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A law of uniform seniority for dependent lives

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ABSTRACT

The law of uniform seniority is an actuarial principle which justifies the replacement of an annuity on joint lives of unequal ages by an annuity on a single life, often computed at a different rate. Gompertz's law of mortality is a prime example of distribution which meets this condition. This paper proposes an extension of this principle to the case of two dependent lives and relates it to aging concepts. In the important special case of a bilinear averaging function, it is shown that the lifetimes have a dependence structure which is Archimedean and marginal distributions from the same scale family. This leads to both functional and stochastic representations for these models, which enjoy closure properties with respect to some common operations. The dependence and aging properties of the models are then discussed. The challenges involved in a multivariate extension are also mentioned.

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1. Introduction

The Law of Uniform Seniority (LUS) is a property of certain univariate life distributions that was initially described by de Morgan (1839) in his study of mortality tables. In his own terms (de Morgan 1859),

“... the chance of two or more lives surviving a given term is always equal to the chance of some one life, older than either, surviving a term of the same length.”

Early references on the subject (Thomas 1906, Goodman 1907, Borland 1915) point to the usefulness of the LUS in actuarial science as it implies that the value of a joint-life annuity on two lives of different ages can be computed from that of an annuity on a single life, possibly at a different rate.

To express this notion in stochastic terms in the case of two lives, let X and Y be non-negative random variables having the same survival function S whose support is $\mathbb{R}_+ = [0, \infty)$ or an interval of the form $[0, w_0]$, so that one has $\Pr(X > w) = \Pr(Y > w) = S(w)$ for all $w \in \mathbb{R}_+$. The LUS then holds if there exists an independent copy Z of X (or Y) and a map $m : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ such that for all possible ages $x, y \in \mathbb{R}_+$ and term $z \in \mathbb{R}_+$, one has both $m(x, y) \geq \max(x, y)$ and

$$\Pr(X > x + z, Y > y + z \mid X > x, Y > y) = \Pr\{Z > m(x, y) + z \mid Z > m(x, y)\}. \quad (1)$$

When the variables X and Y are independent, as appears to have been assumed implicitly in all the literature on the LUS, Condition (1) holds if and only if

$$\frac{S(x+z)S(y+z)}{S(x)S(y)} = \frac{S\{m(x,y)+z\}}{S\{m(x,y)\}} \quad (2)$$

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for all $x, y, z \in \mathbb{R}_+$ for which the denominators don't vanish.

Greville (1956) carried out a systematic investigation of the class of survival functions which satisfy functional equation (2) under weak regularity conditions. As Borland (1915) before him, he found that some well-known life distributions enjoy this property, including Makeham's laws and the so-called arithmetic-geometric, double geometric and trigonometric laws, but most prominently Gompertz's law of mortality, whose survival function is defined, for all $w \in \mathbb{R}_+$, by

$$S(w) = \Pr(X > w) = \exp\{-\theta(e^{\alpha w} - 1)\},$$

where $\theta, \alpha \in (0, \infty)$ are arbitrary parameters. It is indeed easily checked that for this model, Equation (2) holds with m given, for all $x, y \in \mathbb{R}_+$, by $m(x, y) = \alpha^{-1} \ln(e^{\alpha x} + e^{\alpha y})$. Condition (2) also has a natural reliability interpretation in terms of a serial system; see Proposition A.1, p. 370 in Marshall & Olkin (2007).

In practice, however, there are many situations where two lives are neither identically distributed nor stochastically independent. One could think, e.g. of husband and wife. One might then conceive of a bivariate analog of De Morgan's LUS loosely phrased as follows:

“The chance of two lives surviving given terms is always equal to the chance of some two lives of the same age surviving terms of the same lengths.”

More specifically, consider a pair (X, Y) of lives with joint survival function S . Let (U, V) be an independent copy of (X, Y) and let $a: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ be a continuous *averaging* function which is non-decreasing in both of its arguments and such that, for all $x, y, u, v \in \mathbb{R}_+$,

$$\begin{aligned} & \Pr(X > x + u, Y > y + v \mid X > x, Y > y) \\ &= \Pr\{U > a(x, y) + a(u, v), V > a(x, y) + a(u, v) \mid U > a(x, y), V > a(x, y)\}. \end{aligned} \quad (3)$$

One might then say, as it is hereby proposed, that the pair (X, Y) follows the Bivariate Law of Uniform Seniority (BLUS) when Condition (3) holds. This condition differs from the extension originally formulated by the second author in the conclusion of a paper devoted to characterizations of bivariate Gompertz distributions; see Kolev (2016). It also differs from de Morgan's Equation (1) on two accounts:

- (i) on the left-hand side of Equation (3), the lives X and Y are not only of different ages x and y in the conditioning event, but the remaining lifetimes u and v are allowed to differ as well, whereas only the case $u = v = z$ was covered in Equation (1);
- (ii) the right-hand side of Equation (3) involves two random variables U and V , whereas the right-hand side of Equation (1) involved only one, i.e. Z ; nevertheless, the two conditions are more similar than first meets the eye if one imposes, as will be done here, that $a(u, v) = z$ whenever $u = v = z \in \mathbb{R}_+$.

Together with a specific choice of averaging function a , Condition (3) constrains implicitly the dependence between the variables X and Y .

The purpose of this paper is to explore the nature of the association implied by the BLUS in the special case of a bilinear map a , i.e. when there exists a scalar $\pi \in (0, 1)$ such that, for all $x, y \in \mathbb{R}_+$,

$$a(x, y) = \pi x + (1 - \pi)y. \quad (4)$$

Assumption (4), referred to as the bilinear BLUS, or b-BLUS property for short, is a natural starting point because Condition (3) with a bilinear map a holds when the pair (X, Y) has a bivariate Gompertz distribution with parameters $\alpha, \beta \in (0, \infty)$ and $\theta \in [1, \infty)$, i.e. when for all $x, y \in \mathbb{R}_+$,

$$\Pr(X > x, Y > y) = \exp\{-\theta(e^{\alpha x + \beta y} - 1)\}. \quad (5)$$

The appropriate choice of scalar π in Equation (4) is then $\pi = \alpha/(\alpha + \beta) \in (0, 1)$.

In Section 2, a connection is drawn between the general notion of BLUS and two bivariate notions of aging. In Section 3, a functional form is derived for models with the b-BLUS property and it is shown in Section 4 that the underlying survival copula of such models is in fact Archimedean, which leads to a stochastic representation for distributions in this class. Connections between the b-BLUS property and bivariate notions of dependence and aging are then explored in Section 5 and implications for other reliability concepts are discussed in Section 6, where bivariate extensions of the family of distributions due to Teissier (1934) and Chiang & Conforti (1989) are briefly mentioned. Section 7 reports closure properties and stochastic ordering results that could be helpful for actuarial applications, and perspectives on multivariate extensions of these findings are given in Section 8.

2. Connection between BLUS and bivariate notions of aging

Suppose that the joint survival function S of the pair (X, Y) is continuous and strictly decreasing in both of its arguments on its support, which is either \mathbb{R}_+^2 or a rectangle of the form $[0, x_0] \times [0, y_0]$. Then S satisfies Equation (3) if and only if, for all $x, y, u, v \in \mathbb{R}_+$,

$$S(x + u, y + v)S\{a(x, y), a(x, y)\} = S(x, y)S\{a(x, y) + a(u, v), a(x, y) + a(u, v)\}. \quad (6)$$

Alternatively, the functional equation (6) can be expressed in the form

$$S(x + u, y + v) = S(x, y)S(u, v)A(x, y, u, v),$$

where for all $x, y, u, v \in \mathbb{R}_+$ at which it is well defined,

$$A(x, y, u, v) = \frac{S\{a(x, y) + a(u, v), a(x, y) + a(u, v)\}}{S(u, v)S\{a(x, y), a(x, y)\}}. \quad (7)$$

The ratio A may be construed as an ‘aging factor’ in view of its relation to the following notions of bivariate aging considered, e.g. in Mulero & Pellerey (2010) and Pellerey (2008).

Definition 2.1: Let S be a bivariate survival function whose support is in \mathbb{R}_+^2 . Then

- (i) S is said to be bivariate new better than used (BNBU) if and only if $S(x + u, y + v) \leq S(x, y)S(u, v)$ for all $x, y, u, v \in \mathbb{R}_+$;
- (ii) S is said to be bivariate new worse than used (BNWU) if and only if $S(x + u, y + v) \geq S(x, y)S(u, v)$ for all $x, y, u, v \in \mathbb{R}_+$.

It is plain that if a bivariate survival function S satisfies the BLUS, then S is BNBU whenever the ratio A in Equation (7) satisfies $A(x, y, u, v) \leq 1$ for all $x, y, u, v \in \mathbb{R}_+$ wherever it is well defined. Similarly, S is BNWU if $A(x, y, u, v) \geq 1$ for all $x, y, u, v \in \mathbb{R}_+$. The case where A is identically equal to 1 is noteworthy and treated below.

Lemma 2.1: Suppose that a bivariate continuous survival function S satisfying the BLUS is both BNBU and BNWU. Then, for all possible $x, y, u, v \in \mathbb{R}_+$, one has $S(x + u, y + v) = S(x, y)S(u, v)$ and S is the joint survival function of a pair of independent random variables with exponential margins.

Proof: Introduce the map $h : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined, for all $x, y \in \mathbb{R}_+$, by $h(x, y) = \ln\{S(x, y)\}$. Then $h(x + u, y + v) = h(x, y) + h(u, v)$ for all possible $x, y, u, v \in \mathbb{R}_+$, which is the bivariate version of Cauchy’s functional equation, whose only regular solution is of the form $h(x, y) = -\alpha x - \beta y$ for some choices of scalars $\alpha, \beta \in (0, \infty)$, given that h cannot be increasing in either argument; see, e.g. Theorem 5.2 in Castillo et al. (2005). Hence $S(x, y) = \exp(-\alpha x - \beta y)$ for all $x, y \in \mathbb{R}_+$, as claimed. ■

3. Functional representation of survival functions with the b-BLUS property

A functional form will now be derived for a bivariate survival function S with the b-BLUS property under the assumption that S is continuous and strictly decreasing in both of its arguments on its support. By assumption, there exists a scalar $\pi \in (0, 1)$ such that, for all $x, y, u, v \in \mathbb{R}_+$,

$$S(x + u, y + v)S(\pi x + \bar{\pi} y, \pi x + \bar{\pi} y) = S(x, y)S(\pi x + \bar{\pi} y + \pi u + \bar{\pi} v, \pi x + \bar{\pi} y + \pi u + \bar{\pi} v), \tag{8}$$

where $\bar{\pi} = 1 - \pi \in (0, 1)$. Introduce the map $h : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined, for all $x, y \in \mathbb{R}_+$, by

$$h(x, y) = \ln\{S(x, y)\} - \ln\{S(\pi x + \bar{\pi} y, \pi x + \bar{\pi} y)\}.$$

In view of Equation (8), one has $h(x + u, y + v) = h(x, y)$ for all possible $x, y, u, v \in \mathbb{R}_+$. Given that $S(0, 0) = 1$, it is clear that $h(0, 0) = 0$ and hence upon setting $x = y = 0$ in the previous equation, one finds $h(u, v) = h(0, 0) = 0$ for all $u, v \in \mathbb{R}_+$. Equivalently, one has, for all $x, y \in \mathbb{R}_+$,

$$S(x, y) = S(\pi x + \bar{\pi} y, \pi x + \bar{\pi} y). \tag{9}$$

This fact, together with Equation (8), implies that, for all $x, y, u, v \in \mathbb{R}_+$,

$$S(x + u, y + v) = S(\pi x + \bar{\pi} y + \pi u + \bar{\pi} v, \pi x + \bar{\pi} y + \pi u + \bar{\pi} v),$$

from which one can also deduce, upon setting $t = \bar{\pi}/\pi$, that

$$S(x + ty, u/t + v) = S(\pi x + \bar{\pi} y + \pi u + \bar{\pi} v, \pi x + \bar{\pi} y + \pi u + \bar{\pi} v).$$

Combining the two previous displayed equations, one then obtains the following result.

Lemma 3.1: *Let S be a bivariate survival function satisfying the b-BLUS property (8) for some scalar $\pi \in (0, 1)$. Then, for all $x, y, u, v \in \mathbb{R}_+$ and $t = \bar{\pi}/\pi$, one has*

$$S(x + u, y + v) = S(x + ty, u/t + v). \tag{10}$$

The functional equation (10) is a particular case of the generalized bisymmetry equation

$$F\{G(x, y), H(u, v)\} = K\{M(x, u), N(y, v)\}, \tag{11}$$

in which F, G, H, K, M, N are maps such that $F = K = S$ and for all $x, y, u, v \in \mathbb{R}_+$, $G(x, y) = x + ty$, $H(u, v) = u/t + v$, $M(x, u) = x + u$ and $N(y, v) = y + v$.

A general solution to Equation (11) is provided by Theorem 6.4 in Castillo et al. (2005) under the assumption that the map G is invertible in both variables, the maps F and M are invertible in the first variable for a fixed value of the second variable, and the maps H, K , and N are invertible in the second variable for a fixed value of the first. All these conditions are trivially satisfied, provided that the survival function S is continuous and strictly decreasing in both of its arguments on its support.

From Theorem 6.4 in Castillo et al. (2005), whose proof can be found on p. 374 of Aczél (1966), the solution to Equation (11) is given, for all $x, y, u, v \in \mathbb{R}_+$, by

$$S(x + u, y + v) = \psi\{p(x) + q(y) + r(u) + s(v)\} \tag{12}$$

for some non-unique continuous maps ψ, p, q, r, s which are strictly monotonic on their domain except for r . As it can be assumed without loss of generality that $p(0) = q(0) = r(0) = s(0) = 0$, one can deduce that, for all $x, y \in \mathbb{R}_+$,

$$S(x, y) = \psi\{p(x) + q(y)\} = \psi\{r(x) + q(y)\} = \psi\{p(x) + s(y)\} = \psi\{r(x) + s(y)\}, \tag{13}$$

where the map ψ can be taken to be strictly decreasing while p and q are strictly increasing. Moreover, one must have both $\psi(0) = 1$ and $\psi(w) \rightarrow 0$ as $w \rightarrow \infty$.

Let ψ^{-1} denote the inverse of ψ , which is continuous and strictly decreasing on $(0, 1)$ with $\psi^{-1}(0) = \inf\{w \in \mathbb{R}_+ : \psi(w) = 0\}$. One can now deduce from Equation (13) that $p(x) = r(x)$ and $q(y) = s(y)$ for all $x, y \in \mathbb{R}_+$. Moreover, upon setting $y = v = 0$ in Equation (12) and using the first identity in Equation (13), one finds that, for all $x, y, u, v \in \mathbb{R}_+$,

$$S(x + u, 0) = \psi\{p(x + u)\} = \psi\{p(x) + p(u)\}.$$

Therefore, p satisfies Cauchy’s classical functional equation and hence, in view of its continuity and the condition $p(0) = 0$, there exists a scalar $\alpha \in (0, \infty)$ such that $p(x) = \alpha x$ for all x in the support of X . Similarly, there exists a scalar $\beta \in (0, \infty)$ such that $q(y) = \beta y$ for all y in the support of Y . Finally, taking into account relation (9), one finds that, for all such $x, y \in \mathbb{R}_+$,

$$p(x) = p(\pi x) + q(\pi x), \quad q(y) = p(\bar{\pi} y) + q(\bar{\pi} y).$$

Setting $x = ty$ with $t = \bar{\pi}/\pi$ in the first equation above, one then finds that

$$p(x) = q(x/t), \quad q(y) = p(ty),$$

and hence $\alpha = p(1) = q(1/t) = \beta/t = \beta\pi/\bar{\pi}$.

These findings are summarized in the following theorem, which constitutes this paper’s main result.

Proposition 3.1: *Let S be a bivariate survival function which is continuous and strictly decreasing in both of its arguments on its support. If S satisfies the b -BLUS property (8) for some scalar $\pi \in (0, 1)$, then there exist constants $\alpha, \beta \in (0, \infty)$ with $\pi = \alpha/(\alpha + \beta)$ and a continuous, strictly decreasing survival function $\psi : \mathbb{R}_+ \rightarrow [0, 1]$ such that, for all $x, y \in \mathbb{R}_+$ in the support of S ,*

$$S(x, y) = \psi(\alpha x + \beta y). \tag{14}$$

Equation (14) actually holds for all $x, y \in \mathbb{R}_+$ even when the support of S is finite. Indeed, it follows from Proposition 3.1 that if the support of X is finite, say $[0, x_0]$, one must have $\psi(\alpha x_0) = 0$, i.e. the support of the map ψ is the interval $[0, \alpha x_0]$. Given that $S(x_0, 0) = S(0, \alpha x_0/\beta) = 0$, it follows that the support of Y is also finite and given by $[0, \alpha x_0/\beta]$. Therefore, relation (14) can be extended to all of \mathbb{R}_+^2 by setting $\psi(w) = 0$ for all $w \in (\alpha x_0, \infty)$, which is the proper thing to do anyway if ψ is viewed as a survival function. This convention will be implicitly assumed henceforth.

Proposition 3.1 leads rather naturally to the consideration of a new parametric class of survival functions, which may be defined as follows.

Definition 3.1: For any given $\alpha, \beta \in (0, \infty)$, let $\mathcal{L}(\alpha, \beta)$ denote the class of bivariate survival functions S whose support is contained in \mathbb{R}_+^2 and which can be expressed in the form (14) in terms of a continuous, strictly decreasing survival function $\psi : \mathbb{R}_+ \rightarrow [0, 1]$.

The properties of this class of survival functions are explored next.

4. Stochastic representation of distributions in the class $\mathcal{L}(\alpha, \beta)$

Let (X, Y) be a random pair whose joint survival function S is continuous and strictly decreasing in both of its arguments on its support. Further assume that $S \in \mathcal{L}(\alpha, \beta)$, which means that it can be expressed in the form (14) for some specific choice of scalars $\alpha, \beta \in (0, \infty)$ and some map $\psi : \mathbb{R}_+ \rightarrow [0, 1]$ which is continuous and strictly decreasing on its support.

Now introduce scaled versions of the variables X and Y as follows:

$$X^* = \alpha X, \quad Y^* = \beta Y.$$

The joint survival function of the pair (X^*, Y^*) is then given, for all $x, y \in \mathbb{R}_+$, by

$$\Pr(X^* > x, Y^* > y) = \psi(x + y), \quad (15)$$

i.e. the pair (X^*, Y^*) has a Schur-constant joint survival function. This class of models has been extensively studied; see, e.g. Chi et al. (2009), Nelsen (2005), Salminen & Vallois (2005), Ta & Van (2017), and Unnikrishnan Nair & Sankaran (2014). In particular, the following facts hold true:

- (i) The map ψ must be convex in order for Equation (15) to define a bona fide joint survival function.
- (ii) The unique survival copula of the pair (X^*, Y^*) is given, for all $u, v \in (0, 1)$, by

$$C(u, v) = \psi\{\psi^{-1}(u) + \psi^{-1}(v)\}, \quad (16)$$

i.e. C is by definition an Archimedean copula with generator ψ ; see, e.g. Genest & MacKay (1986), Genest et al. (2011), and McNeil & Nešlehová (2009). In fact, any copula that generates a Schur-constant survival function is necessarily Archimedean.

Given that the variables X^* and Y^* are scaled versions of X and Y , respectively, the pairs (X, Y) and (X^*, Y^*) share the same copula; see, e.g. Theorem 2.4.3 in Nelsen (1999). Thus one has the following result.

Proposition 4.1: *Let (X, Y) be a pair of continuous non-negative random variables whose joint survival function $S \in \mathcal{L}(\alpha, \beta)$ satisfies the b-BLUS property for some scalar $\pi \in (0, 1)$. The survival copula of the pair (X, Y) is then Archimedean with generator ψ , as given in Equation (16). Furthermore, the variables $X^* = \alpha X$ and $Y^* = \beta Y$ are identically distributed with convex survival function ψ .*

Proof: The last statement concerning the fact that X^* and Y^* are identically distributed with survival function ψ is obvious from Equation (15). It is also consonant with the fact that X has finite range $[0, x_0]$ if and only if Y has finite range $[0, \alpha x_0/\beta]$, as mentioned at the end of Section 3. ■

Following Corollary 2.5 in Salminen & Vallois (2005) or Proposition 3.3 in McNeil & Nešlehová (2009), identity (15) holds if and only if the pair (X^*, Y^*) has an ℓ_1 -norm symmetric distribution with no atom at the origin. For completeness, the notion of ℓ_1 -norm symmetric distribution, which is due to Fang & Fang (1988), is recalled below.

Definition 4.1: A random pair (X^*, Y^*) is said to have an ℓ_1 -norm symmetric distribution (or equivalently a simplicially contoured distribution) if and only if there exist independent random variables R and Q such that $X^* =_d RQ$, $Y^* =_d R(1 - Q)$, where $=_d$ denotes equality in distribution, R is non-negative and Q is uniformly distributed on the interval $(0, 1)$, denoted $Q \sim \mathcal{U}(0, 1)$.

In the present case, therefore, the variables αX and βY permit the stochastic representation

$$X =_d RQ/\alpha, \quad Y =_d R(1 - Q)/\beta, \quad (17)$$

where the variables R and Q are as specified in Definition 4.1 and R has no atom at zero. It is also easily checked that any random pair (X, Y) with an ℓ_1 -norm symmetric distribution has the b-BLUS property. Therefore, one has the following characterization result.

Proposition 4.2: Let (X, Y) be a pair of non-negative random variables whose joint survival function S is continuous and strictly decreasing in both of its arguments on its support. The following statements are then equivalent.

- (i) The joint survival function S satisfies the b -BLUS property (8) for some scalar $\pi \in (0, 1)$.
- (ii) There exist scalars $\alpha, \beta \in (0, \infty)$ such that $\alpha/\beta = \pi/\bar{\pi}$ and a convex univariate survival function ψ with $\psi(0) = 1$ for which $S(x, y) = \psi(\alpha x + \beta y)$ for all $x, y \in \mathbb{R}_+$, i.e. Equation (14) holds.
- (iii) The survival copula of the random pair (X, Y) is Archimedean with generator ψ and the variables αX and βY are identically distributed with convex survival function ψ such that $\psi(0) = 1$.
- (iv) The pair $(\alpha X, \beta Y)$ has an ℓ_1 -norm symmetric distribution with no atom at the origin.
- (v) The stochastic representation (17) holds, i.e. $\alpha X \stackrel{d}{=} RQ$ and $\beta Y \stackrel{d}{=} R(1 - Q)$ for some independent variables R and Q such that $Q \sim \mathcal{U}(0, 1)$ and $\Pr(R > r) = \psi(r) - r\psi'(r)$ for all $r \in \mathbb{R}_+$.

Proof: The only statement which still needs justification is the formula for the survival function of R , which is part of Theorem 4 in Nelsen (2005) or Theorem 3.1 in McNeil & Nešlehová (2009). Note that the convexity of ψ guarantees the existence and continuity of its right-hand derivative ψ' almost everywhere on \mathbb{R}_+ . ■

The example below, already announced in the Introduction, is a prime illustration of Proposition 4.2.

Example 4.1: The bivariate Gompertz distribution defined through Equation (5) has the b -BLUS property. A simple calculation based on Propositions 4.1 and 4.2 implies that, for all $w \in \mathbb{R}_+$,

$$\psi(w) = \ln\{1 - \ln(w)/\theta\}$$

and that as result, one has, for all $u, v \in (0, 1)$,

$$C(u, v) = uv \exp\{-\ln(u) \ln(v)/\theta\}.$$

This expression defines the Gumbel–Barnett copula with parameter $1/\theta \in (0, 1)$; see Chapter 4 in Nelsen (1999).

As any Archimedean generator ψ gives rise to a survival function with the b -BLUS property, additional examples could be constructed at will using, e.g. Table 4.1 in the book by Nelsen (1999). The few selected examples provided in the subsequent sections are intended to illustrate specific points.

5. Connection between the b -BLUS property and notions of dependence and aging

Given a continuous random pair (X, Y) with joint survival function $S \in \mathcal{L}(\alpha, \beta)$ for some $\alpha, \beta \in (0, \infty)$, it was seen in the previous section that the pair $(\alpha X, \beta Y)$ then has a Schur-constant survival function of the form (15) for some convex univariate survival function ψ with $\psi(0) = 1$. Caramellino & Spizzichino (1994) point out that this condition holds if and only if, for all $x, y \in \mathbb{R}_+$,

$$\Pr(\alpha X > x + t \mid \alpha X > x, \beta Y > y) = \Pr(\beta Y > y + t \mid \alpha X > x, \beta Y > y).$$

As mentioned by these authors and Barlow & Mendel (1993) before them, the above property can be viewed as a no-aging condition, given that the two ‘units’ have the same conditional distribution.

One may thus conclude that a pair (X, Y) of lives has the b-BLUS property if and only if there exist scalars $\alpha, \beta \in (0, \infty)$ such that αX and βY have the same distribution and, for all $x, y, z \in \mathbb{R}_+$,

$$\Pr(X > x + \bar{\pi}z \mid X > x, Y > y) = \Pr(Y > y + \pi z \mid X > x, Y > y),$$

where $\pi = \alpha/(\alpha + \beta)$. This means that under the b-BLUS property, the lives of husband and wife, say, must have distributions from the same scale family and the conditional distributions of their residual life lengths $X-x$ and $Y-y$, too.

To investigate the potential restrictions that the b-BLUS property imposes on the joint modeling of lives X and Y , first note that it is immediate from Definition 2.1 that S is BNBU if and only if $-\ln(\psi)$ is superadditive, i.e. for all $w, w' \in \mathbb{R}_+$.

$$\ln\{\psi(w + w')\} \leq \ln\{\psi(w)\} + \ln\{\psi(w')\}.$$

Given that ψ is the shared survival function of αX and βY , the above condition amounts to saying that these variables, and hence also X and Y , have a new better than used (NBU) distribution; see Barlow & Prochan (1975). Similarly, S is BWBU if and only if $\ln(\psi)$ is superadditive, i.e. if and only if X and Y have new worse than used (NWU) distributions.

As it turns out, the very same conditions on ψ also characterize the following classical notions of dependence for a random pair (X, Y) of lives with survival function S ; see Lehmann (1966).

Definition 5.1: A random pair (X, Y) is said to be positive quadrant dependent (PDQ) if and only if $\Pr(X > x, Y > y) \geq \Pr(X > x) \Pr(Y > y)$ for all $x, y \in \mathbb{R}$. The same pair is said to be negative quadrant dependent (NQD) if and only if $\Pr(X > x, Y > y) \leq \Pr(X > x) \Pr(Y > y)$ for all $x, y \in \mathbb{R}$.

Lemma 5.1: Suppose that a bivariate survival function can be written in the form (14) for some constants $\alpha, \beta \in (0, \infty)$ and a convex survival function ψ with $\psi(0) = 1$. The following facts hold true:

- (i) S is BNBU $\Leftrightarrow S$ is NQD $\Leftrightarrow X$ is NBU $\Leftrightarrow Y$ is NBU $\Leftrightarrow -\ln(\psi)$ is superadditive.
- (ii) S is BNWU $\Leftrightarrow S$ is PQD $\Leftrightarrow X$ is NWU $\Leftrightarrow Y$ is NWU $\Leftrightarrow \ln(\psi)$ is superadditive.

Proof: The conclusion is a simple consequence of Proposition 1 in Avérous & Dortet-Bernadet (2004), which was itself a formalization of findings that were implicit in Capéreaux & Genest (1993) and Genest & MacKay (1986). ■

Therefore, the b-BLUS property can model lives that are either positively or negatively dependent. The following examples illustrate each one of these cases.

Example 5.1: For given scalars $\alpha, \beta, \gamma \in (0, \infty)$, the joint survival function defined at all $x, y \in \mathbb{R}_+$ by

$$S(x, y) = (1 + \alpha x + \beta y)^{-1/\gamma}$$

was introduced by Lindley & Singpurwalla (1986) as a family of bivariate Pareto distributions. It has the b-BLUS property as it is of the form (14) with ψ given by $\psi(w) = (1 + w)^{-1/\gamma}$ for all $w \in \mathbb{R}_+$. This is the generator of a Clayton copula with Kendall's tau equal to $\gamma/(\gamma + 2) \in (0, 1)$. It is easily seen that $\ln(\psi)$ is superadditive, which implies that the model is both BNWU and PQD by Lemma 5.1.

Example 5.2: For given scalars $\alpha, \beta \in (0, \infty)$ and $\rho \in [1, \infty)$, consider the survival function defined at every pair $(x, y) \in \mathbb{R}_+$ such that $\alpha x + \beta y < 1$ by

$$S(x, y) = (1 - \alpha x - \beta y)^\rho.$$

This model, initially considered by Roy (1989), is suitable for variables X and Y whose support is finite, viz. $[0, x_0] = [0, 1/\alpha]$ and $[0, y_0] = [0, 1/\beta]$, respectively. Observe that $y_0 = \alpha x_0/\beta$, as stated

at the end of Section 3. Distributions in this class have the b-BLUS property as they can be expressed in the form (14) with the map ψ given by $\psi(w) = (1 - w)^\rho$ for all $w \in [0, 1]$, and $\psi(w) = 0$ by convention when $w \in (1, \infty)$. Given that $-\ln(\psi)$ is superadditive, this model is both BNBU and NQD by Lemma 5.1. The generator ψ does not correspond to any named copula except when $\rho = 1$, which is the Fréchet–Hoeffding lower bound, i.e. the distribution of a pair $(Q, 1 - Q)$, where $Q \sim \mathcal{U}(0, 1)$. A simple calculation based, e.g. on the formula given in Corollary 5.1.4 in Nelsen (1999) shows that Kendall’s tau for this model is given by $1/(1 - 2\rho) \in [-1, 0)$. See Example 3.1 in McNeil & Nešlehová (2009) for further discussion.

From Lemma 2.1, the only model with the b-BLUS property which is both BNBU and BWBU is the product of two exponential distributions, whose survival function is given, for all $x, y \in \mathbb{R}_+$, by

$$S(x, y) = \exp(-\alpha x - \beta y),$$

where $\alpha, \beta \in (0, \infty)$ are given scalars. In the light of Lemma 5.1, this result is not surprising because PQD and NQD are properties of the underlying copula, and the only copula which is both PQD and NQD is the product copula, which induces independence of the margins. This model is of the form (14) with ψ given, for all $w \in \mathbb{R}_+$, by $\psi(w) = e^{-w}$. This is indeed the only choice of convex survival function ψ with support on \mathbb{R}_+ for which $\pm \ln(\psi)$ are both superadditive.

Examples 5.1–5.2 also show that in terms of degrees of dependence, models with the b-BLUS property impose no restriction, at least when measured in terms of margin-free measures of association such as Kendall’s tau or Spearman’s rho, which can span all possible values between -1 and 1 with suitable choices of generator ψ . However, perhaps surprisingly, the possible range of values for Pearson’s correlation is restricted to the interval $[-1, 1/2)$. This is immediate from the following formula, which is reported both by Caramellino & Spizzichino (1994) and Chi et al. (2009) in the case of Schur-constant models and hence also valid in the present context:

$$\text{corr}(X, Y) = \text{corr}(\alpha X, \beta Y) = \frac{2\kappa^2 - 1}{4\kappa^2 + 1} \in [-1, 1/2), \tag{18}$$

where $\kappa = \sqrt{\text{var}(R)}/E(R)$ is the coefficient of variation associated with the radial variable $R = \alpha X + \beta Y$, assuming that it is finite. Chi et al. (2009) remark that all values in the interval $[-1, 1/2)$ can be achieved by Pearson’s correlation, but neither they nor Caramellino & Spizzichino (1994) offer an explanation for the inability of $\text{corr}(X, Y)$ to take values greater than $1/2$, in spite of the fact that the two Fréchet–Hoeffding bounds are limiting cases of the Archimedean class (Genest & MacKay 1986).

The key to the resolution of this apparent paradox is in the necessity for R to have two finite moments as a prerequisite for a correlation to exist. To illustrate this point, consider the Archimedean generator introduced in Example 5.1, for which $\psi(t) = (1 + t)^{-1/\gamma}$ for any $\gamma \in (0, \infty)$. It then follows from part (v) of Proposition 4.2 that, for any $r \in \mathbb{R}_+$, one has

$$E(R) = \int_0^\infty \Pr(R > r) \, dr = \int_0^\infty \{(1 + r)^{-1/\gamma} - \gamma^{-1} r (1 + r)^{-1/\gamma - 1}\} \, dr = \frac{\gamma(\gamma + 1)}{1 - \gamma}$$

whenever $\gamma \in (0, 1)$ and that the expectation fails to exist if $\gamma \in [1, \infty)$.

Various other concepts of dependence are available in the literature which are more demanding than PQD/NQD; see, e.g. Chapter 5 of the book by Nelsen (1999). For Archimedean copulas, characterizations of these concepts in terms of properties of their generator were derived in Avérous & Dortet-Bernadet (2004) and Capéraà & Genest (1993). These results all have an immediate analog in terms of joint lifetimes with the b-BLUS property.

To give just one more example, consider the notion of stochastic increasingness/decreasingness, whose definition is recalled below.

Definition 5.2: A random variable Y is said to be stochastically increasing in another variable X if and only if $\Pr(Y \leq y \mid X = x') \leq \Pr(Y \leq y \mid X = x)$ for all $x, x', y \in \mathbb{R}$ such that $x < x'$. Similarly, Y is said to be stochastically decreasing in another variable X if and only if $\Pr(Y \leq y \mid X = x') \geq \Pr(Y \leq y \mid X = x)$ for all $x, x', y \in \mathbb{R}$ such that $x < x'$.

Stochastic increasingness, which is more demanding than positive quadrant dependence, is a particularly useful concept as it implies that $E(Y \mid X = x)$ is a decreasing function of x . Proposition 3.3 in Capéraà & Genest (1993) states that an Archimedean copula with generator ψ is stochastically increasing if and only if $\ln(-\psi')$ is convex, and that it is stochastically decreasing if and only if $-\ln(-\psi')$ is convex, which is the same as saying that the distribution function $1 - \psi$ is strongly unimodal.

The following lemma records this fact, along with formulas for the conditional mean and variance of Y given $X = x$ for any random pair (X, Y) with the b-BLUS property.

Lemma 5.2: Let (X, Y) be a random pair with continuous survival function $S \in \mathcal{L}(\alpha, \beta)$ for some $\alpha, \beta \in (0, \infty)$. Then, for all $x \in \mathbb{R}_+$,

$$E(Y \mid X = x) = \frac{1}{\beta h(\alpha x)}, \quad \text{var}(Y \mid X = x) = \frac{1}{\beta^2} \frac{2r(\alpha x)h(\alpha x) - 1}{h^2(\alpha x)},$$

where h and r respectively stand for the hazard and mean residual function of $X^* = \alpha X$, given by

$$h(x) = -\frac{d}{dx} \ln\{\Pr(X^* > x)\} = -\frac{\psi'(x)}{\psi(x)}, \quad r(x) = E(X^* - x \mid X^* > x) = \frac{1}{\psi(z)} \int_z^\infty \psi(w) dw \tag{19}$$

in terms of the Archimedean generator ψ of S . Furthermore, the map $x \mapsto E(Y \mid X = x)$ is decreasing when $\ln(-\psi')$ is convex and increasing when $1 - \psi$ is strongly unimodal.

Proof: The formulas for the conditional mean and variance of Y given $X = x$ easily follow from Theorem 4 in Nelsen (2005) and Proposition 2.1 in Unnikrishnan Nair & Sankaran (2014), upon noting that the pair $(\alpha X, \beta Y)$ is Schur-constant by Proposition 4.2. The expressions for h and r in terms of ψ are also immediate consequences of the fact that, by the same result, the survival function of X^* is ψ . As noted above, the behavior of the map $x \mapsto E(Y \mid X = x)$ follows from Proposition 3.3 in Capéraà & Genest (1993). ■

This result also invites the investigation of the dependence between the components of a pair (X, Y) of continuous lifetimes with the b-BLUS property, conditional on their current ages. In the special case where X and Y are exchangeable, it is known from Proposition 4.2 that the survival function S of (X, Y) is Schur-constant, and the work of Unnikrishnan Nair & Sankaran (2014) can be used to deduce the properties of time-dependent association measures such as the correlation curve and the conditional correlation curve introduced by Bjerve & Doksum (1993), or the nonparametric measures due to Anderson et al. (1992).

Given that the pair $(\alpha X, \beta Y)$ is Schur-constant whenever $S \in \mathcal{L}(\alpha, \beta)$ for some $\alpha, \beta \in (0, \infty)$, it is relatively straightforward to extend the findings of Unnikrishnan Nair & Sankaran (2014) to the present context. To give but one example, consider the measure

$$\varphi(x, y) = \Pr(X > x \mid Y > y) / \Pr(X > x)$$

due to Anderson et al. (1992), which is equal to 1 when X and Y are independent and takes values greater than 1 when X and Y are positively associated. In view of Proposition 4.2, one has, for all

$x, y \in \mathbb{R}_+$,

$$\ln\{\varphi(x, y)\} = \ln\{\psi(\alpha x + \beta y)\} - \ln\{\psi(\alpha x)\} - \ln\{\psi(\beta y)\}.$$

Therefore, $\ln\{\varphi(x, y)\} \geq 0$ for all $x, y \in \mathbb{R}_+$ if and only if $\ln(\psi)$ is superadditive, i.e. when X and Y are NWU. As stated in Lemma 5.1, this is equivalent to S being BNWU or PQD.

6. Relation of the b-BLUS property with other reliability concepts

In this section, a relation will be established between the hazard gradients and the mean residual lifetimes of an absolutely continuous survival function $S \in \mathcal{L}(\alpha, \beta)$ satisfying the b-BLUS property for some scalar $\pi \in (0, 1)$. The definitions of these bivariate analogs of the univariate hazard and mean residual function are first recalled below for completeness; for additional details, see, e.g. Arnold & Zahedi (1988).

Definition 6.1: Let (X, Y) be a non-negative random pair with bivariate survival function S . The components of its hazard gradient (h_1, h_2) are then defined, for all $x, y \in \mathbb{R}_+$, by

$$h_1(x, y) = -\frac{\partial}{\partial x} \ln\{S(x, y)\}, \quad h_2(x, y) = -\frac{\partial}{\partial y} \ln\{S(x, y)\},$$

and the components of its bivariate mean residual lifetime (r_1, r_2) are defined by

$$r_1(x, y) = E(X - x \mid X > x, Y > y) = \frac{1}{S(x, y)} \int_x^\infty S(w, y) \, dw$$

and

$$r_2(x, y) = E(Y - y \mid X > x, Y > y) = \frac{1}{S(x, y)} \int_y^\infty S(x, w) \, dw.$$

Given that Equation (14) holds when $S \in \mathcal{L}(\alpha, \beta)$, it is easily checked that, for all $x, y \in \mathbb{R}_+$,

$$h_1(x, y) = \alpha h(\alpha x + \beta y), \quad h_2(x, y) = \beta h(\alpha x + \beta y),$$

while

$$r_1(x, y) = \frac{1}{\psi(\alpha x + \beta y)} \int_x^\infty \psi(\alpha w + \beta y) \, dw = \frac{1}{\alpha} r(\alpha x + \beta y)$$

and

$$r_2(x, y) = \frac{1}{\psi(\alpha x + \beta y)} \int_y^\infty \psi(\alpha x + \beta w) \, dw = \frac{1}{\beta} r(\alpha x + \beta y),$$

as one can readily verify by making the appropriate change of variables in each of the two integrals.

These calculations lead to the following analog of Proposition 2.4 in Unnikrishnan Nair & Sankaran (2014).

Lemma 6.1: *Let (X, Y) be a pair of non-negative random variables with joint survival function S , and let h and r respectively stand for the hazard rate and the mean residual lifetime of a random variable Z with convex survival function ψ with $\psi(0) = 1$, as defined in Equation (19). Then $S \in \mathcal{L}(\alpha, \beta)$ for some scalars $\alpha, \beta \in (0, \infty)$ if and only if one of the following statements holds:*

- (i) $h_1(x, y)/\alpha = h_2(x, y)/\beta = h(\alpha x + \beta y)$ for all $x, y \in \mathbb{R}_+$;
- (ii) $\alpha r_1(x, y) = \beta r_2(x, y) = r(\alpha x + \beta y)$ for all $x, y \in \mathbb{R}_+$.

Proof: It was already seen that statements (i) and (ii) are both valid when $S \in \mathcal{L}(\alpha, \beta)$. To show the reciprocal statements, it suffices to call on the fact that, for all $x, y \in \mathbb{R}_+$, one has

$$S(x, y) = \exp \left\{ - \int_0^x h_1(w, y) \, dw - \int_0^y h_2(x, w) \, dw \right\}$$

and

$$S(x, y) = \frac{r_1(0, 0)}{r_1(x, 0)} \frac{r_2(x, 0)}{r_2(x, y)} \exp \left\{ - \int_0^x \frac{dw}{r_1(w, 0)} - \int_0^y \frac{dw}{r_2(x, w)} \right\}.$$

The first formula leads to the conclusion that $S(x, y) = \psi(\alpha x + \beta y)$ for all $x, y \in \mathbb{R}_+$ when (i) holds. The second formula leads to the same conclusion when (ii) holds. ■

Lemma 6.1 implies that if a survival function S has the b-BLUS property, then, for all $x, y \in \mathbb{R}_+$,

$$h_1(x, y)r_1(x, y) = h_2(x, y)r_2(x, y).$$

While necessary, the latter requirement is not sufficient to guarantee that S is of the form (14), as shown next.

Example 6.1: Let S be a continuous survival function whose support is \mathbb{R}_+ . Let (h_1, h_2) and (r_1, r_2) be its hazard gradient and bivariate mean residual lifetime, respectively. Assume that for all $x, y \in \mathbb{R}_+$,

$$h_1(x, y)r_1(x, y) = h_2(x, y)r_2(x, y) = 1.$$

It then follows from Theorem 3.1 in Roy & Gupta (1996) that S is the survival function of a bivariate Gumbel exponential distribution with parameters $\alpha, \beta \in (0, \infty)$ and $\delta \in [0, \alpha\beta]$, i.e. for all $x, y \in \mathbb{R}_+$,

$$S(x, y) = \exp(-\alpha x - \beta y - \delta xy).$$

Clearly, this survival function is not of the form (14) unless $\delta = 0$. The latter case corresponds to the case of two independent exponential variables, as already discussed in Lemma 2.1.

Lemma 6.1 provides a different way of generating bivariate lifetime models with the b-BLUS models other than through the selection of an Archimedean copula generator.

Example 6.2: Fix $\alpha, \beta, \theta \in (0, \infty)$ and consider a mean residual lifetime given, for all $w \in \mathbb{R}_+$, by $r(w) = \theta e^{-w}$. The corresponding survival function is then such that, for all $w \in \mathbb{R}_+$,

$$\psi(w) = \exp\{w - \theta(e^w - 1)\}.$$

It is easily checked that ψ is convex and hence an Archimedean generator whenever $\theta \geq (3 + \sqrt{5})/2 \approx 2.618$. The bivariate survival distribution generated via Equation (14) is then such that, for all $x, y \in \mathbb{R}_+$,

$$\alpha r_1(x, y) = \beta r_2(x, y) = r(\alpha x + \beta y).$$

By simple substitution, one gets, for all $x, y \in \mathbb{R}_+$,

$$S(x, y) = \exp\{\alpha x + \beta y - \theta(e^{\alpha x + \beta y} - 1)\}. \tag{20}$$

The same expression can be obtained using the second display in the proof of Lemma 6.1. Distributions in this class are BNWU and PQD by Lemma 5.1, given that $\ln(\psi)$ is superadditive.

In the special case $\alpha = \beta$, the survival function defined in Equation (20) reduces to the bivariate Teissier distribution introduced by Kolev et al. (2017), albeit with a different parametrization. Example 6.2 thus presents a new extension of this distribution that can accommodate non-exchangeable components.

Example 6.3: Fix $\alpha, \beta, \theta \in (0, \infty)$ and consider a mean residual lifetime given, for all $w \in \mathbb{R}_+$, by $r(w) = \theta e^w$. The corresponding survival function is then such that, for all $w \in \mathbb{R}_+$,

$$\psi(w) = \exp\{-w + \theta(e^{-w} - 1)\}.$$

It is easily checked that ψ is always convex and hence an Archimedean generator and that the bivariate survival distribution generated via Equation (14) is given, for all $x, y \in \mathbb{R}_+$, by

$$S(x, y) = \exp\{-\alpha x - \beta y - \theta(e^{-\alpha x - \beta y} - 1)\}.$$

Distributions in this class are BNWU and PQD by Lemma 5.1, given that $\ln(\psi)$ is superadditive.

The model presented in Example 6.3 is, to the authors' knowledge, the first bivariate extension of the Chiang–Conforti family of distributions (Chiang & Conforti 1989).

7. Closure properties and stochastic orderings

The usefulness of Schur-constant models was illustrated in insurance contexts by Chi et al. (2009) for continuous survival lifetimes and by Castañer et al. (2015) in the discretized case. Processes for which the claim inter-arrival periods form can be modeled by a survival function of the form (14) with $\alpha = \beta$ include random payment processes and claim counting processes. As they allow the parameters α and β to differ, models with the b-BLUS property represent a significant extension of this class of models.

In insurance applications, distributional closure properties are often critical to the derivation of closed form solutions. Such properties enjoyed by members of the class $\mathcal{L}(\alpha, \beta)$ are reported below.

Lemma 7.1: *Let $(X, Y), (X_1, Y_1), (X_2, Y_2)$ be mutually independent pairs of non-negative random variables whose distributions belong to the class $\mathcal{L}(\alpha, \beta)$ for some $\alpha, \beta \in (0, \infty)$. Let also B be a Bernoulli random variable independent of (X_1, Y_1) and (X_2, Y_2) . Then the following statements hold true.*

- (i) *The distribution of (cX, cY) is in the class $\mathcal{L}(\alpha, \beta)$ for any $c \in (0, \infty)$.*
- (ii) *The distribution of $(X_\wedge, Y_\wedge) = \min\{(X_1, Y_1), (X_2, Y_2)\}$ is in the class $\mathcal{L}(\alpha, \beta)$.*
- (iii) *The distribution of $(X_\Sigma, Y_\Sigma) = B(X_1, Y_1) + (1 - B)(X_2, Y_2)$ is in the class $\mathcal{L}(\alpha, \beta)$.*

Proof: To prove part (i), let ψ be the Archimedean generator for which $\Pr(X > x, Y > y) = \psi(\alpha x + \beta y)$ for all $x, y \in \mathbb{R}_+$. For arbitrary $t \in \mathbb{R}_+$, set $\psi_c(t) = \psi(t/c)$. It is clear that ψ_c is also an Archimedean generator and that for all $x, y \in \mathbb{R}_+$, one has $\Pr(cX > x, cY > y) = \psi(\alpha x/c + \beta y/c) = \psi_c(\alpha x + \beta y)$. Turning to part (ii), suppose that for all $x, y \in \mathbb{R}_+$,

$$\Pr(X_1 > x, Y_1 > y) = S_1(x, y) = \psi_1(\alpha x + \beta y), \quad \Pr(X_2 > x, Y_2 > y) = S_2(x, y) = \psi_2(\alpha x + \beta y),$$

where the maps $\psi_1, \psi_2 : \mathbb{R}_+ \rightarrow [0, 1]$ are strictly decreasing and convex as per Proposition 4.2. Then, for all $x, y \in \mathbb{R}_+$, one has

$$\Pr(X_\wedge > x, Y_\wedge > y) = S_1(x, y)S_2(x, y) = \psi_1(\alpha x + \beta y) \psi_2(\alpha x + \beta y) = \psi(\alpha x + \beta y), \quad (21)$$

where the map $\psi = \psi_1 \psi_2$ is both strictly decreasing and convex, which shows that the distribution of $\min\{(X_1, Y_1), (X_2, Y_2)\} = (X^*, Y^*)$ is in the class $\mathcal{L}(\alpha, \beta)$.

To establish part (iii), let $b = \Pr(B = 1)$ and for all $t \in \mathbb{R}_+$, set $\psi_\Sigma(t) = b\psi_1(t) + (1 - b)\psi_2(t)$. Then, upon conditioning on the value of B , one finds that, for all $x, y \in \mathbb{R}_+$,

$$\Pr(X_\Sigma > x, Y_\Sigma > y) = b\psi_1(\alpha x + \beta y) + (1 - b)\psi_2(\alpha x + \beta y) = \psi_\Sigma(\alpha x + \beta y),$$

which leads to the desired conclusion given that ψ_Σ is a convex decreasing survival function on \mathbb{R}_+ , and hence an Archimedean generator. ■

Equation (21) also implies that if $(X_1, Y_1), \dots, (X_n, Y_n)$ form a random sample from a distribution in the class $\mathcal{L}(\alpha, \beta)$ with survival function of the form (15) with generator ψ , then for all $x, y \in \mathbb{R}_+$,

$$\Pr\{\min(X_1, \dots, X_n) > x, \min(Y_1, \dots, Y_n) > y\} = \psi^n(\alpha x + \beta y).$$

A simple adaptation of Proposition 2 in Larsson & Nešlehová (2011) or Statement B in Genest & Rivest (1989) then implies that if this distribution has a min-attractor, the latter has a survival function that depends on a parameter $\theta \in [1, \infty)$ and which is defined, for all $x, y \in \mathbb{R}_+$, by

$$S_\infty(x, y) = \exp\{-(\alpha x + \beta y)^\theta\}. \tag{22}$$

In other words, the limit remains within the class $\mathcal{L}(\alpha, \beta)$; it has Weibull margins and the limiting copula is from the Gumbel family with Kendall’s tau equal to $1 - 1/\theta \in [0, 1]$. For this limit to exist, the map $t \mapsto 1 - \psi(1/t)$ must be regularly varying of degree $-1/\theta$. For completeness, this definition is recalled below and the limiting result is formally recorded as a lemma.

Definition 7.1: A measurable function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be regularly varying with index ρ if and only if, for any $x \in \mathbb{R}_+$, $f(tx)/f(x) \rightarrow x^\rho$ as $t \rightarrow \infty$.

Lemma 7.2: Let $S \in \mathcal{L}(\alpha, \beta)$ be a survival function of the form (15) for some $\alpha, \beta \in (0, \infty)$ and some Archimedean generator ψ . Further suppose that the map $t \mapsto 1 - \psi(1/t)$ is regularly varying of degree $-1/\theta \in (0, 1]$. The min-attractor of S is then the member of the class $\mathcal{L}(\alpha, \beta)$ defined by Equation (22).

Yet another useful closure property of the class $\mathcal{L}(\alpha, \beta)$ is given next.

Lemma 7.3: Let (X, Y) be a pair of non-negative random variables whose distribution belongs to the class $\mathcal{L}(\alpha, \beta)$ for some $\alpha, \beta \in (0, \infty)$. Given $x, y \in \mathbb{R}_+$, let $X_x = X - x$ and $Y_y = Y - y$ be the residual lifetimes, conditional on the events $X > x$ and $Y > y$. Then the distribution of the pair (X_x, Y_y) also belongs to the class $\mathcal{L}(\alpha, \beta)$.

Proof: Let ψ be the Archimedean generator for which the survival function S of the pair (X, Y) can be expressed in the form (15). For given $x, y \in \mathbb{R}_+$, define the map ψ_{xy} at every $t \in \mathbb{R}_+$ by $\psi_{xy}(t) = \psi(t + \alpha x + \beta y)/\psi(\alpha x + \beta y)$. It is clear that ψ_{xy} is an Archimedean generator and that, for all $u, v \in \mathbb{R}_+$, one has

$$\begin{aligned} \Pr(X_x > u, Y_y > v \mid X > x, Y > y) &= \frac{\Pr(X > x + u, Y > y + v)}{\Pr(X > x, Y > y)} \\ &= \frac{\psi(\alpha x + \alpha u + \beta y + \beta v)}{\psi(\alpha x + \beta y)} = \psi_{xy}(\alpha u + \beta v). \end{aligned}$$

This concludes the argument. ■

Unfortunately, a closure property for sums is not available for the class $\mathcal{L}(\alpha, \beta)$. That is, if $(X, Y), (X_1, Y_1), \dots, (X_n, Y_n)$ constitute a random sample from a distribution in the class $\mathcal{L}(\alpha, \beta)$

with survival function of the form (15) with generator ψ , then it is not generally true that the distribution of the pair $(X_1 + \dots + X_n, Y_1 + \dots + Y_n)$ has the b-BLUS property. So long as the variables X and Y have finite moments of order 2, the central limit theorem can be invoked to show that the large-sample distribution of the pair is Gaussian with correlation as given in Equation (18).

As a final result involving sums, the following stochastic ordering result, which is easily adapted from Chi et al. (2009), can be useful in comparing risks in an insurance context. This result relies on the definition of standard univariate and bivariate integrals orders that are recalled for completeness.

Definition 7.2: Let (X_1, Y_1) and (X_2, Y_2) be two pairs of random variables, and for each $i \in \{1, 2, 3\}$ and $\ell \in \{1, 2\}$, let $\mathcal{C}_{i\ell}$ be the class of functions of ℓ variables that are increasing ($i = 1$), convex ($i = 2$) or both convex and increasing ($i = 3$). In dimension $\ell = 2$, increasingness and convexity are defined with respect to the ordering \prec which states that, for any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}_+$,

$$(x_1, y_1) \prec (x_2, y_2) \iff x_1 \leq x_2 \text{ and } y_1 \leq y_2.$$

- (i) It is said that X_1 is smaller than X_2 in the order \prec_{i_1} , denoted $X_1 \prec_{i_1} X_2$, if and only if $E\{\phi(X_1)\} \leq E\{\phi(X_2)\}$ for all maps $\phi \in \mathcal{C}_{i_1}$ for which the expectations exist.
- (ii) It is said that (X_1, Y_1) is smaller than (X_2, Y_2) in the order \prec_{i_2} , denoted $(X_1, Y_1) \prec_{i_2} (X_2, Y_2)$, if and only if $E\{\phi(X_1, Y_1)\} \leq E\{\phi(X_2, Y_2)\}$ for all maps $\phi \in \mathcal{C}_{i_2}$ for which the expectations exist.

Lemma 7.4: Let (X_1, Y_1) and (X_2, Y_2) be two pairs of lives whose survival functions belong to the class $\mathcal{L}(\alpha, \beta)$ for some $\alpha, \beta \in (0, \infty)$. Then for $\pi = \alpha/(\alpha + \beta)$ and all $i \in \{1, 2, 3\}$,

$$(X_1, Y_1) \prec_{i_2} (X_2, Y_2) \iff \pi X_1 + \bar{\pi} Y_1 \prec_{i_1} \pi X_2 + \bar{\pi} Y_2.$$

Proof: For arbitrary $i \in \{1, 2, 3\}$, it follows from Theorem 6.B.16 in Shaked & Shanthikumar (2007) that

$$(X_1, Y_1) \prec_{i_2} (X_2, Y_2) \iff (\alpha X_1, \beta Y_1) \prec_{i_2} (\alpha X_2, \beta Y_2)$$

and Proposition 4.1 in Chi et al. (2009) implies that the right-hand statement is equivalent to $\alpha X_1 + \beta Y_1 \prec_{i_1} \alpha X_2 + \beta Y_2$, from which the closure properties of the univariate order can be invoked to conclude. ■

8. Conclusion

This paper proposed a bivariate extension of the Law of Uniform Seniority and explored some of its consequences in the special case where the averaging function a appearing in Equation (3) is bilinear. A functional representation was derived in Proposition 3.1, from which it was concluded that pairs (X, Y) of lives satisfying the b-BLUS have marginal distributions belonging to the same scale family and are bound by a survival Archimedean copula.

Equivalently, it was shown in Proposition 4.1 that if a pair (X, Y) of lives satisfies the b-BLUS, then there exist scalars $\alpha, \beta \in (0, \infty)$ such that αX and βY have the same distribution and the pair $(\alpha X, \beta Y)$ has an ℓ_1 -norm symmetric distribution. A stochastic representation of the pair (X, Y) was also given in Proposition 4.2 in terms of the radial variable $R = \alpha X + \beta Y$ multiplying a pair $(Q, 1 - Q)$ of uniform random variables that are countermonotonic. One could thus think of R as the entire weighted lifetime on a couple, randomly allotted between husband and wife.

In the second part of the paper, the dependence and aging properties of this new class of bivariate lifetime models were briefly explored. Special cases of interest were given in Examples 6.2–6.3, including a non-exchangeable version of the (recently resurrected) Teissier bivariate distribution.

Moreover, various closure properties of the model were mentioned which may prove useful in insurance applications.

Models with the BLUS or b-BLUS property identified here could provide a complement and alternative to existing strategies for valuing contractual guarantees such as annuities based on joint lives. The importance of accounting for the observed dependence between individuals covered by the same policy was well portrayed by Frees et al. (1996), who fitted copula-based joint mortality models to censored residual lifetimes for policyholders in a data set of joint annuity contracts from a large Canadian insurance company. This study was subsequently revisited by many authors; see Gobbi et al. (2019) and references therein. Alternative strategies for modeling joint life insurance contracts can also be found in the work of Gouriéroux & Lu (2015), who use a French data set for illustration purposes.

While this paper focused on the bivariate case for simplicity, most of the material actually extends easily to situations involving not two, but an arbitrary number $d \geq 2$ of lives, at least in the case of a multilinear averaging function a . It has already been reported by Kaminsky (1983) that the standard d -variate version of Gompertz's law of mortality satisfies a multilinear form of LUS.

To illustrate, suppose that there exist scalars $\alpha_1, \dots, \alpha_d \in (0, \infty)$ such that the joint survival function of a vector (X_1, \dots, X_d) of lives can be expressed, for all $x_1, \dots, x_d \in \mathbb{R}_+$, in the form

$$\Pr(X_1 > x_1, \dots, X_d > x_d) = \psi(\alpha_1 x_1 + \dots + \alpha_d x_d).$$

Then the vector $(\alpha_1 X_1, \dots, \alpha_d X_d)$ has an ℓ_1 -norm symmetric distribution in \mathbb{R}^d and it follows from the work of McNeil & Nešlehová (2009) that ψ must be an Archimedean generator in dimension d . All the results derived herein would then apply to the dependence and aging structures of any pair $(\alpha_i X_i, \alpha_j X_j)$ with distinct $i, j \in \{1, \dots, d\}$. A stochastic representation for the weighted vector of lifetimes would also be available from McNeil & Nešlehová (2009) in terms of a radial part $R = \alpha_1 X_1 + \dots + \alpha_d X_d$ and a vector (U_1, \dots, U_d) uniformly distributed on the simplex.

As $d \rightarrow \infty$, however, the resulting model would suffer from the same limitations as copulas in the Archimedean class. In particular, the model would gradually lose the ability to accommodate negative degrees of association between its components, essentially because ψ must be a Laplace transform to guarantee that it generates an Archimedean copula in all dimensions. This is reflected, e.g. in the correlation formula given by Caramellino & Spizzichino (1994), which implies that when the radial variable R has a finite moment of order 2, then for any distinct $i, j \in \{1, \dots, d\}$,

$$\text{corr}(X_i, X_j) = \frac{d \mathbb{E}(R^2) - (d+1)E^2(R)}{2d \mathbb{E}(R^2) - (d+1)E^2(R)},$$

which implies that one must necessarily have $\text{corr}(X_i, X_j) \geq -1/(d-1)$. Along similar lines, McNeil & Nešlehová (2009) show that the value of Kendall's tau between any two components of a d -variate random vector with Archimedean dependence is bounded below by $-1/(2d-3)$. A detailed study of the properties of a multivariate extension of the LUS may be the object of future work.

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