Closed Invariant Subspaces of a Birnbaum-Orlicz Space*

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We begin by summarizing the theory of Birnbaum-Orlicz spaces $L_A(X)$ of functions defined on an arbitrary measure space $(X, \mathcal{M}, \mu)$. In Section 3 we show that if G is a locally compact group and A is a nontrivial generalized Young's function, the space $L_A(G)$ is a left Banach L_1-module and a right Banach $(L_1 \cap L_1^*)$-module.

Finally, in Section 4 we characterize the closed invariant subspaces of $L_A(G)$, where G is a compact group and A satisfies the Δ_2-condition for $u \geq u_0 \geq 0$.

Our notation is as in [2], [3] and [4]. Detailed proofs of all the statements can be found in [1]. Readers who are interested particularly in the material of Sections 1 and 2 are referred to [5] and [6].

§1. Generalized Young's Functions.

(1.1) DEFINITION. A function A on $[0, \infty[$ into $[0, \infty]$ will be called a *generalized Young's function* if

(i) $A(0) = 0$;

(ii) $\dfrac{A(u)}{u}$ is nondecreasing for $u > 0$;

(iii) A is left continuous on $]0, \infty[$.

The zero function and the function which is equal to zero at zero, and equal to ∞ on $]0, \infty[$ are generalized Young's functions. They will be called trivial functions.

Throughout the remaining of this work the letter A will denote a nontrivial generalized Young's function. We also fix a and b as follows: $a = sup\{u : A(u) = 0\}$, $b = inf\{u : A(u) = \infty\}$.

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(1.2) The following properties of A are easily verified:

(i) A is nondecreasing on $[0, \infty[$ and for $a < b$, A is strictly increasing on $[a, b[$;
(ii) $A(\alpha u) \leq \alpha A(u)$ for $0 \leq \alpha \leq 1$ and $0 \leq u < \infty$.

(1.3) DEFINITION. Let p be a nontrivial nondecreasing function on $[0, \infty[$ into $[0, \infty]$. The function B defined by the equality $B(u) = \int_0^u p(t)dt$ for $0 \leq \leq u < \infty$ is called a *Young's function.*

(1.4) THEOREM. *For B as in (1.3) ew have:*

(i) $B(\alpha u + (1-\alpha)v) \leq \alpha B(u) + (1-\alpha)B(v)$ *for all u and v in $[0, \infty[$ and any α in $[0, 1]$;*
(ii) *B is a generalized Young's function.*

(1.5) THEOREM. *Let B be a function on $[0, \infty[$ into $[0, \infty]$ and let $c = inf\{u : B(u) = \infty\}$. Suppose that: $B(0) = 0$, $c > 0$ and B is convex on $[0, c[$. Then B is a Young's function.*

(1.6) DEFINITION. The function A_0 defined by the equality

(i) $A_0(u) = \int_0^u \frac{A(t)}{t} dt$ for $0 \leq u < \infty$ will be called the *regularization* of A.

(1.7) The following properties of A_0 are clear:

(i) A_0 is a Young's function;
(ii) $A_0(u) \leq A(u) \leq A_0(2u)$ for $0 \leq u < \infty$.

(1.8) DEFINITION. The *Young's complement* of A is the function $\bar{A}$ defined on $[0, \infty[$ by the relation

(i) $$\bar{A}(v) = \sup_{0 \leq u < \infty} [uv - A(u)].$$

(1.9) A short computation and an application of (1.5) tell us that $\bar{A}$ is a nontrivial Young's function. It follows immediately from (1.8.i) that

(ii) $$uv \leq A(u) + \bar{A}(v)$$

for all u and v in $[0, \infty[$.

(1.10) THEOREM. *The function A^{-1} defined on $[0, \infty[$ by the equality*

$$A^{-1}(v) = \sup\{u : A(u) \leq v\}$$

is nondecreasing and right continuous. In addition it has the following properties:

(i) $0 < A^{-1}(v) < \infty$ *for* $0 < v < \infty$;
(ii) $A(A^{-1}(v)) \leq v$ *for* $0 < v < \infty$, *the equality holding if A is continuous at* $A^{-1}(v)$;
(iii) $v \leq A^{-1}(A(v))$ *for* $0 \leq v < b$, *the equality holding for* $a \leq v < b$;
(iv) $v \leq A^{-1}(v)\,(\bar{A})^{-1}(v)$ *for* $0 \leq v < \infty$.

(1.11) DEFINITION. The function A is said to satisfy the Δ_2-condition for $u \geq u_0 \geq 0$ if $A(u_0) < \infty$ and there is a positive number c such that $A(2u) \leq cA(u)$ for $u_0 \leq u < \infty$.

§2. Birnbaum-Orlicz Spaces

(2.1) DEFINITION. Let $(X, \mathcal{M}, \mu)$ be an arbitrary measure space montrivial in the sense that $\mu(X) > 0$. The set $L_A(X, \mathcal{M}, \mu)$ of all complex-valued, $\mathcal{M}$-measurable functions defined μ-a.e. on X such that $\int_X A(\alpha|f|)d\mu < \infty$ for some positive number α is called a *Birnbaum-Orlicz space.* Where no confusion seems possible, we will write $L_A(X)$ for $L_A(X, \mathcal{M}, \mu)$.

(2.2) A short computation shows that a Birnbaum-Orlicz space obtained from a Young's function is a complex linear space. Also, taking account of (1.7.ii), we easily find that $L_{A_0}(X) = L_A(X)$. We conclude that every Birnbaum-Orlicz space is a complex linear space.

(2.3) THEOREM. *Consider the function p_A defined on $L_A(X)$ by the equality*

$$p_A(f) = \inf\left\{k \in]0, \infty[: \int_X A\left(\frac{1}{k}|f|\right) d\mu \leq 1\right\}.$$

For f in $L_A(X)$ the following hold:

(i) $0 \leq p_A(f) < \infty$;
(ii) $p_A(f) = 0$ *implies that* $f(x) = 0$ *μ-a.e. on* X;
(iii) $p_A(\beta f) = |\beta| p_A(f)$ *for any complex number* β;
(iv) *if* $p_A(f) > 0$, *then* $\int_X A\left(\frac{1}{p_A(f)}|f|\right) d\mu \leq 1$;
(v) *if A is a Young's function, p_A is a norm;*
(vi) *if* $0 < \mu(E) < \infty$ *then* $\xi_E \in L_A(X)$ *and*

$$p_A(\xi_E) = \frac{1}{A^{-1}\left(\frac{1}{\mu(E)}\right)}.$$

(2.4) It follows from (2.3.v) and (2.2) that p_{A_0} is a norm on $L_A(X)$. We denote:

(i) $||f||_A = p_{A_0}(f)$.

(2.5) THEOREM. *The space $L_A(X)$ with the norm defined in (2.4.i) is a Banach space.*

(2.6) THEOREM. [Hölder's inequality]. *If $f \in L_A(X)$ and $g \in L_{\overline{A}}(X)$, the product fg belongs to $L_1(X)$ and we have*

(i) $$\int_X |fg| \, d\mu \leq 2p_A(f)p_{\overline{A}}(g).$$

(2.7) REMARKS. (i) If $M_\alpha(u) = u^\alpha$ for $0 \leq u < \infty$ and $1 \leq \alpha < \infty$, $L_{M_\alpha}(X)$ is the classical $L_\alpha(X)$ space. A simple computation shows that $p_{M_\alpha}(f) = ||f||_\alpha$ and $||f||_{M_\alpha} = \alpha^{-\frac{1}{\alpha}} ||f||_\alpha$.

(ii) For $M_\infty(u) = \infty \ \xi_{]1,\infty[}(u)$ we have $L_{M_\infty}(X) = L_\infty(X)$ and $p_{M_\infty}(f) = ||f||_{M_\infty} = ||f||_\infty$.

(2.8) THEOREM. *Let A and B be nontrivial generalized Young's functions and suppose that there exists a positive number m such that $A(u) \leq B(mu)$ for $0 \leq u < \infty$. Then $L_B(X) \subset L_A(X)$ and we have*

(i) $p_A(f) \leq m p_B(f)$ *for all* $f \in L_B(X)$.

In particular the following inequalities hold for all $f \in L_A(X)$:

(ii) $$\|f\|_A \leq p_A(f) \leq 2\|f\|_A.$$

(2.9) THEOREM. *If* $\mu(X) < \infty$, $L_A(X)$ *is contained in* $L_1(X)$ *and for all* $f \in L_A(X)$ *we have*

(i) $$\|f\|_1 \leq \frac{4}{(\bar{A})^{-1}\left(\frac{1}{\mu(X)}\right)}\|f\|_A.$$

(2.10) DEFINITION. Let N_A denote the function defined on the class of all complex-valued, $\mathscr{M}$-measurable functions on X by the equality

(i) $N_A(f) = \sup\left\{\int_X |fg|\,d\mu : g \in L_{\bar{A}}(X),\ p_{\bar{A}}(g) \leq 1\right\}$.

(2.11) THEOREM. *The function* N_A *is a seminorm on* $L_A(X)$. *The following inequalities hold for all* f *in* $L_A(X)$:

(i) $N_A(f) \leq 2p_A(f)$;
(ii) $N_{A_0}(f) \leq N_A(f) \leq 2N_{A_0}(f)$.

(2.12) THEOREM. *Let* A *be a nontrivial generalized Young's function. Suppose that* f *is a complex-valued measurable function vanishing outside of a* σ-*finite set and that* $N_A(f) < \infty$. *Then* $f \in L_A(X)$ *and we have*

(i) $\|f\|_A \leq N_A(f)$.

(2.13) THEOREM. *Suppose that* $\mu(X)$ *is finite. Let* (f_n) *be a sequence in* $L_A(X)$ *converging uniformly to a function* $f \in L_A(X)$. *Then we have* $\lim_{n\to\infty} p_A(f_n - f) = 0$.

(2.14) LEMMA. *Let* X *be a locally compact Hausdorff space. Let* μ *be a measure obtained from a nonnegative linear functional on* $\mathbf{C}_{00}(X)$ *as in* §9 *of* [4], *and let* $\mathscr{M}$ *be the* σ-*algebra of all* μ-*measurable subsets of* X. *Then each function* f *in*

$L_A(X)$ can be written as $f_1 + f_2$, where $f_1 = f\xi_F$ for some σ-compact set F, and $|f_2(x)| \le ap_A(f)$ μ-a.e. on X. In particular, if $a = 0$, then f vanishes μ-a.e. outside of a σ-compact set.

(2.15) THEOREM. *Suppose that A satisfies the Δ_2-condition for $u \ge u_0 \ge 0$. Let $(X, \mathcal{M}, \mu)$ be as in (2.14) and let $\mu(X)$ be finite. Then $C(X)$ is $||.||_A$-dense in $L_A(X)$.*

§3. Birnbaum-Orlicz Spaces of Functions on Groups

Let G be a locally compact Hausdorff group and let λ be a left Haar measure on G. We will write $\int_G fd\lambda$ as $\int_G f(x)dx$.

(3.1) THEOREM. *A complex-valued measurable function f belongs to $L_1(G) \cap L_1^*(G)$ if and only if $\max\{1, \frac{1}{\Delta}\} f \in L_1(G)$. The equalities*

(i) $$||f|| = ||f||_1 + ||\frac{1}{\Delta} f||_1,$$

and

(ii) $$|||f||| = ||\max\left\{1, \frac{1}{\Delta}\right\} f||_1$$

define equivalent norms on the linear space $L_1(G) \cap L_1^*(G)$. Precisely, we have

(iii) $|||f||| \le ||f|| \le 2\,|||f|||$ *for all $f \in L_1(G) \cap L_1^*(G)$.*

With either of these two norms, $L_1(G) \cap L_1^*(G)$ is a Banach space.

(3.2) THEOREM. *Let f be a function in $L_A(G)$ and let s be an arbitrary element of G. Then the functions ${}_sf$ and f_s belong to $L_A(G)$ and we have:*

(i) $p_A({}_sf) = p_A(f)$;
(ii) $p_A(f_s) \le \max\{1, \Delta(s^{-1})\}\, p_A(f)$.

The following result is part of (20.7) in the Russian edition of Hewitt and Ross "Abstract Harmonic Analysis", to be published.

(3.3) LEMMA. *Let f be a λ-measurable function on G. The following functions are $\lambda \times \lambda$-measurable on $G \times G$:*

$$(x, y) \longrightarrow f(xy^{-1}), \qquad (x, y) \longrightarrow f(y^{-1}x), \qquad (x, y) \longrightarrow f(x),$$
$$(x, y) \longrightarrow f(x^{-1}), \qquad (x, y) \longrightarrow f(y), \qquad (x, y) \longrightarrow f(y^{-1}).$$

(3.4) THEOREM. *Let f be a function in $L_A(G)$ vanishing outside of a σ-compact set F and let g be a function in $L_1(G)$. The integral*

$$\text{(i)} \qquad g*f(x) = \int_G f(y^{-1}x)g(y)\,dy$$

exists and is finite for almost all x in G. The function $g*f$ is in $L_A(G)$ and we have

$$\text{(ii)} \qquad \|g*f\|_A \leq 4\,\|f\|_A\,\|g\|_1.$$

If $g \in L_1(G) \cap L_1^*(G)$, the integral

$$\text{(iii)} \qquad f*g(x) = \int_G \Delta(y^{-1})f(xy^{-1})g(y)\,dy$$

exists and is finite for λ-almost all x in G. The function $f*g$ is in $L_A(G)$ and we have

$$\text{(iv)} \qquad \|f*g\|_A \leq 4\,\|f\|_A\,\|g\|,$$

where $\|\cdot\|$ is as in (3.1.i).

Theorem (3.4) serves as a lemma for the following general result.

(3.5) THEOREM. *Suppose that $f \in L_A(G)$ and $g \in L_1(G)$. Then the integral*

$$\text{(i)} \qquad g*f(x) = \int_G f(y^{-1}x)g(y)\,dy$$

exists and is finite for λ-almost all x in G. The function $g*f$ is in $L_A(G)$ and we have

$$\text{(ii)} \qquad \|g*f\|_A \leq k\,\|f\|_A\,\|g\|_1,$$

where $k = 4$ *if* $a = 0$ *or if* G *is* σ*-compact, and* $k = 6$ *otherwise.*

If $g \in L_1(G) \cap L_1^*(G)$, *the integral*

(iii) $$f*g(x) = \int_G \Delta(y^{-1}) f(xy^{-1}) g(y)\, dy$$

exists and is finite for λ*-almost all* x *in* G. *The function* $f*g$ *is in* $L_A(G)$ *and we have*

(iv) $$\| f*g \|_A \leq k \| f \|_A \; \| g \|,$$

where k *is as above and* $\| \cdot \|$ *is as in* (3.1.i).

The last two theorems have rather technical proofs, but using (3.5) it is simple to establish the followings results.

(3.6) THEOREM. *The space* $L_1(G) \cap L_1^*(G)$ *is a Banach algebra.*

(3.7) THEOREM. *The Birnbaum-Orlicz space* $L_A(G)$ *is a left Banach* L_1*-module and a right Banach* $(L_1 \cap L_1^*)$*-module.*

§4. Closed Ideals in L (G) where is a Compact Group

Throughout this section we suppose that G is compact. The left Haar measure λ is chosen so that $\lambda(G) = 1$.

(4.1) THEOREM. *Suppose that* f *and* g *are in* $L_A(G)$. *Then the equality*

$$g*f(x) = \int_G f(y^{-1}x) g(y)\, dy$$

defines a function in $L_A(G)$ *and we have*

(i) $$\| g*f \|_A \leq \frac{16}{(\bar{A})^{-1}(1)} \| f \|_A \; \| g \|_A.$$

(4.2) THEOREM. *The space $L_A(G)$ is a Banach algebra under a norm which is a positive constant times* $||\cdot||_A$.

(4.3) THEOREM. *Suppose that A satisfies the Δ_2-condition for $u \geq u_0 \geq 0$. Then the space $\mathfrak{T}(G)$ of trigonometric polynomials on G is $||\cdot||_A$-dense in $L_A(G)$.*

At this point we can see that $L_A(G)$ satisfies the hypothesis of (38.6.a) in [3]. Thus we obtain from (38.22.b) in [3] the following characterization.

(4.4) THEOREM. *Let A be as in* (4.3). *Suppose that S is a closed linear subspace of $L_A(G)$. Then S is a left (right) ideal in $L_A(G)$ if and only if S is closed under the formation of left (right) translates.*

(4.5) THEOREM. *Let A be as in* (4.3). *Then the class of all closed two-sided ideals in $L_A(G)$ is exactly the family $\{(L_A)P: P \subset \Sigma\}$. Distinct subsets of Σ engender distinct ideals.*

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