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Local large deviation principle for Wiener process with random resetting

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We consider a class of Markov processes with resettings, where at random times, the Markov processes are restarted from a predetermined point or a region. These processes are frequently applied in physics, chemistry, biology, economics, and in population dynamics. In this paper, we establish the local large deviation principle (LLDP) for the Wiener processes with random resettings, where the resettings occur at the arrival time

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of a Poisson process. Here, at each resetting time, a new resetting point is selected at random, according to a conditional distribution.

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1. Introduction

Random processes with resettings have recently found their applications in various fields outside of mathematics. We will list some but not all applications of these processes: they are used in random search algorithms [9, 10, 14, 18], in population dynamics [4, 8, 15, 20, 22], and in biological and chemical models [1, 26, 27]. The majority of these applications used the Wiener processes with resettings. A Wiener process with resettings is defined as a solution to the following stochastic equation:

$$\xi(t) = w(t) - \int_0^t \xi(s-)d\nu(s), \tag{1.1}$$

where $w(t)$ denotes a Wiener process and $\nu(t)$ is a Poisson process with rate λ. The processes $w(t)$ and $\nu(t)$ are assumed to be independent. Equation (1.1) corresponds to the case when at each Poisson arrival time, the Wiener process restarts at the origin.

In most of the works where the processes of type (1.1) are considered, the authors concentrate on the analysis of the corresponding stationary distributions or additive and integral functionals. See [13] and references therein. To the best of our knowledge, the paper of Meylahn *et al.* [21] stands out as the only work where the large deviation principle (LDP) was established for integral functionals of the diffusion processes with resettings.

The main objective of this paper is to establish a local large deviation principle (LLDP) for the trajectories of this type of processes. We believe that prior to this work there were no such results proved for the Wiener processes with resettings. Moreover, we prove the LLDP for the case where the resetting point is selected at random.

In what follows, we assume that all random elements considered here are in the probability space $(\Omega, \mathfrak{F} = \mathfrak{B} \cup (\bigcup_{t\geq 0} \mathfrak{F}_t), \mathbf{P})$. Here, $\mathfrak{B}$ is Borel σ-algebra on $\mathbb{R}$, $\mathfrak{F}_t$ is the filtration induced by the trajectories of $\big(w(t), \nu(t)\big)$, where $w(t)$ is the Wiener process, $\nu(t)$ denotes the Poisson process with rate λ, and the processes $w(t)$ and $\nu(t)$ are assumed to be independent. We will examine the following stochastic equation:

$$\xi(t) = w(t) - \int_0^t \zeta(\nu(s-), \xi(s-))d\nu(s), \tag{1.2}$$

where a collection of independent $\mathfrak{B}$-measurable random variables $\zeta(n,x)$, $n \in \mathbb{Z}^+ := \{0\} \cup \mathbb{N}$, $x \in \mathbb{R}$, independent from $(w(t), \nu(t))$.

Note that the Wiener processes with resetting satisfying (1.1) are a special case of the processes evolving according to Eq. (1.2) with $\zeta(n,x) = x$.

In the modern literature on the LDP, various conditions on random processes are considered in order to obtain a rough exponential asymptotics for probabilities of rare events (see [7, 11]). In the studies where LDP have been proved for the solutions of stochastic differential equations containing an integral with respect to a Poisson process measure, a bound on the function $\zeta(n,x)$ is usually required, and is frequently given in the form of the Lipschitz condition (see [19, 23, 25, 16, 5, 6]). In our case $\zeta(n,x)$ is the random function of x, which as we know, does not satisfy the Lipschitz condition, and also is unbounded.

We are interested in establishing the LLDP for positive and negative excursions of a process

$$\xi_T(t) := \frac{\xi(Tt)}{T}, \quad t \in [0,1],$$

where T is an unbounded increasing parameter. This paper extends the ideas of our previous LLDP result for the random walk with catastrophes [17].

The trajectories of the process $\xi_T(\cdot)$ almost surely belong to the set $\mathbb{D}[0,1]$ of cádlág functions (i.e. right continuous and with a left limit). For $f, g \in \mathbb{D}[0,1]$ let

$$\rho(f,g) = \sup_{t\in[0,1]} |f(t) - g(t)|.$$

We recall the definition of LLDP.

Definition 1.1. A family of random processes $\xi_T(\cdot)$ satisfies the LLDP on the set $G \subset \mathbb{D}[0,1]$ with a rate function $I = I(f) : \mathbb{D}[0,1] \to [0,\infty]$ and the normalizing function $\psi(T)$ such that $\lim_{T\to\infty} \psi(T) = \infty$ if for any function $f \in G$, we have

$$\begin{aligned}\lim_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbf{P}(\xi_T(\cdot) \in U_\varepsilon(f))\\ = \lim_{\varepsilon\to 0} \liminf_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbf{P}(\xi_T(\cdot) \in U_\varepsilon(f)) = -I(f), \qquad (1.3)\end{aligned}$$

where $U_\varepsilon(f) := \{g \in \mathbb{D}[0,1] : \rho(f,g) < \varepsilon\}$.

See [2, 3] for more details on the concept of LLDP.

We let $\mathbf{p}_{\zeta(n,x)}(y)$ be the density of the random variable $\zeta(n,x)$. Furthermore, we assume that $\zeta(n,x)$ satisfies the following conditions:

$\mathbf{A}_0$: if $x = 0$, then $\mathbf{P}(\zeta(n,x) = 0) = 1$ for all $n \in \mathbb{Z}^+$;
$\mathbf{A}_+$: if $x > 0$, then $\int_0^x \mathbf{p}_{\zeta(n,x)}(y)dy = 1$ for all $n \in \mathbb{Z}^+$;
$\mathbf{A}_-$: if $x < 0$, then $\int_x^0 \mathbf{p}_{\zeta(n,x)}(y)dy = 1$ for all $n \in \mathbb{Z}^+$;

$\mathbf{B}_+$: if $x > 0$, then there exists $\Delta \geq 1$ such that $\frac{1}{\Delta|x|} \leq \mathbf{p}_{\zeta(n,x)}(y) \leq \frac{\Delta}{|x|}$ holds true for all $n \in \mathbb{Z}^+$, and for almost all $y \in [0, x]$;

$\mathbf{B}_-$: if $x < 0$, then there exists $\Delta \geq 1$ such that $\frac{1}{\Delta|x|} \leq \mathbf{p}_{\zeta(n,x)}(y) \leq \frac{\Delta}{|x|}$ holds true for all $n \in \mathbb{Z}^+$, and for almost all $y \in [x, 0]$.

Note that all of the above conditions hold in the case when $\zeta(n, x)$ is uniformly distributed in the interval between 0 and x whenever $x \neq 0$, and $\zeta(n, 0) = 0$. In this example, $\Delta = 1$.

We use the following notations: $\mathbb{C}_0[0, 1]$ is the set of continuous functions on the interval $[0, 1]$, originating from zero; $\mathbb{VC}_0[0, 1]$ is the subset of functions in $\mathbb{C}_0[0, 1]$ with finite variation; $\mathbb{VC}_0^M[0, 1]$ is the subset of all non-decreasing functions in $\mathbb{VC}_0[0, 1]$; $\mathbb{AC}_0^M[0, 1]$ is the subset of absolutely continuous functions in $\mathbb{VC}_0^M[0, 1]$; $\mathbb{AC}_0^+[0, 1]$ ($\mathbb{AC}_0^-[0, 1]$) is the subset of absolutely continuous functions in $\mathbb{C}_0[0, 1]$, taking positive (negative) values for $t \in (0, 1]$; $\mathrm{V}_a^b(f)$ is the total variation of a function f over the interval $[a, b]$; $\overline{B}$ is complement of the set B; $\mathbb{1}_B(\cdot)$ is the indicator function of the set B; $\lfloor a \rfloor$ is the integer part of a number a.

Further, in Sec. 2, we formulate our main results; in Sec. 3, we prove the LLDP; some auxiliary results are proved in Sec. 4.

2. Main Results

Note that if the conditions $\mathbf{A}_0$, $\mathbf{A}_+$, $\mathbf{A}_-$ hold, then for any $T > 0$ Eq. (1.2) has a solution on the interval $[0, T]$, and it is unique. It is also easy to prove the following result about an asymptotic upper bound for the maximum value of the process.

Theorem 2.1. *Let the conditions* $\mathbf{A}_0$, $\mathbf{A}_+$, $\mathbf{A}_-$ *hold and let the increasing function* $\varphi(T)$ *satisfies*

$$\lim_{T\to\infty} \frac{\varphi(T)}{\sqrt{\ln(\ln T)}} = \infty.$$

Then for any $\varepsilon > 0$

$$\mathbf{P}\left(\lim_{T\to\infty} \sup_{t\in[0,1]} \left| \frac{\xi(Tt)}{\sqrt{T}\varphi(T)} \right| > \varepsilon \right) = 0.$$

The proof of Theorem 2.1 is quite trivial, and we omit it.

To formulate our main result, we recall that any absolutely continuous function starting from zero can be uniquely represented as a difference of functions $f^+ \in \mathbb{AC}_0^M[0, 1]$ and $f^- \in \mathbb{AC}_0^M[0, 1]$ such that

$$\mathrm{V}_0^1(f) = \mathrm{V}_0^1(f^+) + \mathrm{V}_0^1(f^-).$$

The functions f^+ and f^- are called, respectively, positive and negative variations of the function f, see for example [24, Chap. 1, §4].

Theorem 2.2. (LLDP) *Let the conditions* $\mathbf{A}_0$, $\mathbf{A}_+$, $\mathbf{A}_-$, $\mathbf{B}_+$ *hold, then the family of the random processes* $\xi_T(\cdot)$ *satisfies LLDP on the set* $\mathbb{AC}_0^+[0, 1]$ *with normalized* *function* $\psi(T) = T$ *and the rate function*

$$I(f) = \lambda + \frac{1}{2}\int_0^1 (\dot{f}^+(t))^2 dt.$$

Theorem 2.3. (LLDP) *Let the conditions* $\mathbf{A}_0$, $\mathbf{A}_+$, $\mathbf{A}_-$, $\mathbf{B}_-$ *hold, then the family of the random processes* $\xi_T(\cdot)$ *satisfies LLDP on the set* $\mathbb{AC}_0^-[0,1]$ *with normalized function* $\psi(T) = T$ *and the rate function*

$$I(f) = \lambda + \frac{1}{2}\int_0^1 (\dot{f}^-(t))^2 dt.$$

Theorems 2.2 and 2.3 implies the following form of the rate function on the set $\mathbb{AC}_0[0,1]$.

Remark 2.1. Let all the conditions $\mathbf{A}_0$, $\mathbf{A}_+$, $\mathbf{A}_-$, $\mathbf{B}_+$, $\mathbf{B}_-$ hold, then the family of the random processes $\xi_T(\cdot)$ satisfies LLDP on the set $\mathbb{AC}_0[0,1]$ with normalized function $\psi(T) = T$ and the rate function

$$I(f) = \lambda + \frac{1}{2}\int_0^1 (\dot{f}^+(t)\mathbb{1}_{\{f(t)\geq 0\}}(t) + \dot{f}^-(t)\mathbb{1}_{\{f(t)<0\}}(t))^2 dt,$$

where $f(t) = 0$ at finitely many points in $[0,1]$.

Moreover, the proof of the theorems provides the LLDP and the corresponding rate function for the Wiener processes with resetting to the origin. Note that in this case the variables $\zeta(n,x)$ are deterministic functions $\zeta(n,x) = x$, and the conditions $\mathbf{B}_\pm$ do not hold.

Remark 2.2. LLDP for positive and negative excursions of Eq. (1.1) (case of the deterministic resetting at zero) is a trivial task. In this case, the random process $\xi_T(\cdot)$ will stay in the neighborhood of the function $f \in \mathbb{AC}_0^+[0,1]$ or $f \in \mathbb{AC}_0^-[0,1]$ only if the normalized Wiener process will stay in this neighborhood, and the Poisson process will not have jumps on the interval $[0,T]$. Thus, thanks the independence of $w(t)$ and $\nu(t)$ the rate function takes the form

$$I(f) = \lambda + \frac{1}{2}\int_0^1 (\dot{f}(t))^2 dt.$$

Note that we cannot obtain the LDP for the family $\xi_T(\cdot)$ in the metric space $(\mathbb{D}[0,1],\rho_S)$, where ρ_S is Skorohod's metric. Because one can show that the corresponding family of measures is not exponentially tight (see [7, Remark (a), p. 8]).

3. Proof of Theorems 2.2 and 2.3

By (1.2) the process $\xi_T(t)$ can be written as

$$\xi_T(t) = \frac{w(Tt)}{T} - \frac{1}{T}\int_0^{Tt} \zeta(\nu(s-),\xi(s-))d\nu(s) := w_T(t) - \xi_T^-(t). \qquad (3.1)$$

Let us first bound $\mathbf{P}(\xi_T(\cdot) \in U_\varepsilon(f))$ from above. For any $c > 0$ and $\delta \in (0,1)$, we have

$$\begin{aligned}\mathbf{P}(\xi_T(\cdot) \in U_\varepsilon(f)) &\leq \mathbf{P}\left(\sup_{t\in[\delta,1]} |\xi_T(t) - f(t)| < \varepsilon, A_c\right) \\ &\quad + \mathbf{P}\left(\sup_{t\in[\delta,1]} |\xi_T(t) - f(t)| < \varepsilon, \overline{A_c}\right) \\ &:= \mathbf{P}_1 + \mathbf{P}_2,\end{aligned}$$

where $A_c := \{\omega : \nu(T) - \nu(\delta T) \leq cT\}$. We bound $\mathbf{P}_1$ from above. Denote

$$B_f := \{g \in \mathbb{VC}_0[0,1] : \dot{g}^+(t) \geq \dot{f}^+(t) \text{ for almost all } t \in [0,1]\},$$

where $g^+ \in \mathbb{VC}_0^M[0,1]$ is positive variation of the function g. For any $r > 0$ the following inequality holds:

$$\begin{aligned}\mathbf{P}_1 &= \mathbf{P}\left(\sup_{t\in[\delta,1]} |w_T(t) - \xi_T^-(t) - f(t)| < \varepsilon, A_c\right) \\ &\leq \mathbf{P}\left(\sup_{t\in[\delta,1]} |w_T(t) - \xi_T^-(t) - f(t)| < \varepsilon, A_c, w_T \in K_r^\varepsilon\right) + \mathbf{P}(w_T \in \overline{K_r^\varepsilon}) \\ &:= \mathbf{P}_{11} + \mathbf{P}_{12},\end{aligned}$$

where

$$K_r^\varepsilon := \left\{v \in \mathbb{C}_0[0,1] : \inf_{g\in K_r} \sup_{t\in[0,1]} |g(t) - v(t)| \leq \varepsilon\right\}, \quad K_r := \{g : I_1(g) \leq r\},$$

and the functional

$$I_1(g) := \begin{cases} \dfrac{1}{2}\displaystyle\int_0^1 (\dot{g}(t))^2 dt, & \text{if } g \in \mathbb{AC}_0[0,1], \\ \infty, & \text{otherwise.}\end{cases} \tag{3.2}$$

Now, we bound $\mathbf{P}_{11}$ from above. Since the random process $\xi_T^-(t)$ does not decrease on the interval $[\delta, 1]$, and since the set K_r is a compact, from Lemma 4.2 it follows that there exists $\gamma(\varepsilon) > 0$ such that $\gamma(\varepsilon) \to 0$ when $\varepsilon \to 0$ and

$$\mathbf{P}_{11} \leq \mathbf{P}(w_T \in B_f^{\delta,\gamma(\varepsilon)} \cap K_r^\varepsilon, A_c) \leq \mathbf{P}(w_T \in B_f^{\delta,\gamma(\varepsilon)}, A_c),$$

where

$$B_f^{\delta,\gamma(\varepsilon)} := \left\{v \in \mathbb{C}_0[0,1] : \inf_{g\in B_f} \sup_{t\in[\delta,1]} |g(t) - v(t)| \leq \gamma(\varepsilon)\right\}.$$

Thanks of independence of the processes $w(t)$ and $\nu(t)$ we obtain

$$\mathbf{P}_{11} \leq \mathbf{P}(w_T \in B_f^{\delta,\gamma(\varepsilon)}, A_c) = \mathbf{P}(w_T \in B_f^{\delta,\gamma(\varepsilon)})\mathbf{P}(A_c).$$

Thus, for all $r > 0$

$$\begin{aligned}\mathbf{P}_1 &\leq \mathbf{P}(w_T \in B_f^{\delta,\gamma(\varepsilon)})\mathbf{P}(A_c) + \mathbf{P}_{12}\\ &= \mathbf{P}(w_T \in B_f^{\delta,\gamma(\varepsilon)})\mathbf{P}(A_c) + \mathbf{P}(w_T \in \overline{K_r^\varepsilon}).\end{aligned} \tag{3.3}$$

We bound $\mathbf{P}_2$ from above. Denote $\tau_{k_1}, \dots, \tau_{k_{\lfloor cT \rfloor}}$ the first $\lfloor cT \rfloor$ jumps of the process $\nu(Tt)$ which belong to the interval $[\delta, 1]$. Denote

$$\begin{aligned}G_{k_l} &:= \{\omega : \xi(\tau_{k_l}-) \in [T(f(\tau_{k_l}) - \varepsilon); T(f(\tau_{k_l}) + \varepsilon)]\}, \quad 1 \leq l \leq \lfloor cT \rfloor,\\ H_{k_l} &:= \{\omega : \zeta(k_l - 1, \xi(\tau_{k_l}-)) < 2T\varepsilon\}, \quad 1 \leq l \leq \lfloor cT \rfloor.\end{aligned}$$

If a trajectory of the process $\xi_T(t)$ does not leave the set $U_\varepsilon(f)$, then $\zeta(k_l - 1, \xi(\tau_{k_l}-)) < 2T\varepsilon$ for $\tau_{k_l} \in [\delta, 1]$, $1 \leq l \leq \lfloor cT \rfloor$. Therefore, the following inequality holds true:

$$\begin{aligned}\mathbf{P}_2 &= \mathbf{P}\left(\sup_{t\in[\delta,1]} |\xi_T(t) - f(t)| < \varepsilon, \overline{A_c}\right)\\ &\leq \sum_{r=\lfloor cT \rfloor}^{\infty} \mathbf{P}\left(\bigcap_{l=1}^{\lfloor cT \rfloor} H_{k_l}, \bigcap_{l=1}^{\lfloor cT \rfloor} G_{k_l} \,\middle|\, \nu(T) - \nu(\delta T) = r\right)\mathbf{P}(\nu(T) - \nu(\delta T) = r).\end{aligned}$$

Denote $m_\delta := \min_{t\in[\delta,1]} f(t)$. The following inequality holds:

$$\mathbf{P}\left(\bigcap_{l=1}^{\lfloor cT \rfloor} H_{k_l}, \bigcap_{l=1}^{\lfloor cT \rfloor} G_{k_l} \,\middle|\, \nu(T) - \nu(\delta T) = r\right) \leq \left(\frac{2\varepsilon\Delta}{m_\delta - \varepsilon}\right)^{\lfloor cT \rfloor}. \tag{3.4}$$

We prove it separately in Sec. 4, see Sec. 4.1. Thus,

$$\mathbf{P}_2 \leq \sum_{r=\lfloor cT \rfloor}^{\infty} \left(\frac{2\varepsilon\Delta}{m_\delta - \varepsilon}\right)^{\lfloor cT \rfloor} \mathbf{P}(\nu(T) - \nu(\delta T) = r) \leq \left(\frac{2\varepsilon\Delta}{m_\delta - \varepsilon}\right)^{\lfloor cT \rfloor}.$$

For the sufficiently small ε the inequality $m_\delta > \sqrt{\varepsilon}$ holds, therefore

$$\mathbf{P}_2 \leq \left(\frac{2\varepsilon\Delta}{m_\delta - \varepsilon}\right)^{\lfloor cT \rfloor} \leq \left(\frac{2\sqrt{\varepsilon}\Delta}{1 - \sqrt{\varepsilon}}\right)^{\lfloor cT \rfloor}. \tag{3.5}$$

From (3.5) it follows that for any $c > 0$:

$$\lim_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{1}{T} \ln \mathbf{P}_2 \leq c \lim_{\varepsilon\to 0} \ln\left(\frac{2\sqrt{\varepsilon}\Delta}{1 - \sqrt{\varepsilon}}\right) = -\infty. \tag{3.6}$$

It is known (see, for example, [12, Theorems 2.1 and 2.2]) that Weiner process satisfies the LDP on the metric space $(\mathbb{D}[0,1],\rho)$, where ρ is the uniform metric, with the rate function (3.2). It implies that for any $\varepsilon > 0$

$$\lim_{r\to\infty}\limsup_{T\to\infty}\frac{1}{T}\ln\mathbf{P}(w_T\in\overline{K_r^\varepsilon}) = -\infty. \tag{3.7}$$

Thus, using (3.3), (3.6), (3.7) and the fact that the set $B_f^{\delta,\gamma(\varepsilon)}$ is the closed set for any $c\in(0,1)$, $\delta\in(0,1)$, we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to0}\limsup_{T\to\infty}\frac{1}{T}\ln\mathbf{P}(\xi_T(\cdot)\in U_\varepsilon(f))\\
&\quad\le\lim_{\varepsilon\to0}\limsup_{T\to\infty}\frac{1}{T}\ln(\mathbf{P}_{11}+\mathbf{P}_{12}+\mathbf{P}_2)\\
&\quad\le\lim_{\varepsilon\to0}\limsup_{T\to\infty}\frac{1}{T}\ln(3\max\{\mathbf{P}_{11},\mathbf{P}_{12},\mathbf{P}_2\})\\
&\quad\le\lim_{\varepsilon\to0}(-I_1(B_f^{\delta,\gamma(\varepsilon)})-\lambda(1-\delta)+\lambda(1-\delta)c-c\ln c)\\
&\quad=-I_1(B_f^\delta)-\lambda(1-\delta)+\lambda(1-\delta)c-c\ln c,
\end{aligned}$$

where

$$B_f^\delta := \left\{v\in\mathbb{C}_0[0,1] : \inf_{g\in B_f}\sup_{t\in[\delta,1]}|g(t)-v(t)|=0\right\},$$

and in the last inequality we applied the following simple inequality:

$$\mathbf{P}(\nu(T)-\nu(\delta T)\le cT)\le\exp\{-\lambda(1-\delta)T+\lambda(1-\delta)cT-Tc\ln c\}. \tag{3.8}$$

Taking the limits $\delta\to0$ and $c\to0$, we obtain

$$\lim_{\varepsilon\to0}\limsup_{T\to\infty}\frac{1}{T}\ln\mathbf{P}(\xi_T(\cdot)\in U_\varepsilon(f))\le -I_1(B_f)-\lambda=-I_1(f^+)-\lambda.$$

To complete the proof, we bound now $\mathbf{P}(\xi_T(\cdot)\in U_\varepsilon(f))$ from below. We have

$$\begin{aligned}
\mathbf{P}_3 &:= \mathbf{P}\left(\sup_{t\in[0,1]}|w_T(t)-\xi_T^-(t)-f(t)|<\varepsilon\right)\\
&\ge \mathbf{P}(w_T(\cdot)\in U_{\frac{\varepsilon}{2}}(f^+),\xi_T^-(\cdot)\in U_{\frac{\varepsilon}{2}}(f^+-f)).
\end{aligned}$$

Note that $f^+-f\in\mathbb{AC}_0^M[0,1]$. If $f^+-f\equiv0$, then

$$\mathbf{P}(w_T(\cdot)\in U_{\frac{\varepsilon}{2}}(f^+),\xi_T^-(\cdot)\in U_{\frac{\varepsilon}{2}}(f^+-f))\ge\mathbf{P}(w_T(\cdot)\in U_{\frac{\varepsilon}{2}}(f^+),\nu(T)=0).$$

Therefore, since $w(t)$ and $\nu(t)$ are independent we obtain

$$\mathbf{P}_3\ge\mathbf{P}(w_T(\cdot)\in U_{\frac{\varepsilon}{2}}(f^+))e^{-\lambda T}. \tag{3.9}$$

Let $f^+ - f \not\equiv 0$. Define

$$n(\varepsilon) := \min\left\{n \in \mathbb{N} : \frac{M}{n} \leq \frac{\varepsilon}{8}\right\},$$

where $M := \max_{t\in[0,1]}(f^+(t) - f(t)) = f^+(1) - f(1)$.

Since $f^+ - f$ is continuous and non-decreasing function, then there exists a finite set of points $0 = t_0 < t_1 < \cdots < t_{n(\varepsilon)} = 1$ such that the following equalities hold true:

$$f^+(t_1) - f(t_1) = \frac{M}{n(\varepsilon)}, \quad f^+(t_2) - f(t_2) = \frac{2M}{n(\varepsilon)}, \ldots, \quad f^+(t_{n(\varepsilon)}) - f(t_{n(\varepsilon)}) = M.$$

Therefore, if the random process $\nu(Tt)$ has no jumps on $[0, t_1]$ and has only one jump in each of intervals $[t_{k-1}, t_k]$, $2 \leq k \leq n(\varepsilon)$, and if random variables $\zeta(k-1, \xi(\tau_k-))$ takes values from the interval

$$\left(\frac{TM}{n(\varepsilon)} - 2T\varepsilon^3; \frac{TM}{n(\varepsilon)} - T\varepsilon^3\right),$$

then for sufficiently small ε the inequality

$$\sup_{t\in[0,1]} |\xi^-(t) - (f^+(t) - f(t))| < \frac{\varepsilon}{2}$$

holds. Hence, for ssufficiently small ε the inequality

$$\begin{aligned} \mathbf{P}_3 &\geq \mathbf{P}(w_T(\cdot) \in U_{\varepsilon^3}(f^+), \xi_T^-(\cdot) \in U_{\frac{\varepsilon}{2}}(f^+ - f)) \\ &\geq \mathbf{P}\left(w_T(\cdot) \in U_{\varepsilon^3}(f^+), \bigcap_{k=1}^{n(\varepsilon)} A_k, \bigcap_{k=1}^{n(\varepsilon)-1} B_k\right), \end{aligned} \tag{3.10}$$

holds, where

$$A_1 := \{\omega : \nu(Tt_1) = 0\}, \quad A_k := \{\omega : \nu(Tt_k) - \nu(Tt_{k-1}) = 1\}, \quad 2 \leq k \leq n(\varepsilon),$$

$$B_k := \left\{\omega : \zeta(k-1, \xi(\tau_k-)) \in \left(\frac{TM}{n(\varepsilon)} - 2T\varepsilon^3; \frac{TM}{n(\varepsilon)} - T\varepsilon^3\right)\right\}, \quad 1 \leq k \leq n(\varepsilon) - 1.$$

From the inequality (3.10) it follows that

$$\mathbf{P}_3 \geq \mathbf{P}\left(\bigcap_{k=1}^{n(\varepsilon)-1} B_k \,\middle|\, w_T(\cdot) \in U_{\varepsilon^3}(f^+), \bigcap_{k=1}^{n(\varepsilon)} A_k\right) \mathbf{P}\left(w_T(\cdot) \in U_{\varepsilon^3}(f^+), \bigcap_{k=1}^{n(\varepsilon)} A_k\right).$$

The following inequality we prove in Sec. 4:

$$\mathbf{P}\left(\bigcap_{k=1}^{n(\varepsilon)-1} B_k \,\middle|\, w_T(\cdot) \in U_{\varepsilon^3}(f^+), \bigcap_{k=1}^{n(\varepsilon)} A_k\right) \geq \left(\frac{\varepsilon^3}{\Delta(f^+(1) + \varepsilon^3)}\right)^{n(\varepsilon)-1}. \tag{3.11}$$

Thanks (3.11) it follows that

$$\mathbf{P}_3 \geq \left(\frac{\varepsilon^3}{\Delta(f^+(1)+\varepsilon^3)}\right)^{n(\varepsilon)-1} \mathbf{P}\left(w_T(\cdot) \in U_{\varepsilon^3}(f^+), \bigcap_{k=1}^{n(\varepsilon)} A_k\right).$$

Since $w_T(t)$ and $\nu(Tt)$ are independent, then

$$\begin{aligned}\mathbf{P}_3 &\geq \left(\frac{\varepsilon^3}{2(f^+(1)+\varepsilon^3)}\right)^{n(\varepsilon)-1} \mathbf{P}(w_T(\cdot) \in U_{\varepsilon^3}(f^+))\mathbf{P}\left(\bigcap_{k=1}^{n(\varepsilon)} A_k\right) \\ &= \left(\frac{\varepsilon^3}{2(f^+(1)+\varepsilon^3)}\right)^{n(\varepsilon)-1} \mathbf{P}(w_T(\cdot) \in U_{\varepsilon^3}(f^+)) \\ &\quad \times (\lambda T)^{n(\varepsilon)-1} e^{-\lambda T} \prod_{k=2}^{n(\varepsilon)} (t_k - t_{k-1}). \end{aligned} \tag{3.12}$$

Using inequalities (3.9), (3.12), we obtain

$$\liminf_{T\to\infty} \frac{1}{T} \ln \mathbf{P}(\xi_T(\cdot) \in U_\varepsilon(f)) \geq -\lambda - I_1(U_{\varepsilon^3}(f^+)).$$

Since Weiner process satisfies LDP with rate function (3.2) we obtain

$$\lim_{\varepsilon\to 0} \liminf_{T\to\infty} \frac{1}{T} \ln \mathbf{P}(\xi_T(\cdot) \in U_\varepsilon(f)) \geq \lim_{\varepsilon\to 0}(-\lambda - I_1(U_{\varepsilon^3}(f^+))) = -\lambda - I_1(f^+).$$

The proof of Theorem 2.3 is similar, where instead of the condition $\mathbf{B}_+$ we work with the condition $\mathbf{B}_-$.

4. Auxiliary Results

The next technical lemma will be useful for the proof of Lemma 4.2. The proof of Lemma 4.1 is quite trivial and will be omitted.

Lemma 4.1. *Let the function $f \in \mathbb{AC}_0^+[0,1]$ is represented in the form*

$$f(t) = g_1(t) - g_2(t), \tag{4.1}$$

where $g_1 \in \mathbb{C}_0[0,1]$ and $g_2 \in \mathbb{VC}_0^M[0,1]$. Then the function $g_1(t)$ has the finite variation and for almost all $t \in [0,1]$ the inequality

$$\dot{g}_1^+(t) \geq \dot{f}^+(t), \tag{4.2}$$

holds true, where $g_1^+(t)$ is the positive variation of the function $g_1(t)$.

Denote $(\mathbb{C}[0,1],\rho)$ the space of continuous functions on the interval $[0,1]$ with given uniform metric ρ. Let $\mathbb{D}_0^M[0,1]$ be the set of cádlág functions (continuous from the right and has a limit from the left) starting from the zero which are non-decreasing on the interval $[0,1]$.

Consider the family of functions $u_T(t)$, $t \in [0,1]$, $T > 0$ which can be represented in the form $u_T(t) := \tilde{u}_T(t) - \hat{u}_T(t)$, where $\hat{u}_T \in \mathbb{D}_0^M[0,1]$, and $\tilde{u}_T \in \mathbb{C}_0[0,1] \cap K_{(\mathbb{C}[0,1],\rho)}$, and $K_{(\mathbb{C}[0,1],\rho)} \subset (\mathbb{C}[0,1],\rho)$ is some compact set.

Lemma 4.2. *Let for a function $f \in \mathbb{AC}_0^+[0,1]$ the following holds true*:

$$\lim_{T\to\infty} \sup_{t\in[0,1]} |u_T(t) - f(t)| = 0. \tag{4.3}$$

Then

$$\lim_{T\to\infty} \inf_{g\in B_f} \sup_{t\in[0,1]} |\tilde{u}_T(t) - g(t)| = 0.$$

Proof. Proof by contradiction. Suppose not. Then, there exists $\gamma > 0$ such that for any $M > 0$ there exists $T > M$ and

$$\inf_{g\in B_f} \sup_{t\in[0,1]} |\tilde{u}_T(t) - g(t)| \geq \gamma. \tag{4.4}$$

Since the family $\tilde{u}_T$ is contained in some compact set, then, if the inequality (4.4) holds, then there exists subsequence T_M and continuous function $\tilde{g}$ such that

$$\lim_{M\to\infty} \sup_{t\in[0,1]} |\tilde{u}_{T_M}(t) - \tilde{g}(t)| = 0, \quad \inf_{g\in B_f} \sup_{t\in[0,1]} |\tilde{g}(t) - g(t)| \geq \gamma.$$

Therefore, from (4.3) it follows that

$$\lim_{M\to\infty} \sup_{t\in[0,1]} |\hat{u}_{T_M}(t) - (\tilde{g}(t) - f(t))| = 0.$$

Wherein due to $\hat{u}_T \in \mathbb{D}_0^M[0,1]$ the function $\hat{g}(t) := \tilde{g}(t) - f(t)$ should belong to the set $\mathbb{VC}_0^M[0,1]$. Thus, $f(t) = \tilde{g}(t) - \hat{g}(t)$, where $\tilde{g} \notin B_f$, $\hat{g} \in \mathbb{VC}_0^M[0,1]$, which contradicts Lemma 4.1. □

4.1. *Proof of inequality* (3.4)

Let $G_{k_0} := \Omega$, $H_{k_0} := \Omega$. We show that the inequality

$$\mathbf{P}_l := \mathbf{P}\left(H_{k_l}, G_{k_l} \,|\, \nu(T) - \nu(\delta T) = r, \bigcap_{d=0}^{l-1} G_{k_d}, \bigcap_{d=0}^{l-1} H_{k_d}\right) \leq \frac{2\varepsilon\Delta}{m_\delta - \varepsilon} \tag{4.5}$$

holds for $1 \leq l \leq \lfloor cT \rfloor$. We estimate from above $\mathbf{P}_l$.

We note that, by definition, a family of random variables $\zeta(k_l - 1, m_{k_l})$, $m_{k_l} \in \mathbb{R}$ not depends on $w(t)$ and $\nu(t)$, $\zeta(k_{l-1} - 1, m_{k_{l-1}})$, $m_{k_{l-1}} \in \mathbb{R}, \ldots, \zeta(k_1 - 1, m_{k_1})$, $m_{k_1} \in \mathbb{R}$, and hence on $\xi(\tau_{k_1}-), \ldots, \xi(\tau_{k_l}-)$. Therefore, the next inequality

$$\mathbf{P}\left(H_{k_l}, G_{k_l} \,|\, \nu(T) - \nu(\delta T) = r, \bigcap_{d=0}^{l-1} G_{k_d}, \bigcap_{d=0}^{l-1} H_{k_d}\right)$$

$$\leq \int_0^{2T\varepsilon} \left(\int_{T(m_\delta - \varepsilon)}^{T(M_1+\varepsilon)} \mathbf{p}_{\zeta(k_l,x)}(y) d\tilde{F}(x) \right) dy$$

holds, where $M_1 := \max_{t\in[0,1]} f(t)$,

$$\tilde{F}(x) := \mathbf{P}\left(\xi(\tau_{k_l}-) < x \,|\, \nu(T) - \nu(\delta T) = r, \bigcap_{d=0}^{l} G_{k_d}, \bigcap_{d=0}^{l-1} H_{k_d}\right).$$

Using the condition $\mathbf{B}_+$, we get for sufficiently small ε

$$\begin{aligned}
\mathbf{P}_l &\le \int_0^{2T\varepsilon}\left(\int_{T(m_\delta-\varepsilon)}^{T(M_1+\varepsilon)} \mathbf{p}_{\zeta(k_l-1,x)}(y)d\tilde{F}(x)\right)dy \\
&\le \int_0^{2T\varepsilon}\left(\int_{T(m_\delta-\varepsilon)}^{T(M_1+\varepsilon)} \frac{\Delta}{|x|}d\tilde{F}(x)\right)dy \\
&\le \int_0^{2T\varepsilon}\left(\int_{T(m_\delta-\varepsilon)}^{T(M_1+\varepsilon)} \frac{\Delta}{T(m_\delta-\varepsilon)}d\tilde{F}(x)\right)dy \\
&\le \frac{2T\varepsilon\Delta}{T(m_\delta-\varepsilon)} = \frac{2\varepsilon\Delta}{m_\delta-\varepsilon}.
\end{aligned}$$

Thus, the inequality (4.5) is proved. Using the inequality (4.5), we obtain

$$\mathbf{P}\left(\bigcap_{l=1}^{\lfloor cT\rfloor} H_{k_l}, \bigcap_{l=1}^{\lfloor cT\rfloor} G_{k_l} \,\middle|\, \nu(T)-\nu(\delta T) = r\right) = \prod_{l=1}^{\lfloor cT\rfloor}\mathbf{P}_l \le \left(\frac{2\varepsilon\Delta}{m_\delta-\varepsilon}\right)^{\lfloor cT\rfloor}.$$

4.2. *Proof of inequality* (3.11)

We show that, for $1 \le k \le n(\varepsilon)-1$ the inequality

$$\mathbf{P}_k := \mathbf{P}\left(B_k \,|\, w_T(\cdot) \in U_{\varepsilon^3}(f^+), \bigcap_{r=1}^{n(\varepsilon)} A_r, B_1, \ldots, B_{k-1}\right) \ge \frac{\varepsilon^3}{\Delta(f^+(1)+\varepsilon^3)} \tag{4.6}$$

holds. If events $\{\omega : w_T(\cdot) \in U_{\varepsilon^3}(f^+)\}$, $\bigcap_{r=1}^{n(\varepsilon)} A_r$, $B_1, \ldots, B_{k-1}$ have occurred, then

$$\begin{aligned}
T(f^+(1)+\varepsilon^3) &> T(f^+(\tau_k)+\varepsilon^3) > \xi(\tau_k-) \\
&\ge T\left(w_T(\tau_k-) - (k-1)\left(\frac{M}{n(\varepsilon)} - \varepsilon^3\right)\right) \\
&> T\left(f^+(\tau_k) - \varepsilon^3 - (k-1)\left(\frac{M}{n(\varepsilon)} - \varepsilon^3\right)\right) \\
&> T\left(f^+(t_k) - \varepsilon^3 - (k-1)\left(\frac{M}{n(\varepsilon)} - \varepsilon^3\right)\right) \\
&> T\left(f^+(t_k) - f(t_k) - \varepsilon^3 - (k-1)\left(\frac{M}{n(\varepsilon)} - \varepsilon^3\right)\right) \\
&> \frac{TM}{n(\varepsilon)} - T\varepsilon^3.
\end{aligned} \tag{4.7}$$

We note that, by definition, the family of random variables $\zeta(k-1, m_k)$, $m_k \in \mathbb{R}$ not depends on $w(t)$ and $\nu(t)$, $\zeta(k-2, m_{k-1})$, $m_{k-1} \in \mathbb{R}, \ldots, \zeta(0, m_1)$, $m_1 \in \mathbb{R}$, and hence on $\xi(\tau_k-), \ldots, \xi(\tau_1-)$. Therefore, using inequality (4.7), we obtain

$$\begin{aligned}\mathbf{P}_k &= \mathbf{P}\left(B_k \,|\, w_T(\cdot) \in U_{\varepsilon^3}(f^+), \bigcap_{r=1}^{n(\varepsilon)} A_r, B_1, \ldots, B_{k-1}\right)\\ &= \mathbf{P}\Bigg(\zeta(k-1, \xi(\tau_k-)) \in \left(\frac{TM}{n(\varepsilon)} - 2T\varepsilon^3; \frac{TM}{n(\varepsilon)} - T\varepsilon^3\right) \Bigg| \, w_T(\cdot) \in U_{\varepsilon^3}(f^+),\\ &\qquad \bigcap_{r=1}^{n(\varepsilon)} A_r, \bigcap_{r=1}^{k-1} B_r\Bigg)\\ &= \int_{\frac{TM}{n(\varepsilon)}-2T\varepsilon^3}^{\frac{TM}{n(\varepsilon)}-T\varepsilon^3} \left(\int_{\frac{TM}{n(\varepsilon)}-T\varepsilon^3}^{T(f^+(1)+\varepsilon^3)} \mathbf{p}_{\zeta(k-1,x)}(y) dF(x)\right) dy,\end{aligned}$$

where

$$F(x) := \mathbf{P}\left(\xi(\tau_k-) < x \,|\, w_T(\cdot) \in U_{\varepsilon^3}(f^+), \bigcap_{r=1}^{n(\varepsilon)} A_r, \bigcap_{r=1}^{k-1} B_r\right).$$

Using the condition $\mathbf{B}_+$, we get for sufficiently large T

$$\begin{aligned}\mathbf{P}_k &\geq \int_{\frac{TM}{n(\varepsilon)}-2T\varepsilon^3}^{\frac{TM}{n(\varepsilon)}-T\varepsilon^3} \left(\int_{\frac{TM}{n(\varepsilon)}-T\varepsilon^3}^{T(f^+(1)+\varepsilon^3)} \frac{1}{\Delta|x|} dF(x)\right) dy \geq \int_{\frac{TM}{n(\varepsilon)}-2T\varepsilon^3}^{\frac{TM}{n(\varepsilon)}-T\varepsilon^3} \frac{1}{\Delta T(f^+(1)+\varepsilon^3)} dy\\ &= \frac{T\varepsilon^3}{\Delta T(f^+(1)+\varepsilon^3)} = \frac{\varepsilon^3}{\Delta(f^+(1)+\varepsilon^3)}.\end{aligned}$$

Thus, the inequality (4.6) is proved. Using the inequality (4.6), we get

$$\mathbf{P}\left(\bigcap_{k=1}^{n(\varepsilon)-1} B_k \,\middle|\, w_T(\cdot) \in U_{\varepsilon^3}(f^+), \bigcap_{k=1}^{n(\varepsilon)} A_k\right) = \prod_{k=1}^{n(\varepsilon)-1} \mathbf{P}_k \geq \left(\frac{\varepsilon^3}{\Delta(f^+(1)+\varepsilon^3)}\right)^{n(\varepsilon)-1}.$$

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