Nesting problems with continuous rotations: a survey

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RESUMO

Problemas de *nesting*, também conhecidos como problemas de corte e empacotamento de peças irregulares, é um problema de otimização amplamente estudado na literatura. A maior parte das publicações os resolve por meio de heurísticas, seguidas de métodos exatos e híbridos (combinação de heurísitcas com métodos exatos). No entanto, a maioria desses estudos não consideram a rotação contínua/livre das peças, o que é uma característica importante presente em aplicações do mundo real. Este documento é uma proposta de análise de trabalhos que lidam com os problemas de *nesting* com rotações livres, com foco nas principais estratégias e algoritmos utilizados para resolver o problema. O objetivo é organizar as contribuições presentes até o momento e indicar suas vantagens, desvantagens e tendências.

PALAVRAS CHAVE. Problemas de nesting. Rotações livres. Otimização.

Tópicos: Otimização Combinatória; Programação Matemática.

ABSTRACT

The nesting problem, also known as the irregular cutting and packing problem, is a widely studied optimization problem in the literature. Most of the papers consider heuristic approaches, followed by exact, and hybrid methods (combination of exact and heuristic methods). However, the majority of these studies do not work with the continuous/free rotation of the pieces, even though it is an important characteristic present in real-world applications. We propose an analysis of the articles that deal with the nesting problem with continuous rotation, identifying their main features and algorithms adopted. The goal is to summarize the contributions performed so far, indicating their strengths, weaknesses, and future tendencies.

KEYWORDS. Nesting problems. Continuous rotations. Optimization.

Paper topics: Combinatorial Optimization; Mathematical Programming.

1. Introduction

The nesting problem, also known as irregular cutting and packing problem, appears in industrial contexts, from textile companies through its two-dimensional version, to 3D printing or container ship loading problem. The problem consists of packing/allocating all the pieces into some container, aiming to find the best layout that minimizes the waste of material or space.

Besides its NP-Hardness [Chazelle et al., 1989; Fasano, 2007], there are two main challenges when working with nesting problems in general: finding the best way to represent the pieces and verifying if they overlap. For this reason, Bennell and Oliveira [2008] show compilations of geometry techniques and insightful algorithm designs.

According to Wäscher et al. [2007], there are many variations of the problem, and one example is the irregular strip packing problem (ISPP). In the ISPP, the container (a strip or a board) has a fixed height and a variable length. The objective is to place all the pieces on the board, minimizing the used length. Figure 1 shows an example of a feasible solution of the two-dimensional irregular strip packing problem.

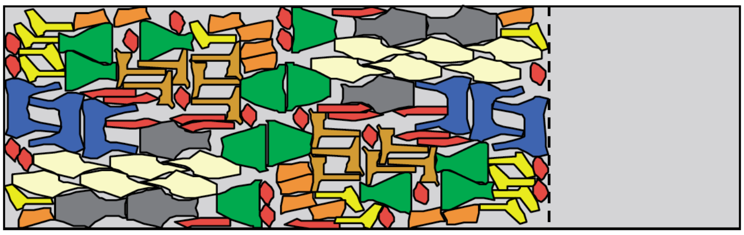


Figure 1: A cutting pattern of an irregular strip packing problem.
Source: Bennell and Oliveira [2008].

Leao et al. [2020] present a review of mathematical models and show some gaps in the literature of nesting problem models, like three-dimensional nesting, matheuristics, irregular containers, clusterization of the pieces, and continuous rotations of the pieces. Concerning this last issue, we note that even though it is a crucial characteristic present in real contexts, few papers study the advantages to allow pieces to rotate freely.

The purposes of this paper are the following: *i)* summarize the most important publications on the literature of nesting problems with continuous rotations; *ii)* briefly discuss and point out their advantages and disadvantages; and *iii)* identify new trends to deal with continuous rotations.

The remainder of this paper is organized as follows. Section 2 summarizes the main techniques for piece representation and their way to avoid overlapping between the pieces. Section 3 analyzes and summarizes two-dimensional nesting problems with continuous rotations. This section is divided into heuristic and exact algorithms to solve the problem. Section 4 analyzes the few works related to three-dimensional nesting problems with continuous rotations. Conclusions and future work proposals are stated in Section 5. Table 1, at the end, summarizes the works studied on this paper.

2. Problem geometry and how to avoid overlap

Geometry techniques are the most critical part of nesting problems as we deal with convex and nonconvex pieces. Avoiding pieces overlap is a mainly geometrical process and is directly associated with how pieces are represented. Here we will address six geometrical methods for representing a piece and their features to avoid overlapping. They are the raster method, circle covering, separating hyperplanes, direct trigonometry, no-fit polygon, and phi-functions.

The most straightforward technique is the raster method, which discretizes pieces in pixels (in a matrix arrangement), then, for overlap checking, we need to verify if two pixels are located at the same point. A three-dimensional piece can be represented in the same way using voxels (a pixel in three-dimensions). An obstacle with this method is that memory complexity grows as the number of pixels increases. Another disadvantage of the raster method is when we want to represent curves: the more accurate the curve, the more pixels are demanded, and therefore, the need for memory space increases.

Another technique to represent a piece and avoid overlapping is by using circles to represent a piece. In the technique, each piece is composed of a set of circles. If two circles of different sets are overlapping, then two pieces are overlapping (Figure 2). This solving method can be extended to the three-dimensional case by using spheres instead of circles. The circles need to cover the entire area of the piece to obtain the optimality of a solution. For this purpose, the work of Rocha et al. [2014] is useful. In that paper, the authors find the best circle covering (complete circle covering) of a piece by using the minimal number of circles. This results in a trade-off between the quality of the piece representation, and the number of circles (the complexity is proportional to the number of circles).

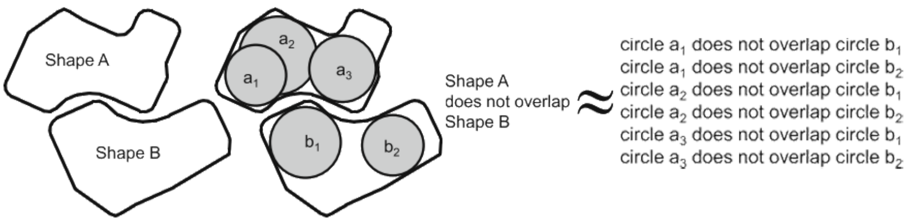


Figure 2: Approximating the nonoverlap constraint using inscribed circles
Source: Jones [2013].

Another intuitive way to represent a piece is by using a polygon. A polygon is defined by a set of vertices and a set of edges defined by those vertices. Like the raster method, polygonal representation does not preserve the geometry of the piece because of its deficiency to represent curves. A curved edge can be refined by adding more vertices to smooth the curvature. Like the raster method, this increases the number of vertices, and therefore, the computational complexity also increases. The most common representation of polyhedra is a mesh. A mesh is a set of vertices in $\mathbb{R}^3$, and a set o faces defined with these vertices. Four approaches to avoid overlap that uses the polygon representation are direct trigonometry, separating hyperplanes, no-fit polygon, and phi-function.

Direct trigonometry works by verifying the intersection between the edges of the pieces. If two polygons overlap, then their enclosing rectangles must overlap (Figures 3a, 3b and 3c) and if two edges intersect, then the enclosing rectangles of the edges must intersect (Figures 3d and 3e). Additional tests are used to verify cases that cannot be covered by the edge intersection tests, e.g., when a piece is inside another piece (Figure 3f).

Kallrath [2008] presented the concept of separating hyperplanes. The idea beyond this concept is, if a line separates two convex polygons, then all points of each polygon are located on opposite sides of the line (Figure 4). If one polygon is nonconvex, it can be partitioned into convex polygons. Separating hyperplanes are very common in modeling nonoverlapping constraints and can be extended to three-dimensions.

One of the most popular methods used to solve the two-dimensional nesting problem is the

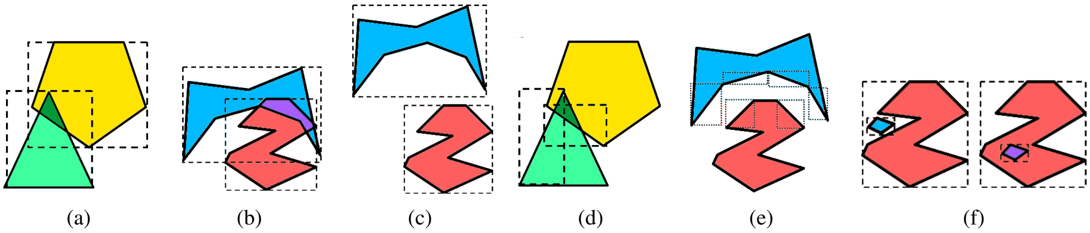


Figure 3: Direct trigonometry.
Source: Bennell and Oliveira [2008] (adapted).

No-Fit Polygon (NFP). The essence of the NFP is a polygon derived from the combination of two other polygons, where its interior represents the points in which the two polygons combined would be overlapping. Figure 5 shows a triangle (P_j) and a hexagon (P_i) with its respective reference points. The triangle "slides" around the hexagon and the path formed by the triangle reference points represent the NFP of P_i by P_j (NFP(P_i, P_j)). The NFPs can be precomputed and used as an entry for solving methods, but an NFP for each pair of polygons should be computed. The advantage of using NFP is computational efficiency for verify intersection.

The Φ-function (from now on, referred as phi-function) is a technique that appeared at the beginning of the 2000s, and today is the dominant technique in exact algorithms for solving irregular packing problems. The first apparition of phi-functions in the literature was in Stoyan et al. [2002], to describe the interaction between two simple two-dimensional objects (phi-objects). Phi-functions represent this interaction by a combination (union, intersection, and complement) of primary shapes (like lines, curves, circles). This representation allows us to describe simple pieces with accuracy (Figure 6). Stoyan et al. [2004] constructed phi-functions but for complex two-dimensional objects. Convex decomposition is also possible with phi-functions. Although phi-functions describe the pieces by a union, intersection, and complement between primary shapes, it is hard to represent complex pieces. The concept of phi-functions for two-dimensional objects can be extended naturally to the three-dimensional case based on the definition of primary three-dimensional objects.

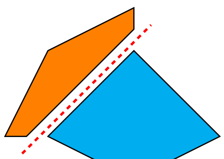


Figure 4: Separating line of two polygons.

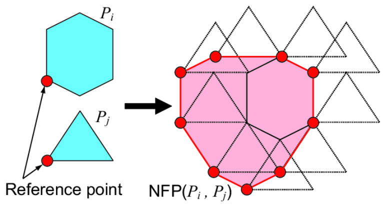
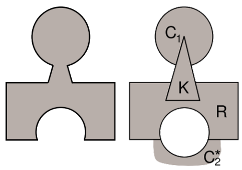


Figure 5: NFP(P_i, P_j).
Source: Umetani et al. [2006].



$C_1 \cup K \cup (R \cap C_2^\star)$

Figure 6: An example of a composed phi-object.
Source: Chernov et al. [2010].

With phi-functions, it is also possible to measure how much two pieces overlap. Let Φ^{AB} be the phi-function for the phi-objects A and B. Then: *i)* $\Phi^{AB} < 0$ if A overlaps B; *ii)* $\Phi^{AB} = 0$ if A touches B[1] and; *iii)* $\Phi^{AB} > 0$ if A does not overlap B.

Phi-functions are present in the majority of exact algorithms, as nonlinear programming

[1]The NFP is the same case when the value of a phi-function is zero.

models, to solve nesting problems. The features of phi-functions allow designing models that admit continuous rotations, and for this reason, many works proposed to solve the problem using them.

3. Two-dimensional nesting with continuous rotations

Works that study nesting problems (in two or three dimensions) with continuous rotation are rare in the literature, but there are a variety of papers that allow several specific angles (45^o, 90^o, 180^o). This last case will not be addressed in this survey because it does not consider continuous rotations. The following works presented here address continuous rotations.

3.1. Heuristic methods

Liao et al. [2016] proposed an algorithm to pack irregular pieces into a rectangular container by using physical simulation (Rubber Band Packing Algorithm - RBPA). A rubber band wraps the pieces, and the tension of the rubber band keeps the pieces close to each other (Figure 7) until the minimum length of the rubber band or the convex hull formed by the rubber band is equal to the container, or there is no possible movement due to the rubber band tension. This simulation permits the pieces to translate and rotate freely. A decomposition algorithm was applied to the nonconvex pieces. The algorithm obtained a satisfactory solution in a acceptable amount of time.

The paper from Abeysooriya et al. [2018] studies the two-dimensional irregular packing problem with multiple homogeneous bins (2DIBPP), using iterated jostle heuristics. The NFP controls the occurrence of overlapping between pieces. For that purpose, the algorithms calculate the NFP for each orientation of the piece during its execution. Two pieces calculate the NFP: the first one is the union of all the pieces already placed, and the other one is the next piece to be placed. The rotation problem was solved by setting a set of predefined angles, e.g., 0^o, 90^o, 180^o, 270^o. This phase is the finite rotation approach. For the continuous rotation phase, instead of calculating all the angles, which is impossible, the authors developed a mechanism: when two pieces are touching each other, this touchpoint defines two new angles of rotation, and the pieces can rotate clockwise in one angle or counterclockwise in the other (Figure 8). The authors show that the proposed algorithms can be applied to different variants of the problem and generate significantly better results.

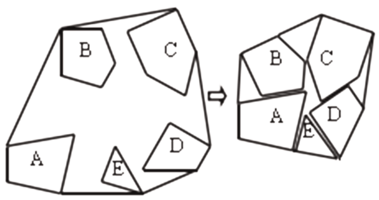


Figure 7: Example of rubber band physical movement.
Source: Liao et al. [2016].

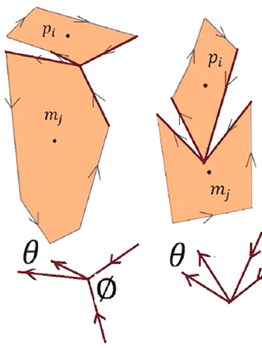


Figure 8: Two angles obtained by a touchpoint.
Source: Abeysooriya et al. [2018] (adapted).

Beyond the existence of heuristic and exact solutions, it is worth emphasizing that there are matheuristics too. A matheuristic, often called model-based heuristic, is interoperation between metaheuristics and mathematical programming algorithms. The only available is Martinez-Sykora et al. [2017] that deals with the Two-Dimensional Irregular Bin Packing Problem (2DIBPP). This paper describes a constructive algorithm that explicitly considers the two aspects of the optimization problem; the assignment of pieces to bins and the arrangement of assigned pieces in the bin. The authors proposed several integer programming models to determine the association between pieces and bins, and then a MIP model for placing the pieces into the bins. The computational results show that the algorithm obtains high-quality results in a variety of instances, including artificial instances and real-world industrial data.

3.2. Hybrid methods

Pankratov et al. [2019] deal with packing ellipses into a convex polygonal container with a given shape. The objective is to find the minimum scaling coefficient for the polygon, still containing the set of ellipses. Phi-functions were used for modeling the containment constraints and quasi-phi-functions to describe the nonoverlapping constraints. For solving the problem, a local search strategy composed of four stages was proposed. A nonlinear programming model was built to find the local maxima and local minima of the feasible parameters obtained in the later stages. The article provides instances for benchmark, and the computational results demonstrate the efficiency of the approach.

Plankovskyy et al. [2020] considered a problem of cutting irregular pieces from a rectangular (metal) sheet. The nonlinear programming model proposed for the problem supports a lot of technical requirements to geometrical constraints (such as minimal allowable distances, prohibited areas, range of the possible object rotations, changing shapes of pieces by adding auxiliary circular zones). The pieces are represented by phi-objects bounded by line segments and circular arcs. The solution to the problem is a combination of heuristics and local search optimization. The proposed algorithm works sufficiently fast for complex instances and uses a multistart strategy. The first part is to generate feasible starting points by using rectangular approximations for the pieces. Then, the algorithm performs a local search procedure, and a system of inequalities is applied to each piece, providing an arrangement for the object. The third step chooses the best of local minima obtained at the second step, and use it for the set of feasible starting points as a solution to the problem.

3.3. Exact algorithms

Kallrath [2008] has introduced an exact nesting algorithm for packing circles, rectangles, and convex polygons in rectangular containers. The author presented the idea of separating hyperplanes to model the nonoverlapping constraints. His method reaches the optimality for a small number of polygons, but it struggles for a large number. It is also hard to solve instances with non-rectangular polygons.

Chernov et al. [2010] have been studying the cutting and packing problem for decades and proposed a mathematical model used to solve some three-dimensional nesting with the usage of the phi-functions. The article presents the construction of phi-functions for two and three-dimensional pieces. They presented for two circles, two spheres, two rectangles, two boxes, two cylinders, and two convex and nonconvex polygons (and polyhedra). Then phi-functions are presented for a rectangle and a circle, a convex polygon and a circle, a "pill" and a circle, two circulars segments, and for more general objects. Phi-functions with specific rotational angles were presented as well. The article shows noticeable results and how the use of phi-functions and mathematical programming can improve the performance of cutting and packing algorithms.

The problem solved by Birgin and Lobato [2010] is the opposite: through an mixed integer nonlinear programming model (MINLP), instead of allocating irregular pieces inside a rectangular container, they assign identical rectangles within an arbitrary convex shape, not necessarily using orthogonal rotation, as seen in Figure 9. Regarding that, the pieces are rectangles, the model only needs a rotation angle between 0^o and 90^o, and an extra 90^o to change the orientation. To solve the formulation, the authors presented a combination of a branch and bound and active-set strategies for bound-constrained minimization of smooth functions and showed that the solution method applied to the problem was reliable.

Fasano [2012] presented solutions for solving the packing of three-dimensional *tetris*-like items (a cluster of mutually orthogonal boxes) and the packing of convex and nonconvex polygons with continuous rotations inside a polygonal shaped container. The author presented a mixed integer programming model (MIP) to solve the first problem. The heuristic consists of modules, each one performing one job to solve the problem. The author presented an MINLP model to solve the second problem and demonstrated how difficult it is to deal with overlapping in situations like this. MINLP models work more efficiently if the initial solutions are favorable. Therefore, the decomposition of the polygon in a *tetris*-like piece, used in the first problem, was suggested (Figure 10) to solve the second problem. This decomposition allows working with the piece at every rotation by decomposing the piece again quickly. A solution to deal with a piece with holes was presented further.

In Jones [2013] the nesting problem is solved by using the inner circle covering technique. The nonoverlapping constraints of the algorithm that allow the pieces to rotate freely were relaxed. The author formulated the problem as a quadratic programming problem and solved it by using many nonlinear global solvers. However, the exactness of the method relies on the number of circles used. Better approximations use more circles.

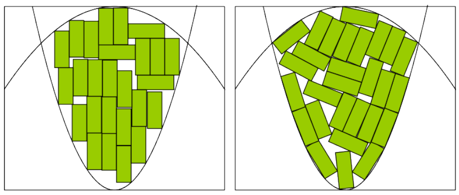


Figure 9: Packing identical rectangles within a convex region, orthogonally and freely. Source: Birgin and Lobato [2010].

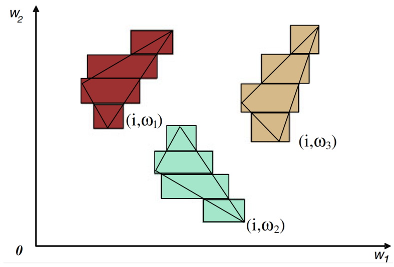


Figure 10: *Tetris*-like decomposition. Source: Fasano [2012]

Bennell et al. [2014] considers convex polygonal containers of different sizes and two irregular objects bounded by circular arcs and/or line segments, that can be continuously translated and rotated. The objective is to allocate these two polygons in a container in a way that the container reaches its smallest possible area (or perimeter or homothetic coefficient). Additional constraints like the minimum allowable distance between objects and between the container frontier may be imposed. The paper presents a generic nonlinear mathematical programming model, whose objective function is polynomial, and the solution strategy is based on the concept of phi-functions. Computational experiments with nonlinear optimization solvers demonstrate the effectiveness of the methodology.

Stoyan et al. [2016] developed an exact solution for the irregular packing problem. They proposed an NLP model for packing pieces into rectangular or circular containers. In the model, the pieces were delimited by arcs or line segments. The model admits prohibited area and minimal distance allowed constraints. A fast algorithm was proposed to generate feasible points and then, feasible subregions. The main algorithm reduces the problem to subproblems and considerably reduces the number of inequalities in those subproblems. The article provided instances that were used in benchmarks of many other publications.

Cherri et al. [2018] implemented a mixed integer programming model to solve the irregular strip packing problem with continuous rotations. It can handle convex and nonconvex polygon intersection by using direct trigonometry. In the model, the pieces are allocated on the board using a reference point (x_i, y_i), and its location is given by the translation and rotation θ_i of the piece i. The computational effort to calculate the rotation increases due to the usage of sine and cosine functions that are nonconvex. The model was solved in three different global optimization solvers, and the results were competitive with the ones in the state-of-the-art literature.

Peralta et al. [2018] developed a nonlinear programming model using separating lines to control overlapping between two-dimensional pieces. The idea is based on the separating planes by Kallrath [2008], but this work has some improvements: it uses fewer variables than such study, and it is applied to general polygons, not only circles and convex polygons. The proposed model uses a heuristic with predefined rotations to build an initial solution. Afterwards, an interior point algorithm is used to improve the solutions. The results found for the model were compared with other works in the literature, which consider the continuous rotation of the pieces [Stoyan et al., 2016; Liao et al., 2016]. The authors obtained better solutions in some instances (better than in Liao et al. [2016]) and a significant improvement in the model resolution time.

4. Three-dimensional nesting with continuous rotations

There is a plethora of three-dimensional problems within the class of nesting problems. The most common is the container loading problem. It consists of packing small rectangular boxes orthogonally into a larger rectangular box called container. There are many variations of container loading problems, each one admitting different constraints, e.g., weight limits, weight distribution, loading priority, and orientation constraints. For further reading about container loading problems, we suggest Kurpel et al. [2020] and Bortfeldt and Wäscher [2013]. Here we will discuss only one work of container loading problems since it is not the main subject of this paper.

Besides this, three-dimensional nesting problems are characterized by packing irregular three-dimensional objects in some container, not necessarily a rectangular container as usual (Figure 11).

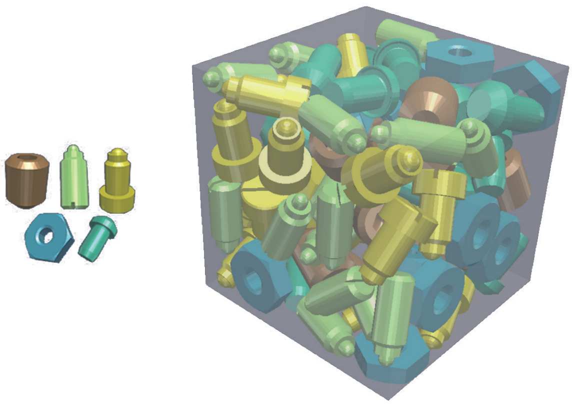


Figure 11: Example of three-dimensional nesting with continuous rotations.
Source: Ma et al. [2018].

The three-dimensional nesting problem with continuous rotations, in contrast to the container loading problems, is that the pieces can be complex and irregular, i.e., convex, nonconvex, with or without curves, and even with holes. This version admits continuous rotations, any degree of freedom in any direction. Works with these characteristics, equipped with three-dimensional pieces, are rare in the literature. We will present some of these works below.

Ma et al. [2018] propose the packing of intricate pieces inside containers that can also be complex. The proposed algorithm creates separate spaces (cells) for each piece to be placed. The piece must be placed entirely in the cell to avoid overlapping. The algorithm is a hybrid optimization heuristic was used: combinatorial optimization to perform hole filling and swap or replace a piece, and continuous optimization to compute the best position and orientation of each piece. The continuous optimization, in this case, works as local optimization. It consists of shrinking a piece and then adjusting its position and rotations until the piece returns to its original size while avoiding overlapping. However, this continuous optimization leads quickly to local optimum, and combinatorial optimization is used to overcome this issue. The algorithm was able to pack complex objects in a variety of arbitrarily shaped containers.

Kallrath [2015] is a work that handles the packing of ellipsoids (three-dimensional ellipses) into a box, minimizing the box's volume. The article presents NLP formulations based on purely algebraic approaches to represent rotated and shifted ellipsoids. The usage of separating hyperplanes manages the nonoverlapping.

Romanova et al. [2018] study the problem of packing convex and concave polyhedra (nonconvex pieces) into a cuboid container. Besides the rotation constraints, minimal allowable distances between the pieces are taken into account. They proposed an exact mathematical model using quasi-phi-functions to describe nonoverlapping, and this results in a nonlinear programming formulation with smooth functions. The solution uses a fast starting point algorithm and compaction procedures that reduce the problem to a sequence of considerably smaller nonlinear programming subproblems. New instances came with the article for benchmark, and the results show superior performance compared to instances in the literature. In Romanova et al. [2019], the continuation of the article, the authors used the same technique and considered the usage of different shaped containers such as elliptical, cylinder, and spherical.

5. Final remarks

This survey shows the most notable works on nesting problems with continuous rotations. Publications approaching two and three-dimensional nesting with continuous rotations were presented and briefly discussed. It is easy to perceive that there exists a lack of studies with continuous rotations, even though it is a critical characteristic present in real contexts. One reason can be the difficulty in modeling continuous variables, which brings non-linearity to the problem.

Despite the phi-functions' difficulty in representing complex pieces, they are used in most of the exact algorithms to solve problems with continuous rotations, even in three-dimensional nesting. The constraints based on phi-functions are compelling when using mathematical expressions to define them, in contrast with direct trigonometry seen in other works.

Matheuristics (model-based heuristics) are trending on optimization problems. As described in Section 3, there is only one that deals with continuous rotations. As a suggestion, the opportunity to create matheuristics based on phi-functions is an unexplored field. Another technique to avoid overlapping that could be used is the No-Fit Raster, used in Mundim et al. [2017]. No-Fit Raster represents the points where a polygon can be placed around another polygon keeping a feasible solution. The technique can be useful if a grid of points represents the container. Although the paper does not admit continuous rotations, there is a possibility of recalculating the no-fit raster, as we saw on Abeysooriya et al. [2018], or use the precaulculated NFP for each rotation angle.

In Table 1, we present the works discussed here. The table contains information related to the problem's dimensions, the solving method (exact algorithms or heuristics), how the pieces are represented, item and container shape, and some comments if needed.

Table 1: Summary of the publications about nesting problems with continuous rotations.

Nº	Publication	Problem	Dimension	Solving method	Piece representation	Item shape	Container shape	Overlapping verification method
1	Kallrath [2008]	GN	2D	E ^{ab}	P	C/EL	R	Separating lines
2	Birgin and Lobato [2010]	Isotropic Convex Region	2D	E	P	R	C	Continuous and differentiable constraints
3	Chernov et al. [2010] [c]	GN	2D/3D	H/E	PO	C/N/WH	R	Phi-function
4	Fasano [2012]	CLP / GN	2D	E ^{bd}	P	C/N/WH	C	Integer constraints[d]
5	Jones [2013]	SPP	2D	E	CC	C/N/WH	R	Circle covering
6	Bennell et al. [2014][c]	Two Item Nesting	2D	E	PO	C/N	C/CI	Phi-function
7	Kallrath [2015]	Ellipsoid packing	3D	E ^{a}	P	EL	R	Separating hyperplane
8	Stoyan et al. [2016][c]	SPP	2D	E	PO	C/N/WH	R/CI/EL	Phi-function
9	Liao et al. [2016]	SPP	2D	H	P	C/N	R	Rubber Band Packing Algorithm
10	Martinez-Sykora et al. [2017]	IBPP	2D	Ma	P	C/N	R	No-fit polygon
11	Ma et al. [2018]	GN	3D	H	Me	C/N/WH	C/N/WH	Individual cells
12	Cherri et al. [2018]	SPP	2D	E	P	C/N	R	Direct trigonometry
13	Abeysooriya et al. [2018]	IBPP	2D	H	NFP	C/N/WH	R	No-fit polygon
14	Peralta et al. [2018]	SPP	2D	E	P	C/N	R	Separating lines
15	Romanova et al. [2018][c]	GN	3D	E ^{a}	PO	C/N/WH	R	Phi-function
16	Romanova et al. [2019][c]	GN	2D/3D	E ^{a}	PO	C/N/WH	R/C/EL	Phi-function
17	Pankratov et al. [2019][c]	Elliptical Nesting	2D	H+E	PO	EL	C	Phi-function
18	Plankovskyy et al. [2020][c]	SPP	2D	H+E	PO	C/N/WH	R	Phi-function

[a]NLP
[b]MINLP
[c]Publications that uses Phi-function. Cf. Stoyan et al. [2002].
[d]MIP

Caption:

Problem type

SPP	strip packing problem
GN	general nesting
CLP	container loading problem
IBPP	irregular bin packing problem

Solving method

E	exact
H	heuristic
Ma	matheuristic

Representation

PO	phi-object
P	polygonal
Me	mesh
CC	circle covering
NFP	no-fit polygon

Piece format

C	convex polygon
N	nonconvex polygon
R	rectangular
EL	elliptical
CI	circular
WH	with holes

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