatory Lemmas for the proof of Theorem 2.

Lemma 2.2: The set G_m of m-tuples of V in general position is open in V^m.

Proof: Indeed, $(v_1,\dots,v_m)$ is in general position if and only if for every $1 \le i_1 < \dots < i_n \le m$, $\det(v_{i_1},\dots,v_{i_n}) \neq 0$. That is, G_m is the complement in $V^m = F^{nm}$ of a union of finitely many hypersurfaces. //

If A is a matrix, A_i denotes its i-th column.

Lemma 2.3: Let $H \subseteq G_m$ be a product, $H = H_1 \times \ldots \times H_m$ where each $H_i \subseteq V$. Choose $1 \leq i_1 < \ldots < i_n \leq m$, and let A be a matrix whose j-th column comes from H_{i_j}, $j = 1,\ldots,n$. Then, for any $x \in H_r$, where r is not one of the i_j 's, all components of $A^{-1}x$ are nonzero.

Proof: Note, first that the columns of A must be linearly independent, hence A^{-1} exists. Since H is a product, the (n+1)-tuple $(A_1,\ldots,A_n,x)$ is in general position; thus x cannot be a linear combination of fewer than n A_i's. As $x = \sum_{i=1}^{n} A_i (A^{-1}x)_i$, it follows that $(A^{-1}x)_i \neq 0$, for i=1,...,n. //

Proof of Theorem 2: Choose a point q which does not lie on any hyperplane spanned by columns of matrices in W. Let $h: M_n^k \times V \to (V^n)^k \times V = V^{nk+1}$ be the map which identifies each matrix with its ordered set of columns. Thus $\omega = h(W,q) \in G_{nk+1}$; choose a positive real γ such that $\mathcal{B}(\omega,\gamma) \subseteq G_{nk+1}$ ($\mathcal{B}(s.\alpha) \equiv \{x \mid d(x,s) \leq \alpha\}$, where s is either a point or a set in whatever metric space which is clear from the context) - that choice is possible because of Lemma 2.2. Choose now positive ε, δ, such that

$$\mathcal{B}(\mathcal{B}(\omega,\delta), \varepsilon) \subset \mathcal{B}(\omega,\delta).$$

Of course, it is enough that $\delta + \varepsilon \leq \gamma$, but if the valuation is non-archimedean, it is more convenient to have $\varepsilon = \gamma = \delta$. Define now, for s=1,...,k, $B_s = \mathcal{B}(W_1^{(s)}, \gamma)$, $C_s = (W_n^{(s)}, \gamma)$,

$A^{(s)} = B(W^{(s)}, \delta)$. Define also $L_j = \{\mathrm{diag}(\lambda_1,\ldots,\lambda_n) \mid \lambda_j = 0,$ $|\lambda_i| \leq 1$ for $i=1,\ldots,n\} \subsetneq D_n$.

Let $i,j \in \{1,\ldots,k\}$, $i \neq j$ and $r=1$ or n. Define $\beta_{i,j,r} = \sup\{ \| (ADA^{-1}x)/(A^{-1}x)_r \| \mid A \in A^{(i)}, x \in B_j \cup C_j \cup \{p\}, D \in L_r\}$.

To see that $\beta_{i,j,r}$ are well defined real numbers, note first that if $A \in A^{(i)}$ and $x \in B_j \cup C_j \cup \{p\}$ they satisfy the conditions of Lemma 2.3, since $B(\omega,\gamma)$ is the product of its projections into V. Hence $(A^{-1}x)_r$ is never zero. It follows that $\|(ADA^{-1}x)/(A^{-1}x)_r\|$ is continous over its domain. Since that domain is compact, $\beta_{i,j,r}$ is well defined.

Set now

$$\mu_i = \sup_{j \neq i} \left\{ \frac{\beta_{i,j,1}}{\varepsilon}, \frac{\beta_{i,j,n}}{\varepsilon}, 1 \right\}.$$

Finally, let

$$D^{(i)} = \{\mathrm{diag}(\lambda_1,\ldots,\lambda_n) \mid \mu_i^{-1} |\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_{n-1}| > \mu_i |\lambda_n| > 0\}.$$

We assert that $D = D^{(1)} \times \ldots \times D^{(k)}$ and $A = A^{(1)} \times \ldots \times A^{(k)}$ satisfy the requirements of Theorem 2.

It remains to be shown that $\phi_k(A \times D)$ consists only of free k-tuples.

Let $P_i = \{x \in V \mid x \in B_i \cup C_i$ for some $\lambda \in F\}, i=1,\ldots,k$. Given $g \in \phi_k(A \times D)$ we shall show that it, together with $P_1,\ldots,P_k$ and the point q, chosen at the beginning of the proof, satisfy the conditions of Lemma 2.1.

We show first that if $i \neq j$, then $g_i(P_j \cup \{p\} \subseteq P_i$.

Since P_i is closed under multiplication by nonzero scalars, it suffices to show that, given $x \in P_j$, there is an $\alpha \in F^*$ such that $g_i(\alpha x) \in B_i \subseteq P_i$.

We may write $g_i = ADA^{-1}$, with $A \in \mathcal{A}^{(i)}$, $D \in \mathcal{D}^{(i)}$. Let $\alpha = (D_{11}.(A^{-1}x)_1)^{-1}$. Then:

$g_i(\alpha x) = (ADA^{-1}x)/(D_{11}(A^{-1}x)_1) = A\bar{D}A^{-1}\bar{x}$, where $\bar{D}=D_{11}^{-1}D=\text{diag}(1,\lambda_2,\ldots,\lambda_n)$, $\lambda_i = D_{ii}/D_{11}$, and $\bar{x} = x/(A^{-1}x)_1$. Note that $|\lambda_2| \le \mu_i^{-1}$ and $(A^{-1}\bar{x})_1=1$. Hence

$d(g_i(\alpha x),A_1)=||g_i(\alpha x)-A_1||=||A\bar{D}A^{-1}\bar{x} - A.\text{diag}(1,0,\ldots,0)A^{-1}\bar{x}||$

$= ||A\bar{\bar{D}}A^{-1}\bar{x}||$, where $\bar{\bar{D}} = \text{diag}(0,\lambda_2,\ldots,\lambda_n)$

$= |\lambda_2|\;||A\tilde{D}A^{-1}\bar{x}||$, where $\tilde{D} = \lambda_2^{-1}\bar{\bar{D}} \in \mathcal{L}_1$

$< \mu_i^{-1}\beta_{i,j,1} \le \varepsilon$.

But, as $A \in \mathcal{A}^{(i)}$, $d(A_1,W_1^{(i)}) \le \delta$ and it follows that $g_i(\alpha x) \in \mathcal{B}(\mathcal{B}(W_1^{(i)},\delta),\varepsilon) \subseteq \mathcal{B}(W_1^{(i)},\gamma) = B_i$.

Similarly, replacing α by $\alpha' = \lambda_n/(A^{-1}x)_n$, $W_1^{(i)}$ by $W_n^{(i)}$, and so on, one shows that $g_i^{-1}(\alpha' x) \in C_i$, whence $g_i^{-1}(P_j \cup \{p\}) \subseteq P_i$.

Finally, notice that since $\mu_i \ge 1$, if $D \in \mathcal{D}^{(i)}$, all its powers D^m, $m \in Z^*$ are in $\mathcal{D}^{(i)}$. Hence, if g is in $\phi_k(\mathcal{A} \times \mathcal{D})$, all k-tuples $(g_1^m,\ldots,g_k^m)$ are also there, hence $g_i^m(P_j \cup \{p\}) \subseteq P_i$, $m \in Z^*$. Thus, the conditions of Lemma 2.1 are fulfilled, and the Theorem is proved.

One should note that if in the definition of $\mathcal{D}^{(i)}$ one allows $|\lambda_1|=\mu_i|\lambda_2|$, $|\lambda_{n-1}|=\mu_i|\lambda_n|$, the ensuing argument would still carry over to the end of the proof. But there is a glitch: if $\mu_i = 1$, one would have I_n as member of a free k-tuple. This ridiculous conclusions shows that actually $\mu_i > 1$.

3. Examples

We shall follow here the proof of Theorem 2 in order to construct examples of open sets of free pairs, first over the reals, second over $\mathbb{Q}_p$, the p-adic completion of the field of rational numbers. The simplification suggested by Corollary 1.3 is to be followed so that we obtain an 8-parameter family of free pairs, where each parameter is allowed to vary so that its absolute value lies within a certain interval. A computation of appropriately sized intervals, which ensure that the resulting pairs are free is the gist of the calculation.

3.1 - Let $F = \mathbb{R}$, $n = k = 2$

$$W = \left(\begin{array}{cc|cc} 1 & 0 & 1/2 & 1 \\ 0 & 1 & 1 & 1/2 \end{array}\right),$$

$q = (1,1)$, $\gamma=1/6$. One readily verifies that $B(W,1/6) \subseteq G_5$. In the notation of Corollary 1.3, with ξ suitably adapted to P,

$$P = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right), \quad \xi(B(W,1/6)) = \left(\begin{array}{cc|cc} 1 & x_2 & \frac{1}{2}+x_3 & 1 \\ x_1 & 1 & 1 & \frac{1}{2}+x_4 \end{array}\right)$$

where $|x_1|$, $|x_2| \le 1/5$, $-\frac{3}{14} \le x_3, x_4 \le \frac{3}{10}$. Recall that $\xi(B_1)$, $\xi(C_1)$, $\xi(B_2)$, $\xi(C_2)$ are the projections of $\xi(B(W,1/6))$ onto the first through fourth column, respectively. See figure.

Choosing $\delta = 1/11$, $\varepsilon = 5/66$ (so that $\varepsilon + \delta = 1/6$), we have, with $A = B(W,\delta)$,

$$\xi(A) = \left\{ \begin{pmatrix} 1 & \varepsilon_2 & \vdots & \frac{1}{2}+\varepsilon_3 & 1 \\ \varepsilon_1 & 1 & \vdots & 1 & \frac{1}{2}+\varepsilon_4 \end{pmatrix} \;\middle|\; \begin{matrix} |\varepsilon_1|, |\varepsilon_2| \le 1/10, \\ -\frac{1}{8} \le \varepsilon_3, \varepsilon_4 \le \frac{3}{20} \end{matrix} \right\}$$

Note that $L_1 = \{ \lambda E_{22} \mid |\lambda| \le 1 \}$

If $A \in \xi(A^{(1)})$,

$$A = \begin{pmatrix} 1 & \varepsilon_2 \\ \varepsilon_1 & 1 \end{pmatrix}, (\det A)..A^{-1} = \begin{pmatrix} 1 & -\varepsilon_2 \\ -\varepsilon_1 & 1 \end{pmatrix}, (\det A).A\,E_{22}\,A^{-1} = \begin{pmatrix} -\varepsilon_1\varepsilon_2 & \varepsilon_2 \\ \varepsilon_1 & 1 \end{pmatrix}$$

If $x \in B_2$,

$\det A.A\,E_{22}\,A^{-1}x = (\varepsilon_2(-\varepsilon_1(\frac{1}{2} + x_3) + 1), \ \varepsilon_1(\frac{1}{2} + x_3) + 1)$, hence

$|(\det A).A\,E_{22}\,A^{-1}x| \le \frac{1}{2}\left(\frac{1}{2} + \frac{3}{10}\right) + 1 = \frac{108}{100}$;

also $|\det(A).(A^{-1}x)_1| = |\frac{1}{2} + x_3 - \varepsilon_2| \ge \frac{1}{2} - \frac{3}{14} - \frac{1}{10} = \frac{13}{70}$.

Hence $\sup \left\{ \left| \dfrac{ADA^{-1}x}{(A^{-1}x)_1} \right| \;\middle|\; A \in \xi(A^{(1)}),\ D \in L_1,\ x \in B_2 \right\} \le$

$\frac{108}{100} / \frac{13}{70} = \frac{378}{65}$. (A slightly more complicated exact calculation gives sup = 72/13).

Similarly, one finds that $\sup \left| \dfrac{ADA^{-1}x}{(A^{-1}x)_1} \right|$ is bounded above by 90/92 when $x \in C_2$, and by 11/9 when $x = q$.

It follows that $\beta_{1,2,1} \le 378/65$, and the same bound holds for $(\beta_{1,2,2}$ (no coincidence - this reflects simmetries of the chosen set A).

Thus one gets $\mu_1 = \sup \left\{ \frac{\beta_{1,2,1}}{\varepsilon}, \frac{\beta_{1,2,2}}{\varepsilon}, 1 \right\} \leq \frac{24948}{325} < 77.$

Similar calculations yeld $\mu_2 \leq \frac{14916}{175} < 86$ (additional effort yields the exact values $\mu_1 = 4752/65$, $\mu_2 = 407/5$).

In conclusion, Theorem 2 guarantees that, $\phi_2(A^{(1)}, A^{(2)}, D^{(1)}, D^{(2)})$ is a free pair whenever

$$A^{(1)} = \begin{pmatrix} 1 & \varepsilon_2 \\ \varepsilon_1 & 1 \end{pmatrix} \quad |\varepsilon_1|, |\varepsilon_2| \leq 1/10 \,,$$

$$A^{(2)} = \begin{pmatrix} \frac{1}{2} + \varepsilon_2 & 1 \\ 1 & \frac{1}{2} + \varepsilon_4 \end{pmatrix} \quad -\frac{1}{8} \leq \varepsilon_3, \varepsilon_4 \leq \frac{3}{20} \,,$$

$$D^{(1)} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \quad |\alpha_1| > \mu_1 \; |\alpha_2| > 0 \,,$$

$$D^{(2)} = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}, \quad |\beta_1| > \mu_2 \; |\beta_2| > 0 \,,$$

and by Corollary 1.3 the image of the interior of this domain is an open set.

3.2 - Let $F = \mathbb{Q}_p$. We indicate one set of parameters; the calculations, following the first example as paradigm, are easily completed due to the ultrametric property of $|\,|_p$, yielding exact values for μ_1 and μ_2.

Let

$$W = \left(\begin{array}{cc|cc} 1 & 0 & p & 1 \\ 0 & 1 & 1 & p \end{array}\right), \quad q = (1,1),$$

with $\gamma = p^{-2}$. We take $\varepsilon = \delta = \gamma$. Then

$$\xi(B(W,\gamma)) = \xi(A) = \left(\begin{array}{cc|cc} 1 & \varepsilon_2 & p+\varepsilon_3 & 1 \\ \varepsilon_1 & 1 & 1 & p+\varepsilon_4 \end{array}\right), \quad |\varepsilon_i|_p \leq p^2.$$

One gets $\mu_1 = \mu_2 = p^3$. Thus one gets an open set of free pairs of $PGL(2,Q_p)$ as $\phi_2(A^{(1)},A^{(2)},D^{(1)},D^{(2)})$ (sort of), where

$$A^{(1)} = \begin{pmatrix} 1 & p^2d_2 \\ p^2d_1 & 1 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} p+p^2d_3 & 1 \\ 1 & p+p^2d_4 \end{pmatrix} \quad |d_i|_p \leq 1, \quad 1 \leq i \leq 6 .$$

$$D^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & p^2d_5 \end{pmatrix} \qquad D^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & p^3d_6 \end{pmatrix} .$$

4. What about Zariski topology?

Open sets in the topology hitherto considered are not very large; indeed, it was conjectured in [4] that there exists a dense open set of free k-tuples in GL_n^k. An even stronger conjecture was suggested by B. Hartley (private communication): the existence of a Zariski-open set of free k-tuples. However, often enough, these sets do not exist (Corollary 4.3).

Here F will be a field with <u>infinitely many roots of unity</u>. The only topology considered herewith in the several spaces involved is the Zariski topology ascribed to them by their apparent identification as a constructible subset of some F^m.

Theorem 4.1: The set $T_n = \{A \in GL_n \mid A^s = I \text{ for some } s \in \mathbb{N}^*\}$ is dense in GL_n.

Corollary 4.2: T_n^k is dense in GL_n^k .

Corollary 4.3: The set of free k-tuples has empty interior in GL_n^k.

Since the product of dense sets is dense in the Zariski product topology, Theorem 4.1 implies the first corollary, and this entails the second one immediately. Notice that the existence of infinitely many roots of unity is also a necessary condition for Theorem 4.1 to hold: if a field has only r roots of 1, every member of T_n is a root of the polynomial (in n^2 variables) $(\det A)^r - 1$. However, one feels that Corollary 4.3 is true whenever F is infinite.

The proof of 4.1 requires the two lemmas belows:

Lemma 4.4: The set $T_n(F)$ is dense in the set of semisimple matrices of $GL(n,K)$, where K is the algebraic closure of F.

Proof: Consider the morphism of varieties

ϕ: $GL(n,K) \times D_n(K) \to GL(n,K)$, given by the usual formula $\phi(A,D) = ADA^{-1}$. Its image is the set of semisimple matrices. The set $T_n(F)$ contains the image of $GL(n,F) \times D_n(R)$, where R is the set of roots of 1 in F, $D_n(R)$ is the set of diagonal matrices all whose diagonal entries are in R. Since R is infinite, it is dense in K, hence $D_n(R)$ is dense in $D_n(K)$; essentially by the same token $GL(n,F)$ is dense in $GL(n,K)$. It follows that $GL(n,K) \times D_n(R)$ is dense in the domain of ϕ; by continuity, its image is dense in the image of ϕ.

□

Lemma 4.5: Let K be an algebraically closed field.

The set of matrices with only simple eigenvalues is open in $M_n(K)$.

Proof: Let $X = (x_{ij})$ be a matrix of indeterminates over R, and let $d(x_{ij}) \in K[x_{ij}]$ be the discriminant of the caracteristic polynomial p_X of X. A polynomial has multiple roots if and only if its discriminant is zero. Hence a matrix A has multiple eigenvalues if and only if it is a zero of $d(x_{ij})$.

□

Proof of Theorem 4.1: Combining the two lemmas, $T_n(F)$ is dense in a subset of $GL(n,K)$ which contains an open set. As open sets are dense, $T_n(F)$ is dense in $GL(n,K)$; a fortiori, $T_n(F)$ is dense in $GL(n,F)$, since this has the topology induced from $GL(n,K)$.

□

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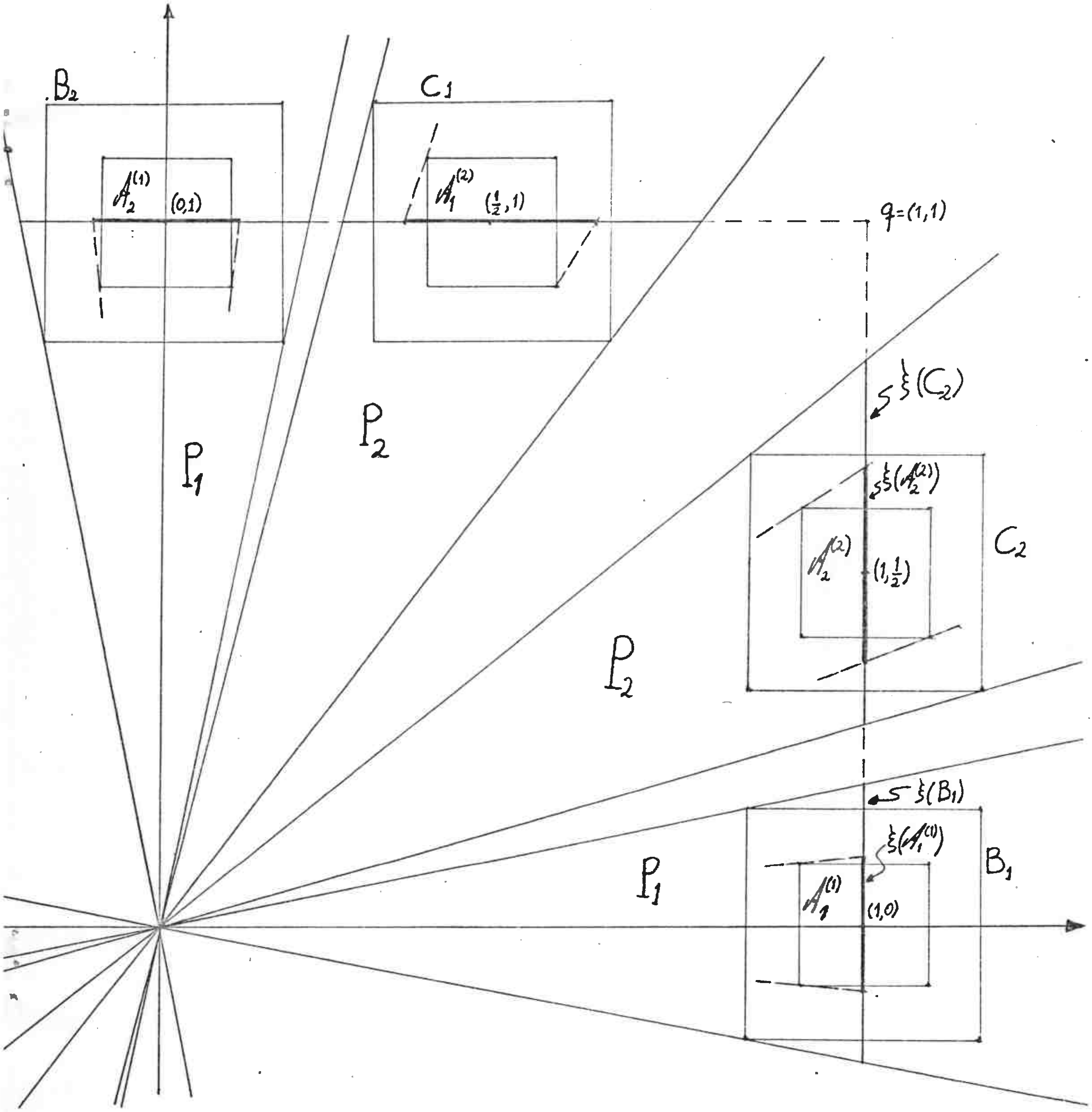

B₂
C₁
A₂⁽¹⁾
(0,1)
A₁⁽²⁾
(½,1)
q=(1,1)
ξ(C₂)
ξ(A₂⁽²⁾)
A₂⁽²⁾
(1,½)
C₂
P₁
P₂
P₂
ξ(B₁)
ξ(A₁⁽¹⁾)
B₁
P₁
A₁⁽¹⁾
(1,0)

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