JOURNAL OF FUNCTIONAL ANALYSIS 122, 361–401 (1994)

Circle Actions on C^*-Algebras, Partial Automorphisms, and a Generalized Pimsner–Voiculescu Exact Sequence

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Communicated by A. Connes

Received February 15, 1993

We introduce a method to study C^*-algebras possessing an action of the circle group, from the point of view of their internal structure and their K-theory. Under relatively mild conditions our structure theorem shows that any C^*-algebra, where an action of the circle is given, arises as the result of a construction that generalizes crossed products by the group of integers. Such a generalized crossed product construction is carried out for any partial automorphism of a C^*-algebra, where by a partial automorphism we mean an isomorphism between two ideals of the given algebra. Our second main result is an extension to crossed products by partial automorphisms, of the celebrated Pimsner–Voiculescu exact sequence for K-groups. The representation theory of the algebra arising from our construction is shown to parallel the representation theory for C^*-dynamical systems. In particular, we generalize several of the main results relating to regular and covariant representations of crossed products.

1. INTRODUCTION

Given a group action on a manifold M, one of the main goals of the Dynamical Systems specialist is to describe M in terms of the elementary components singled out by the presence of the action as, for example, orbits and fixed points. The simplest case in which such a description can be thoroughly carried out is that of a free action of a compact group: M can be then described as a principal bundle over the quotient space.

While the interaction between Operator Algebras and Dynamical Systems has been very intense in the last several decades, little has been accomplished in addressing the above question from the Operator Algebras point of view. The C^*-Dynamical Systems counterpart of that program would be to describe the structure of a C^*-algebra, where a group action

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0022-1236/94 $6.00

is given, in terms of elementary data which, supposedly should be extracted from the action.

Among the few cases in which the above mentioned task was successfully carried out, is a result by Paschke (Theorem 2.3 in [14]) in which it is shown that, under certain circumstances, a C^*-algebra carrying an action of the circle group can be described as the crossed product of its fixed point subalgebra (the counterpart of the quotient space) by an action of the integers. The main hypothesis in that theorem is that the action have "large spectral subspaces" (see [14, 11]), a condition that, at least in the case of the circle, singles out the non-commutative analog of free actions.

If one goes as far as to accept that a crossed product is the non-commutative version of a principal bundle, then Paschke's result can be considered as a generalization of the well known fact about free actions of compact groups mentioned above.

The circle of ideas around the notion of "large spectral subspaces" has now a long history. Without attempting a comprehensive account, we should mention that a version of this notion appeared in a paper by Fell [8] in 1969 under te name of homogeneous actions and later in [9] under the name of saturated actions. Related notions were also studied by Phillips [16]. Recent work of Rieffel [20] contains yet another version of this concept, providing a far reaching generalization of the Takesaki–Takai duality (Corollary 1.7 of [20], see also [11]).

But, in the same way that free actions are not the rule, the conditions so far alluded to, exclude many actions of S^1 which one would still like to investigate.

The purpose of the present work is to introduce a method which allows for a description of the structure of C^*-algebras carrying circle actions which are not supposed to have large spectral subspaces. While our method does not include all possible circle actions, since we assume our actions to be semi-saturated (see below), the gain in generality is very significant in the sense that a wealth of new examples becomes tractable using our theory. A typical such example is the action of S^1 on the Toeplitz algebra, given by conjugation by the diagonal unitaries $\operatorname{diag}(1, z, z^2, \ldots)$, for $z \in S^1$. The condition of having large spectral subspaces fails for this action.

Our method consists in first, introducing a construction, inspired on the crossed product construction, which produces a C^*-algebra equipped with a circle action. We then go on to show that any C^*-algebra having an action of S^1 arises as the result of our construction as long as the action satisfies some relatively mild restrictions. In other words we provide a means to "disassemble" a C^*-algebra possessing a circle action, reverting, in a sense, the crossed product construction.

While a crossed product by $\mathbf{Z}$ depends on an automorphism of the given C^*-algebra A (i.e., an action of $\mathbf{Z}$), our construction requires a partial automorphism. Precisely speaking, a partial automorphism of A is a triple $\Theta = (\theta, I, J)$ where I and J are ideals in A and $\theta: I \to J$ is a C^*-algebra isomorphism. Given a partial automorphism, we construct its "covariance algebra" which we denote by $C^*(A, \Theta)$. When both ideals agree with A, then our construction becomes the usual crossed product construction, that is, $C^*(A, \Theta)$ becomes $A \times_\theta \mathbf{Z}$. For that reason, $C^*(A, \Theta)$ should be considered as the crossed product of A by the partial automorphism Θ.

Our second major objective, accomplished by Theorem (7.1), is a generalization of the celebrated Pimser–Voiculescu exact sequence [17]. Precisely, we get the following exact sequence of K-groups

$$\begin{array}{ccccc}
K_0(J) & \xrightarrow{i_* - \theta_*^{-1}} & K_0(A) & \xrightarrow{i_*} & K_0(C^*(A, \Theta)) \\
\uparrow & & & & \downarrow \\
K_1(C^*(A, \Theta)) & \xleftarrow{i_*} & K_1(A) & \xleftarrow{i_* - \theta_*^{-1}} & K_1(J)
\end{array}$$

As in most proofs of the Pimsner–Voiculescu exact sequence and related results [4–6, 1], we derive our exact sequence from the K-theory exact sequence for a suitable Toeplitz extension. The crucial step, as it is often the case, is to show that A has the same K-groups as the Toeplitz algebra. We do so by showing that these algebras are, in fact, KK-equivalent.

My first attempt at proving the exactness of the sequence above was, of course, by trying to deduce it from the well known result of Pimsner and Voiculescu. After failing in doing so, I am now tempted to believe that this cannot be done. Our proof is done from scratch and, given a rather involving use of KK-theory, it turns out considerably longer then the available proofs of the original result.

We thank Bill Paschke for bringing to our attention his paper [1] with J. Anderson, where they generalize a result, from unpublished lecture material of Arveson's, as well as from Proposition (5.5) in [6], from which the crucial step in [17] follows. Our generalization of these ideas plays a central role in the proof of our result.

One interesting aspect, central in our use of KK-theory, is worth mentioning here. If A and B are C^*-algebras, then $KK(A, B)$ may be described, as was shown by Cuntz [7], by the set of homotopy classes of homomorphisms from qA to the multiplier algeba of $B \otimes K$. Nevertheless the KK-theory elements that we need to introduce have no easy description in such terms. Instead we exhibit these elements by replacing $B \otimes K$, above, by an algebra which contains B as a full corner and hence is stably isomorphic to B (at least in the separable case).

We feel that our structure theorem, used in conjunction with the generalized Pimsner–Voiculescu exact sequence above, adds to the growing collection of powerful tools, developed since the early eighties, designed to compute K-groups for C^*-algebras.

Several attempts to generalize crossed products have been made by many authors. Among those we should mention is Kumjian's work on Localizations [12]. Another well known example is the theory of crossed products by endomorphisms developed by Paschke in [13]. Paschke's crossed products carry a circle action, which can easily be made to fall under our restrictions. So, in a certain sense, his algebras are special cases of our construction. In addition, we have recently learned that L. Brown has developed a theory which formalizes the concept of crossed products by imprimitivity bimodules.

Our construction should be regarded as a generalization of crossed products, only as long as the group $\mathbf{Z}$, of integers, is concerned. With some more work, it appears to me that one could attempt to widen the present methods to include a larger class of groups. It would be largely desirable, although quite likely very difficult, to be able to study along the present lines, actions of non-compact groups such as the group of real numbers. Florin Pop pointed out to me recently that the definition of a "partial action" of a discrete group, as well as that of the corresponding covariance algebra, could be obtained by trivial modifications of our definitions.

As we already indicated, our motivation, rather than to produce new classes of C^*-algebras, is to attempt a description of the structure of C^*-algebras possessing a circular symmetry, represented by an action of the circle. A partial automorphism arises from a given action α of S^1 on a C^*-algebra B, in the following way. First, one lets B_1 be the first spectral subspace of α, i.e., $B_1 = \{b \in B : \alpha_z(b) = zb,\ z \in S^1\}$. The set $B_1^* B_1$ (meaning, according to convention (2.2) adopted throughout this paper, the closed linear span of the set of products) is an ideal of the subalgebra B_0 of fixed points for α and the same is true with respect to $B_1 B_1^*$. These ideals are strongly Morita equivalent [18, 19] with B_1 playing the role of the imprimitivity bimodule. Morita equivalent C^*-algebra are quite often isomorphic to each other and indeed, under the assumption that the algebras be stable with strictly positive elements, they are forcibly isomorphic [3]. The two isomorphic ideals are thus the ingredients of our partial automorphism.

Our structure Theorem (4.21) states that B is (isomorphic to) the covariance algebra of that partial automorphism as long as B is generated, as a C^*-algebra, by the union of B_0 and B_1. This condition, which we call semi-saturation, is a weakening of the condition of having large spectral subspace and is the point of departure for our theory. As opposed to what happens to the latter, even when absent, our condition can be forced upon the action by restricting one's attention to the subalgebra of B that $B_0 \cup B_1$ happen to generate.

The algebraic formalism developed here seems to flow with such a naturality that it is perhaps a bit surprising that it has been overlooked until now. Nevertheless, it does not seem possible to extend our methods to rings not possessing a C^*-algebra structure since we make extensive use of the existence of approximate units and facts like ideals of ideals of a C^*-algebra are, themselves, ideals of that algebra, or that the intersection of two ideals equals their product.

Permeating most of our techniques is a concept we have not tried to formalize, but I think the effort to do so seems worthwhile. The reader is invited to compare the definition of multipliers of C^*-algebras on one hand, and (4.11) and (4.13) on the other, and he will likely see the rudiments of a concept which deserves the name of partial multipliers.

After a short Section 2, intended mainly to fix some notation, we describe in Section 3 our construction of the covariance algebra associated to a partial automorphism. Section 4 is where our structure Theorem (4.21) is proved. The fifth section deals with the representation theory for our covariance algebras and to some extent can be regarded as the taming of the algebraic properties of partial isometric operators on Hilbert's space. With surprising ease, partial isometries are made to play the role usually played by implementing unitaries for representations of crossed product algebras.

In Section 6 we introduce the Toeplitz algebra associated to a partial automorphism and describe it, within our theory, as a covariance algebra, as well. In doing so, we are able to obtain a crucial universal property, Lemma (6.7), characterizing representations of the Toeplitz algebra. The seventh and last section is where our main K-theoretical work is developed and where we prove the existence of the generalized Pimsner–Voiculescu exact sequence (Theorem (7.1)).

Our notation is reasonably standard except, possibly, for Definitions (2.2) and (5.4) as well as for our use of the symbol $\oplus$ after (6.3). The reader is advised to go over these immediatly, in order to avoid possible surprises.

This work is an extended version of a paper by the author, entitled "The Structure of Actions of the Circle Group on C^*-Algebras."

2. Spectral Subspaces

This section is concerned with some preliminaries about C^*-dynamical systems based on the circle group. Let B be a fixed C^*-algebra and α an action of S^1 on B.

(2.1) DEFINITION. For each $n \in \mathbf{Z}$ the nth spectral subspace for α is defined by

$$B_n = \{b \in B : \alpha_z(b) = z^n b \text{ for } z \in S^1\}.$$

It is an easy matter to verify that $B_n B_m \subseteq B_{n+m}$ and that $B_n^* = B_{-n}$. Regarding the product $B_n B_m$ just mentioned, we adopt the following convention:

(2.2) DEFINITION. If X and Y are subsets of a C^*-algebra then XY denotes the *closed* linear span of the set of products xy with $x \in X$ and $y \in Y$.

A simple fact we make extensive use of, is the following.

(2.3) PROPOSITION. *For each $n \in \mathbf{Z}$ one has that $B_n^* B_n$ is a closed two sided ideal of the fixed point subalgebra B_0.*

(2.4) DEFINITION. The nth spectral projection for the action α is the transformation

$$P_n : B \to B$$

defined by

$$P_n(b) = \int_{S^1} z^{-n} \alpha_z(b)\, dz, \qquad b \in B.$$

It is well known that P_n is a contractive projection whose image is B_n.

(2.5) PROPOSITION. *Let $b \in B$ and ϕ be a continuous linear functional on B.*

(a) *If $\phi(P_n(b)) = 0$ for all $n \in \mathbf{Z}$, then $\phi(b) = 0$.*

(b) *If $P_n(b) = 0$ for all $n \in \mathbf{Z}$, then $b = 0$.*

(c) *$\bigoplus_{n \in \mathbf{Z}} B_n$ is dense in B.*

Proof (cf. [14, Proposition 2.1]). For the first statement it is enough to note that the nth Fourier coefficient of $z \in S^1 \to \phi(\alpha_z(b))$ is given by $\phi(P_n(b))$. Hence, if $\phi(P_n(b)) = 0$ for all n we have that $\phi(\alpha_z(b))$ is identically zero, as a function of z and, in particular, $\phi(b) = 0$. Finally (b) and (c) follow from (a) and the Hahn–Banach Theorem. ∎

(2.6) PROPOSITION. *If $(e_\lambda)_{\lambda \in \Lambda}$ is an approximate identity for $B_n^* B_n$ then, for each $x \in B_n$, we have $x = \lim_\lambda x e_\lambda$.*

Proof. We have

$$\begin{aligned}\|x-xe_\lambda\|^2 &= \|(x-xe_\lambda)^*\,(x-xe_\lambda)\|\\ &= \|x^*x-x^*xe_\lambda-e_\lambda x^*x+e_\lambda x^*xe_\lambda\|\\ &\leq \|x^*x-x^*xe_\lambda\|+\|e_\lambda\|\;\|x^*x-x^*xe_\lambda\|\to 0. \quad \blacksquare\end{aligned}$$

An immediate consequence is:

(2.7) COROLLARY. *For each $n\in\mathbf{Z}$ one has $B_nB_n^*B_n=B_n$.*

Since the product of two ideals in a C^*-algebra equals their intersection we have:

(2.8) PROPOSITION. *If n and m are integers then $B_n^*B_nB_m^*B_m=B_m^*B_mB_n^*B_n$.*

The following concludes our preparations.

(2.9) PROPOSITION. *Let B and B' be C^*-algebras and let α and α' be actions of S^1 on B and B', respectively. Suppose $\psi\colon B\to B'$ is a covariant homomorphism. If the restriction of ψ to the fixed point subalgebra B_0 is injective, then ψ itself is injective.*

Proof. Assume $b\in B$ is such that $\psi(b)=0$. Then, denoting by P_n and P'_n the respective spectral projections, we have

$$\psi(P_n(b)\,P_n(b)^*)=\psi(P_n(b))\,\psi(P_n(b))^*=P'_n(\psi(b))\,P'_n(\psi(b))^*=0.$$

If one now notes that $P_n(b)\,P_n(b)^*\in B_0$, the hypothesis is seen to imply that $P_n(b)=0$ and so, by (2.5)(b), we have $b=0$. ∎

3. PARTIAL AUTOMORPHIMS AND THEIR COVARIANCE ALGEBRAS

In this section we describe a generalization of the concept of crossed products by an automorphism. For that purpose, let A be a C^*-algebra considered fixed throughout the present section.

(3.1) DEFINITION. A partial automorphism of A is a triple $\Theta=(\theta, I, J)$ where I and J are ideals in A (always assumed closed and two sided) and $\theta\colon I\to J$ is a C^*-algebra isomorphism.

If such a partial automorphism is given, we let, for each integer n, D_n denote the domain of θ^{-n} with the convention that $D_0=A$ and θ^0 is the identity automorphism of A. The domain of θ^{-n} is clearly the image of θ^n so this provides an equivalent definition of D_n.

Alternatively, we can give an inductive definition for these objects by letting $D_0 = A$,

$$D_{n+1} = \{a \in J : \theta^{-1}(a) \in D_n\}$$

for $n \geqslant 0$, and

$$D_{n-1} = \{a \in I : \theta(a) \in D_n\}$$

for $n \leqslant 0$.

According to this, one clearly has $D_1 = J$ and $D_{-1} = I$. Of course, unless I and J have a substantial intersection, the sets D_n would be rather small. The extreme case in which $I \cap J = \{0\}$ will see D_n being the singleton $\{0\}$ for all n, except for $n = -1, 0, 1$.

One of the simplest examples of partial automorphisms is obtained when one lets $A = \mathbf{C}^m$, $I = \{(x_i) \in \mathbf{C}^m : x_m = 0\}$, $J = \{(x_i) \in \mathbf{C}^m : x_1 = 0\}$, and θ be the forward shift

$$\theta(x_1, ..., x_{m-1}, 0) = (0, x_1, ..., x_{m-1}).$$

In this case D_n becomes the set of all m-tuples having n leading zeros when $n \geqslant 0$ or $|n|$ trailing zeros if $n \leqslant 0$.

As it will turn out, the covariance algebra for this example is isomorphic to the algebra of $m \times m$ complex matrices.

(3.2) PROPOSITION. *For each integer n, D_n is an ideal in A.*

Proof. By definition the assertion is obvious for $n = -1, 0, 1$. Arguing by induction assume that $n \geqslant 0$ and that D_n is an ideal in A. Then D_{n+1} is clearly an ideal in J, being the inverse image of D_n under θ^{-1}. Since an ideal of an ideal of a C^*-algebra is always an ideal of that C^*-algebra (by existence of approximate identities) we have that D_{n+1} is an ideal in A. A symmetric argument yields the result for negative values of n. ∎

Let us agree to call by the name of chain any finite sequence $(a_0, a_1, ..., a_n)$ of elements in A such that $a_0 \in I$, $a_n \in J$, and $a_i \in I \cap J$ for $i = 1, ..., n-1$, satisfying $\theta(a_{i-1}) = a_i$ for $i = 1, 2, ..., n$. The integer n will be called the length of said chain.

The concept of a chain can be used for giving yet another definition of D_n. Namely, D_n is the set of elements a in A which admit a chain of length $|n|$, ending in a in case $n \geqslant 0$ or, beginning in a if $n \leqslant 0$.

The following proposition can be easily proven if one thinks in terms of chains. It is nevertheless crucial for what follows.

(3.3) PROPOSITION. *If n and m are integers then $\theta^{-n}(D_n \cap D_m) \subseteq D_{m-n}$. In addition, if x is in $D_n \cap D_m$ then $\theta^{n-m}(\theta^{-n}(x)) = \theta^{-m}(x)$.*

Denote by L the subspace of $l_1(\mathbf{Z}, A)$ formed by all summable sequences $(a(n))_{n\in\mathbf{Z}}$ such that $a(n) \in D_n$ for each n. We propose to equip L with an involutive Banach algebra structure. For that purpose we define, for a and b in L,

$$(a * b)(n) = \sum_{k=-\infty}^{\infty} \theta^k(\theta^{-k}(a(k))\, b(n-k))$$

$$(a^*)(n) = \theta^n(a(-n)^*)$$

$$\|a\| = \sum_{n=-\infty}^{\infty} \|a(n)\|.$$

We next verify some of the axioms of involutive Banach algebras for the multiplication, involution, and norm defined above. But, before that, we should not that our multiplication is well defined since, for each k, $\theta^{-k}(a(k))$ is in D_{-k} while $b(n-k)$ is in D_{n-k}. The product $\theta^{-k}(a(k))\, b(n-k)$ is therefore in the intersection $D_{-k} \cap D_{n-k}$. By Proposition (3.3) it follows that the kth summand in our definition of the product in fact lies in $D_{(n-k)-(-k)} = D_n$. Similarly, note that a^* is an element of L for each a in L.

(3.4) PROPOSITION. *The product defined above is associative.*

Proof. If a is in D_n we denote by $a\delta_n$ the element of L given by $(a\delta_n)(m) = \delta_{n,m}a$, where $\delta_{n,m}$ is the Kronecker symbol.

It is readily seen that the associativity of our product follows from the identity

$$(a_n\delta_n * a_m\delta_m) * a_p\delta_p = a_n\delta_n * (a_m\delta_m * a_p\delta_p),$$

where $a_i \in D_i$ for $i = n, m, p$, which we now propose to prove. Using Proposition (3.3), the left hand side above becomes

$$\begin{aligned}(\theta^n(\theta^{-n}(a_n)a_m)\, \delta_{n+m}) * a_p\delta_p &= \theta^{n+m}(\theta^{-n-m}(\theta^n(\theta^{-n}(a_n)a_m))a_p)\, \delta_{n+m+p} \\ &= \theta^{n+m}(\theta^{-m}(\theta^{-n}(a_n)a_m)a_p)\, \delta_{n+m+p}.\end{aligned}$$

On the other hand, the right hand side of our identity equals

$$a_n\delta_n * (\theta^m(\theta^{-m}(a_m)a_p)\, \delta_{m+p}) = \theta^n(\theta^{-n}(a_n)\, \theta^m(\theta^{-m}(a_m)a_p))\, \delta_{n+m+p}.$$

Note that the term within the outermost parentheses, to the right of the last equal sign, is in $D_{-n} \cap D_m$, so that the coefficient of δ_{n+m+p} above is in D_{m+n} by (3.3). Thus, proving our identity amounts to verifying that

$$\theta^{-m}(\theta^{-n}(a_n)a_m)a_p = \theta^{-n-m}(\theta^n(\theta^{-n}(a_n)\,\theta^m(\theta^{-m}(a_m)a_p)))$$

or, again by (3.3), that

$$\theta^{-m}(\theta^{-n}(a_n)a_m)a_p = \theta^{-m}(\theta^{-n}(a_n)\,\theta^m(\theta^{-m}(a_m)a_p)).$$

Let $(u_i)_i$ be an approximate identity for D_{-m}. So the left hand side above can be written as

$$\begin{aligned}\lim_i \theta^{-m}(\theta^{-n}(a_n)a_m)\,u_i a_p &= \lim_i \theta^{-m}(\theta^{-n}(a_n)\,a_m\theta^m(u_i a_p))\\ &= \lim_i \theta^{-m}(\theta^{-n}(a_n)\,\theta^m(\theta^{-m}(a_m)\,u_i a_p))\\ &= \theta^{-m}(\theta^{-n}(a_n)\,\theta^m(\theta^{-m}(a_m)a_p)),\end{aligned}$$

concluding the proof. ∎

(3.5) Proposition. *For a and b in L one has $(ab)^* = b^*a^*$.*

Proof. Arguing as in the beginning of the previous proof it is enough to verify that

$$(a_n\delta_n * a_m\delta_m)^* = (a_m\delta_m)^* * (a_n\delta_n)^*$$

for $a_n \in D_n$ and $a_m \in D_m$. The left hand side equals, by definition and by (3.3),

$$\begin{aligned}(\theta^n(\theta^{-n}(a_n)a_m)\,\delta_{n+m})^* &= \theta^{-n-m}(\theta^n(\theta^{-n}(a_n)a_m)^*)\,\delta_{-n-m}\\ &= \theta^{-m}(\theta^{-n}(a_n)a_m)^*\,\delta_{-n-m}\\ &= \theta^{-m}((a_m)^*\,\theta^{-n}(a_n)^*)\,\delta_{-n-m}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(a_m\delta_m)^* * (a_n\delta_n)^* &= \theta^{-m}(a_m)^*\,\delta_{-m} * \theta^{-n}(a_n)^*\,\delta_{-n}\\ &= \theta^{-m}(\theta^m(\theta^{-m}(a_m)^*)\,\theta^{-n}(a_n)^*)\,\delta_{-m-n}\\ &= \theta^{-m}((a_m)^*\,\theta^{-n}(a_n)^*)\,\delta_{-m-n}.\end{aligned}$$ ∎

The main difficulties being overcome, we now have:

(3.6) Theorem. *L is an involutive Banach algebra with the above defined multiplication, involution, and norm.*

(3.7) DEFINITION. *The covariance algebra for the partial automorphism* $\Theta = (\theta, I, J)$ *is the* C^**-algebra* $C^*(A, \Theta)$ *obtained by taking the enveloping* C^**-algebra of* L.

Elementary examples of this construction are standard crossed products by the group of integers and the algebra of $n \times n$ complex matrices. The latter is obtained as the covariance algebra for the partial automorphism mentioned after (3.1).

A slightly more elaborate example is what one gets by taking $A = c_0(\mathbf{N})$, $I = A$, J the ideal formed by sequences with a leading zero, and, finally, θ the forward shift. The covariance algebra, in this case, can be shown to be the algebra of compact operators on $l_2(\mathbf{N})$. If, instead, we took A to be the unitization of $c_0(\mathbf{N})$, $I = A$, J the set of elements in A with a leading zero coordinate, and θ the forward shift, then the covariance algebra becomes the Toeplitz algebra, that is, the C^*-algebra generated by the forward shift on $l_2(\mathbf{N})$. The above statements follow as easy corollaries of our structure Theorem (4.21), once one considers the circle action given, in each case, by conjugation by $\text{diag}(1, z, z^2, \ldots)$, for $z \in S^1$. With respect to the Toeplitz algebra, see also (6.6).

The major example we would like to present is related to actions of the circle group on C^*-algebras. As we shall see, any C^*-algebra having an action of S^1, under relatively mild hypothesis on the action, is the covariance algebra for a certain partial automorphism of the algebra of fixed points under S^1. This will be the subject of Section 4.

(3.8) PROPOSITION. *Let* $(e_i)_i$ *be a (bounded) approximate identity for* A. *Then* $(e_i\delta_0)_i$ *is an approximate identity for* L *and hence also for* $C^*(A, \Theta)$.

Proof. For $a_n \in D_n$ we have

$$\lim_i (e_i\delta_0) * (a_n\delta_n) = \lim_i e_i a_n \delta_n = a_n \delta_n.$$

By taking adjoints if follows that $\lim_i (a_n\delta_n) * (e_i\delta_0) = a_n\delta_n$. Since $\sup_i \|e_i\delta_0\| < \infty$ the above implies the conclusion. ∎

(3.9) PROPOSITION. *The map* $E\colon a \in L \to a(0)\delta_0$ *is a contractive positive conditional expectation* [18] *from* L *onto the subalgebra* $A\delta_0$ *of* L.

Proof. To prove positivity let $a \in L$. Then

$$\begin{aligned}(a * a^*)(0) &= \sum_{k=-\infty}^{\infty} \theta^k(\theta^{-k}(a(k))\, a^*(-k)) \\ &= \sum_k \theta^k(\theta^{-k}(a(k))\, \theta^{-k}(a(k)^*)) = \sum_k a(k)\, a(k)^* \geqslant 0.\end{aligned}$$

The remaining statements can be easily verified and are left to the reader. ∎

(3.10) COROLLARY. *The obvious inclusion of A into L, composed with the map from L into $C^*(A, \Theta)$, gives an isometric *-homomorphism of A into the latter.*

Proof. Let $a \in A$ and let f be a state on A such that $f(a^*a) = \|a\|^2$. Identifying A and its copy $A\delta_0$ within L provides us with a state f on $A\delta_0$ such that $f((a\delta_0)^* * (a\delta_0)) = \|a\|^2$. If that state is composed with the conditional expectation of Proposition (3.9) we get a state on L. By (3.8), L has an approximate identity so one is allowed to use the GNS construction, which provides a representation π of L, and hence of the covariance algebra, having a cyclic unit vector ξ and which satisfies

$$\langle \pi(a\delta_0)\xi, \pi(a\delta_0)\xi \rangle = f(a^*a) = \|a\|^2.$$

Thus, the norm $\|a\delta_0\|$, computed in the covariance algebra, is no less than $\|a\|$. The converse inequality follows from the fact that the map mentioned in the statement is a C^*-algebra homomorphism, hence contractive. ∎

We would now like to define the dual action, a concept closely related to dual actions for crossed products.

Let, for every $z \in S^1$, α_z be the transformation of L defined by

$$(\alpha_z(a))(n) = z^n a(n) \qquad \text{for} \quad a \in L, \quad z \in S^1.$$

The reader can easily verify that each α_z is a *-automorphism of L which, in turn, extends to a *-automorphism of $C^*(A, \Theta)$. The resulting map $z \to \alpha_z$ becomes an action of S^1 on $C^*(A, \Theta)$ which we call the dual action.

(3.11) PROPOSITION. *For each n, let B_n be the nth spectral subspace for the dual action. Then the map $\phi_n: x \in D_n \to x\delta_n \in C^*(A, \Theta)$ is a linear isometry onto B_n.*

Proof. It is clear that $x\delta_n \in B_n$ for every $x \in D_n$. Note that for such an x

$$(x\delta_n) * (x\delta_n)^* = (x\delta_n) * (\theta^{-n}(x^*)\delta_{-n}) = \theta^n(\theta^{-n}(x)\,\theta^{-n}(x^*))\delta_0 = xx^*\delta_0.$$

Thus, in order to show that ϕ_n is an isometry, it suffices to consider the case $n = 0$. But this is just the conclusion of (3.10). It now remains to show that the image of ϕ_n is all of B_n. So let $y \in B_n$ and write $y = \lim_k y_k$ where each y_k belongs to L (or rather, the dense image of L in $C^*(A, \Theta)$). Note that L is invariant under the spectral projections of the covariance algebra and also that $y = P_n(y) = \lim_k P_n(y_k)$. So, we may assume that the y_k's belong to the nth spectral subspace for the corresponding action of S^1 on L. That spectral subspace is obviously $D_n\delta_n$ which, by our previous remarks, embeds isometrically into the covariance algebra. This implies that $(y_k)_k$ is a Cauchy sequence with respect to the norm of L and hence that $y = \lim y_k \in D_n\delta_n$. ∎

4. The Structure of Actions of the Circle Group

In this section we intend to prove that any C^*-algebra admitting an action of S^1 is isomorphic to a covariance algebra, as described above, provided the action satisfies two conditions which we now describe.

(4.1) Definition. An action α of S^1 on a C^*-algebra B is called semi-saturated if B is generated, as a C^*-algebra, by the union of the fixed point algebra B_0 and the first spectral subspace B_1 (compare [8, 9, 11, 14, 16, 20]).

(4.2) Definition. An action α of S^1 on a C^*-algebra B is said to be stable if there exists an action α' on a C^*-algebra B' such that $B \simeq B' \otimes K$ and α is the tensor product of α' by the trivial S^1-action on the algebra K of compact operators on a separable, infinite dimensional Hilbert space.

From now on we shall mainly be concerned with semi-saturated stable actions on separable C^*-algebras, but we would like to argue that the above restrictions are quite mild ones. First of all any action gives rise to a stable action by tensoring the old action with the trivial action on K. Moreover, if the old action was semi-saturated then so will be the tensor product action.

Of course not all actions of S^1 are semi-saturated but this difficulty could be circumvented in some cases. That is, given an action α on a C^*-algebra B, it follows from (2.5) that B is generated by the union of all its spectral subspaces.

Now, if we let, for each positive integer n, $B^{(n)}$ be the sub-C^*-algebra of B generated by B_0 and B_n then $B^{(n)}$ is invariant under α and the formula

$$\alpha_z^{(n)} = \alpha_{z^{1/n}} \qquad \text{for} \quad z \in S_1$$

provides a well defined semi-saturated action on $B^{(n)}$.

With some luck, one could put together the information obtained about each $B^{(n)}$ to learn something of interest about the original action.

(4.3) PROPOSITION. *Let α be a stable action of S^1 on a separable C^*-algebra B. Then there exists an isomorphism $\theta: B_1^* B_1 \to B_1 B_1^*$ (see (2.2)) and a linear isometry λ from B_1^* onto $B_1 B_1^*$ such that for $x, y \in B_1$, $a \in B_1^* B_1$, and $b \in B_1 B_1^*$*

(i) $\lambda(x^*b) = \lambda(x^*)b$

(ii) $\lambda(ax^*) = \theta(a)\lambda(x^*)$

(iii) $\lambda(x^*)^* \lambda(y^*) = xy^*$

(iv) $\lambda(x^*)\lambda(y^*)^* = \theta(x^*y)$.

Proof. Everything will follow from [3, Theorem (3.4)] after we verify that both $B_1^* B_1$ and $B_1 B_1^*$ are stable C^*-algebras with strictly positive elements. Now, stability follows at once from our assumption that the action be stable while the existence of strictly positive elements is a consequence of our separability hypothesis. ∎

The conclusions of the last proposition will be among our main tools in what follows. In fact the only reason we consider stable actions on separable C^*-algebras is to obtain the conclusions of (4.3).

If, for any other reason, the existence of θ and λ as above is guaranteed in a specific example, we may discard stability of the action α and separability of the algebra B without hurting the results of this section.

In fact it is quite common to find examples in which the conclusions of (4.3) hold, but still the action is not stable. The simplest such example is provided by the action of S^1 on $B = M_2(\mathbf{C})$ given by conjugation by the unitary matrices $\left(\begin{smallmatrix} 1 & 0 \\ 0 & z \end{smallmatrix}\right)$ for z in S^1. In this case we have

$$B_0 = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbf{C} \right\},$$

$$B_1 = \left\{ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} : y \in \mathbf{C} \right\},$$

$$B_1^* B_1 = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} : x \in \mathbf{C} \right\},$$

and

$$B_1 B_1^* = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} : y \in \mathbf{C} \right\}.$$

The maps θ and λ defined by

$$\theta \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \quad \text{for } x \in \mathbf{C}$$

and

$$\lambda\begin{pmatrix}0 & x\\ 0 & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & x\end{pmatrix} \quad \text{for} \quad x\in \mathbf{C}$$

satisfy the conclusions of (4.3), as the reader may easily verify, although our action is obviously not stable.

(4.4) DEFINITION. An action of S^1 on a C^*-algebra B will be called regular if the conclusions of (4.3) hold. That is, there should exist an isomorphism θ: $B_1^* B_1 \to B_1 B_1^*$ and a linear isometry λ from B_1^* onto $B_1 B_1^*$ satisfying (i)–(iv) of (4.3).

We obviously have:

(4.5) COROLLARY. *Stable circle actions on separable C^*-algebras are regular.*

We also note the following fact.

(4.6) PROPOSITION. *Let α be the dual action on the covariance algebra of a partial automorphism. Then α is regular.*

Proof. The partial automorphism is built into the picture and λ is given by

$$\lambda((x\delta_1)^*)=x^*\delta_0 \qquad \text{for} \quad x\in D_1. \quad \blacksquare$$

Also, there is no hope for an action of S^1 to be equivalent to a dual action unless it is semi-saturated as we now demonstrate.

(4.7) PROPOSITION. *If $\Theta=(\theta, I, J)$ is a partial automorphism of the C^*-algebra A, then the dual action α on the covariance algebra $B=C^*(A, \Theta)$ is semi-saturated.*

Proof. By (2.5) and by induction it is enough to show that, for $n\geqslant 1$, $B_{n+1}\subseteq B_1 B_n$. If $x\in B_{n+1}$ is of the form $x=a\delta_{n+1}$ for $a\in D_{n+1}$, note that $\theta^{-1}(a)\in D_{-1}\cap D_n$. Without loss of generality we may assume that $\theta^{-1}(a)=bc$ where $b, c\in D_{-1}\cap D_n$ (the reduction to this case being made by an approximation argument). Put $b_1=\theta(b)$ and observe that

$$(b_1\delta_1)*(c\delta_n)=\theta(\theta^{-1}(b_1)c)\,\delta_{n+1}=\theta(bc)\,\delta_{n+1}=a\delta_{n+1}=x.$$

This shows that B_{n+1}, which equals $D_{n+1}\delta_{n+1}$ by (3.11), is contained in $B_1 B_n$. $\blacksquare$

One of the crucial properties of semi-saturated actions is obtained in our following result. It also provides an alternate characterization of semi-saturation.

(4.8) PROPOSITION. *An action* α *of* S^1 *on* B *is semi-saturated if and only if for each* $n>0$ *one has* $B_n=(B_1)^n$. *In this case, if* $n, m \geq 0$ *then* $B_{n+m}=B_n B_m$.

Proof. Let B' be the dense *-subalgebra of B generated by $B_0 \cup B_1$. Any element of B' is a sum of "words" of the form $w=x_1 x_2 \cdots x_k$ where $x_i \in B_0 \cup B_1 \cup B_1^*$.

A word, such as above, represents an element of B which belongs to some spectral subspace, the precise determination of which is obtained by subtracting the number of x_i's belonging to B_1^* from the number of those in B_1. For this reason, to apply the spectral projection P_n to a sum of words is to eliminate all summands, but those which belong to B_n.

If $b \in B_n$ let $b=\lim_j b_j$ with $b_j \in B'$. Thus $b=P_n(b)=\lim_j P_n(b_j)$, so we may assume (upon replacing b_j by $P_n(b_j)$) not only that $b_j \in B' \cap B_n$, but also that each b_j is a sum of words each of which belongs to B_n. A minute of reflection will show that all words in B_n are in $(B_1)^n$ for $n>0$. The converse follows from (2.5). ▮

Let us now fix, for the remainder of this section, a C^*-algebra B where a regular semi-saturated action α of S^1 is defined.

(4.9) DEFINITION. If λ and θ are as in (4.4) let $\rho: B_1^* \to B_1^* B_1$, $\lambda^{\dagger}: B_1 \to B_1^* B_1$, and $\rho^{\dagger}: B_1 \to B_1 B_1^*$ be the isometries defined by $\rho=\theta^{-1} \circ \lambda$, $\lambda^{\dagger}(x)=\rho(x^*)^*$, and $\rho^{\dagger}(x)=\lambda(x^*)^*$ for all $x \in B_1$.

(4.10) PROPOSITION. *For* $a \in B_1^* B_1$, $b \in B_1 B_1^*$, *and* $x, y \in B_1$ *one has*

(i) $\rho(ax^*)=a\rho(x^*)$

(ii) $\rho(x^*b)=\rho(x^*)\,\theta^{-1}(b)$

(iii) $\rho(x^*)\,\rho(y^*)^*=x^*y$

(vi) $\rho(x^*)^*\,\rho(y^*)=\theta^{-1}(xy^*)$

(v) $\lambda^{\dagger}(bx)=\theta^{-1}(b)\,\lambda^{\dagger}(x)$

(vi) $\lambda^{\dagger}(xa)=\lambda^{\dagger}(x)a$

(vii) $\lambda^{\dagger}(x)^*\,\lambda^{\dagger}(y)=x^*y$

(viii) $\lambda^{\dagger}(x)\,\lambda^{\dagger}(y)^*=\theta^{-1}(xy^*)$

(ix) $\rho^{\dagger}(bx)=b\rho^{\dagger}(x)$

(x) $\rho^\dagger(xa)=\rho^\dagger(x)\,\theta(a)$

(xi) $\rho^\dagger(x)\,\rho^\dagger(y)^*=xy^*$

(xii) $\rho^\dagger(x)^*\,\rho^\dagger(y)=\theta(x^*y)$.

Proof. This follows by routine computations. ▮

(4.11) PROPOSITION. *The maps λ, ρ, $\lambda^\dagger$, and $\rho^\dagger$ extend to isometries*

$$\lambda\colon B_1^*B\to B_1B$$
$$\rho\colon BB_1^*\to BB_1$$
$$\lambda^\dagger\colon B_1B\to B_1^*B$$
$$\rho^\dagger\colon BB_1\to BB_1^*$$

satisfying

(i) $\lambda(sb)=\lambda(s)b$ *for* $s\in B_1^*B_1$, $b\in B$

(ii) $\lambda(s)^*\,\lambda(t)=s^*t$ *for* $s,t\in B_1^*B$

(iii) $\rho(bs)=b\rho(s)$ *for* $s\in BB_1^*$, $b\in B$

(iv) $\rho(s)\,\rho(t)^*=st^*$ *for* $s,t\in BB_1^*$

(v) $\lambda^\dagger(sb)=\lambda^\dagger(s)b$ *for* $s\in B_1B$, $b\in B$

(vi) $\lambda^\dagger(s)^*\,\lambda^\dagger(t)=s^*t$ *for* $s,t\in B_1B$

(vii) $\rho^\dagger(bs)=b\rho^\dagger(s)$ *for* $s\in BB_1$, $b\in B$

(viii) $\rho^\dagger(s)\,\rho^\dagger(t)^*=st^*$ *for* $s,t\in BB_1$.

Moreover

(a) *If $X\subseteq B_1^*B$ is a closed subspace, invariant under left multiplication by $B_1^*B_1$, then $\lambda(X)=B_1X$.*

(b) *If $X\subseteq BB_1^*$ is closed and right $B_1B_1^*$-invariant then $\rho(X)=XB_1$.*

(c) *If $X\subseteq B_1B$ is closed and left $B_1B_1^*$-invariant then $\lambda^\dagger(X)=B_1^*X$.*

(d) *If $X\subseteq BB_1$ is closed and right $B_1^*B_1$-invariant then $\rho^\dagger(X)=XB_1^*$.*

Finally, $\rho^\dagger=\rho^{-1}$ and $\lambda^\dagger=\lambda^{-1}$ hold for the extended maps.

Proof. If $s\in B_1^*B$ is of the form $s=\sum_{i=1}^n x_i^*b_i$ where $x_i\in B_1$ and $b_i\in B$, put $\lambda(s)=\sum_{i=1}^n\lambda(x_i^*)b_i$. Observe that

$$\left\|\sum_{i=1}^n\lambda(x_i^*)b_i\right\|^2=\left\|\sum_{ij}b_j^*\lambda(x_j^*)^*\,\lambda(x_i^*)b_i\right\|=\left\|\sum_{ij}b_j^*x_jx_i^*b_i\right\|=\left\|\sum_{i=1}^n x_i^*b_i\right\|^2,$$

so that λ is well defined, clearly satisfies (i) and (ii) and is norm preserving. The extensions of ρ, $\lambda^\dagger$, and $\rho^\dagger$ are handled in a similar way.

Let X be a closed subspace of $B_1^* B$, invariant under left multiplication by $B_1^* B_1$. Then by (2.6), $X = B_1^* B_1 X$ so

$$\lambda(X) = \lambda(B_1^*)\, B_1 X = B_1 B_1^* B_1 X = B_1 X.$$

Similar proofs apply to the next three statements.

Finally, let $x, y \in B_1$ and note that

$$\theta(y^* x) = \lambda(y^*)\, \lambda(x^*)^* = \lambda(y^* \lambda(x^*)^*) = \lambda(y^* \rho^\dagger(x)),$$

so that

$$y^* x = \theta^{-1}(\lambda(y^* \rho^\dagger(x))) = \rho(y^* \rho^\dagger(x)) = y^* \rho(\rho^\dagger(x)).$$

Therefore, by (2.6), $x = \rho\rho^\dagger(x)$.

For $s = bx$ with $b \in B$ and $x \in B_1$ we have $\rho\rho^\dagger(bx) = \rho(b\rho^\dagger(x)) = b\rho\rho^\dagger(x) = bx$ so the composition

$$BB_1 \xrightarrow{\rho^\dagger} BB_1^* \xrightarrow{\rho} BB_1$$

is the identity. Since ρ is isometric, hence injective, we get $\rho^\dagger = \rho^{-1}$. The proof that $\lambda^\dagger = \lambda^{-1}$ goes along identical lines. ∎

(4.12) Proposition. *The composition*

$$B_1^* BB_1 \xrightarrow{\lambda} B_1 BB_1 \xrightarrow{\rho^\dagger} B_1 BB_1^*$$

is a C^*-algebra isomorphism which extends θ and whose inverse is given by the composition

$$B_1 BB_1^* \xrightarrow{\rho} B_1 BB_1 \xrightarrow{\lambda^\dagger} B_1^* BB_1.$$

Proof. For $x, y \in B_1$ and $a \in B$ we have

$$\rho^\dagger(\lambda(x^* ay)) = \rho^\dagger(\lambda(x^*)\, ay) = \lambda(x^*)\, a\rho^\dagger(y).$$

Thus if $x_1, y_1 \in B_1$ and $a_1 \in B$

$$\begin{aligned}\rho^\dagger(\lambda(x^* ay))\, \rho^\dagger(\lambda(x_1^* a_1 y_1)) &= \lambda(x^*)\, a\rho^\dagger(y)\, \lambda(x_1^*)\, a_1 \rho^\dagger(y)\\ &= \lambda(x^*)\, a\lambda(y^*)^*\, \lambda(x_1^*)\, a_1 \rho^\dagger(y)\\ &= \lambda(x^*)\, ayx_1^* a_1 \rho^\dagger(y) = \rho^\dagger(\lambda(x^* ayx_1^* a_1 y_1)),\end{aligned}$$

proving $\rho^\dagger \circ \lambda$ to be multiplicative.

We also have

$$\rho^\dagger(\lambda(x^* ay))^* = \rho^\dagger(y)^*\, a^* \lambda(x^*)^* = \lambda(y^*)\, a^* \rho^\dagger(x) = \rho^\dagger(\lambda(y^* a^* x)),$$

which shows that $\rho^\dagger \circ \lambda$ is star preserving.

To check the statement about extending θ note that for $x, y \in B_1$ we have

$$\rho^\dagger(\lambda(x^*y)) = \lambda(x^*)\,\rho^\dagger(y) = \lambda(x^*)\,\lambda(y^*)^* = \theta(x^*y).$$

The fact that $\lambda^\dagger\rho$ inverts $\rho^\dagger\lambda$ follows from (4.11). ∎

Our next result resembles one of the main axioms for multipliers of C^*-algebras.

(4.13) PROPOSITION. *If $s \in BB_1^*$ and $t \in B_1^* B$ then $\rho(s)t = s\lambda(t)$.*

Proof. It clearly suffices to verify the case in which $s = x^*$ and $t = y^*$ with $x, y \in B_1$. We have

$$\rho(x^*)\,y^* = \rho(x^*)\,\rho(\rho^\dagger(y))^* = x^*\rho^\dagger(y)^* = x^*\lambda(y^*). \quad ∎$$

(4.14) PROPOSITION. *Regarding the partial automorphism $(\theta, B_1^* B_1, B_1 B_1^*)$ of B_0 we have* $\mathrm{Dom}(\theta^n) = B_n^* B_n$ *for all integers n.*

Proof. This is clearly true for $n = -1, 0, 1$. For $n \geqslant 1$, using induction, we have

$$\mathrm{Dom}(\theta^{n+1}) = \{x \in \mathrm{Dom}(\theta) : \theta(x) \in \mathrm{Dom}(\theta^n)\} = \{x \in B_1^* B_1 : \theta(x) \in B_n^* B_n\}.$$

Thus, given x in $\mathrm{Dom}(\theta^{n+1})$

$$\theta(x) \in \mathrm{Im}(\theta) \cap \mathrm{Dom}(\theta^n) = \mathrm{Im}(\theta) \cdot \mathrm{Dom}(\theta^n) = B_1 B_1^* B_n^* B_n B_1 B_1^*,$$

so

$$\begin{aligned} x = \theta^{-1}(\theta(x)) &= \lambda^\dagger\rho(\theta(x)) \in \lambda^\dagger(\rho(B_1 B_1^* B_n^* B_n B_1 B_1^*)) \\ &= B_1^* B_1 B_1^* B_n^* B_n B_1 B_1^* B_1 = B_{n+1}^* B_{n+1}. \end{aligned}$$

This shows that $\mathrm{Dom}(\theta^{n+1}) \subseteq B_{n+1}^* B_{n+1}$. Conversely, if $x \in B_{n+1}^* B_{n+1}$ then clearly $x \in \mathrm{Dom}(\theta)$ and

$$\theta(x) = \rho^\dagger\lambda(x) \in \rho^\dagger(\lambda(B_{n+1}^* B_{n+1})) = B_1 B_{n+1}^* B_{n+1} B_1^* \subseteq B_n^* B_n = \mathrm{Dom}(\theta^n).$$

So $x \in \mathrm{Dom}(\theta^{n+1})$. ∎

Our main goal, as we already indicated, is to describe B as the covariance algebra for the partial automorphism $\Theta = (\theta, B_1^* B_1, B_1 B_1^*)$ of the fixed point algebra B_0. The conclusions of (4.14) suggest that, if we are to succeed, then $B_n B_n^*$, being essentially the nth spectral subspace of the covariance algebra $C^*(B_0, \Theta)$, should correspond to B_n. This leads us to the following definitions.

(4.15) DEFINITION. Let $i_0: B_0 \to B_0$ denote the identity map of B_0 and $i_n: B_n \to B_0$ be defined, inductively, by $i_n(x) = i_{n-1}(\rho^\dagger(x))$ when $n \geqslant 1$ or by $i_n(x) = i_{n+1}(\rho(x))$ when $n \leqslant -1$.

It is clear that i_n is an isometry for each n (although not necessarily surjective).

(4.16) PROPOSITION. *For all integers m and n and for any $x_n \in B_n$ and $y_m \in B_m$ we have*

$$i_{n+m}(x_n y_m) = i_n(x_n i_m(y_m)).$$

Proof. Assume initially that $n = 1$. If $m = 0$ there is nothing to prove and if $m > 0$, using induction, we have

$$\begin{aligned} i_{1+m}(x_1 y_m) &= i_m(\rho^\dagger(x_1 y_m)) = i_m(x_1 \rho^\dagger(y_m)) \\ &= i_1(x_1 i_{m-1} \rho^\dagger(y_m)) = i_1(x_1 i_m(y_m)). \end{aligned}$$

If $m = -1$, still assuming $n = 1$, we have

$$i_1(x_1 i_{-1}(y_{-1})) = \rho^\dagger(x_1 \rho(y_{-1})) = \rho^\dagger \rho(x_1 y_{-1}) = x_1 y_{-1} = i_0(x_1 y_{-1}).$$

Now let $m < -1$ and observe that, by induction

$$\begin{aligned} i_{1+m}(x_1 y_m) &= i_{1+(m+1)}(\rho(x_1 y_m)) = i_{1+(m+1)}(x_1 \rho(y_m)) \\ &= i_1(x_1 i_{m+1} \rho(y_m)) = i_1(x_1 i_m(y_m)). \end{aligned}$$

If $n > 1$ we may assume, without loss of generality, by (4.8) or (2.6) that $x_n = x_1 x_{n-1}$ where $x_1 \in B_1$ and $x_{n-1} \in B_{n-1}$. So, by induction once more,

$$\begin{aligned} i_{n+m}(x_n y_m) &= i_{1+(n-1)+m}(x_1 x_{n-1} y_m) = i_1(x_1 i_{n-1+m}(x_{n-1} y_m)) \\ &= i_1(x_1 i_{n-1}(x_{n-1} i_m(y_m))) = i_n(x_1 x_{n-1} i_m(y_m)). \end{aligned}$$

The case $n = 0$ is trivial and the proof for $n < 0$ is done in a similar way. ∎

(4.17) PROPOSITION. *For every integer n one has $i_n(B_n) = B_n B_n^*$.*

Proof. The statement follows by definition when $n = -1, 0, 1$. If $n \geqslant 1$, by induction, (2.7), (4.8), and (4.11) one has

$$\begin{aligned} i_{n+1}(B_{n+1}) &= i_1(B_1 i_n(B_n)) = \rho^\dagger(B_1 B_n B_n^*) \\ &= \rho^\dagger(B_1 B_1^* B_1 B_n B_n^*) = \rho^\dagger(B_1 B_n B_n^* B_1^* B_1) \\ &= B_1 B_n B_n^* B_1^* B_1 B_1^* = B_{n+1} B_{n+1}^*. \end{aligned}$$

As usual, the proof for negative values of n is omitted. ∎

(4.18) PROPOSITION. *If* $n \in \mathbf{Z}$, $a \in B_n^* B_n$, *and* $x_n \in B_n$ *then* $i_n(x_n a) = i_n(x_n)\, \theta^n(a)$.

Proof. For $n = -1, 0, 1$ the statement is obvious. For $n > 1$ assume, as before, that $x_n = x_1 x_{n-1}$, the subscripts indicating the spectral subspace each factor lies in. So, by induction,

$$\begin{aligned} i_n(x_n a) &= i_n(x_1 x_{n-1} a) = i_1(x_1 i_{n-1}(x_{n-1} a)) \\ &= \rho^\dagger(x_1 i_{n-1}(x_{n-1})\, \theta^{n-1}(a)) \\ &= \rho^\dagger(x_1 i_{n-1}(x_{n-1}))\, \theta^n(a) = i_n(x_n)\, \theta^n(a). \end{aligned}$$

For $n < -1$ the proof is similar. ▮

(4.19) PROPOSITION. *For all integers* n *and* m *and for all* $x_n \in B_n$ *and* $x_m \in B_m$

$$\theta^{-n}(i_{n+m}(x_n y_m)) = \theta^{-n}(i_n(x_n))\, i_m(x_m).$$

Proof. Let $\{e_i\}$ be an approximate identity for $B_n^* B_n$. We have

$$\begin{aligned} \theta^{-n}(i_{n+m}(x_n y_m)) &= \theta^{-n}(i_n(x_n i_m(y_m))) = \lim_j \theta^{-n}(i_n(x_n e_j i_m(y_m))) \\ &= \lim_j \theta^{-n}(i_n(x_n)\, \theta^n(e_j i_m(y_m))) \\ &= \lim_j \theta^{-n}(i_n(x_n))\, e_j i_m(y_m) = \theta^{-n}(i_n(x_n))\, i_m(y_m). \end{aligned}$$ ▮

(4.20) PROPOSITION. *For any* n *and for any* $x_n \in B_n$ *one has* $\theta^{-n}(i_n(x_n)) = i_{-n}(x_n^*)^*$.

Proof. This is obvious for $n = 0$. If $n = 1$ we have

$$\theta^{-1}(i_1(x_1)) = \lambda^\dagger \rho \rho^\dagger(x_1) = \lambda^\dagger(x_1) = \rho(x_1^*)^* = i_{-1}(x_1^*)^*.$$

If $n \geqslant 1$ assume, without loss of generality, that $x_n = x_1 x_{n-1}$ where $x_1 \in B_1$ and $x_{n-1} \in B_{n-1}$. So, by induction, we have

$$\begin{aligned} \theta^{-n}(i_n(x_n)) &= \theta^{-(n-1)} \theta^{-1}(i_1(x_1 i_{n-1}(x_{n-1}))) \\ &= \theta^{-(n-1)}(i_{-1}(i_{n-1}(x_{n-1})^* x_1^*))^* \\ &= \theta^{-(n-1)}(i_{n-1}(x_{n-1})^*\, i_{-1}(x_1^*))^* \\ &= \theta^{-(n-1)}(i_{-1}(x_1^*)\, i_{n-1}(x_{n-1})) \\ &= \theta^{-(n-1)}(i_{n-1}(i_{-1}(x_1^*)\, x_{n-1})) = i_{-(n-1)}(x_{n-1}^* i_{-1}(x_1^*)^*)^* \\ &= i_{-n}(x_{n-1}^* x_1^*)^* = i_{-n}(x_n^*)^*. \end{aligned}$$

The proof for negative values of n is similar. ▮

We are now prepared to present our first main result.

(4.21) THEOREM. *Let α be a semi-saturated regular action of S^1 on a C^*-algebra B. If $\Theta = (\theta, B_1^* B_1, B_1 B_1^*)$ is the partial automorphism of the fixed point algebra B_0 as in* (4.4), *then there exists an isomorphism*

$$\phi: C^*(B_0, \Theta) \to B$$

which is covariant with respect to the dual action.

Proof. Recall that L is the Banach *-algebra formed by summable sequences $(a(n))_{n \in \mathbf{Z}}$ with $a(n) \in \mathrm{Dom}(\theta^{-n}) = B_n B_n^*$. Define

$$\phi: a \in L \to \sum i_n^{-1}(a(n)) \in B.$$

To show that ϕ is multiplicative, let $a_n \in B_n B_n^*$, $b_m \in B_m B_m^*$, and put $x_n = i_n^{-1}(a_n)$ and $y_m = i_m^{-1}(b_m)$. We must then verify that

$$\phi(a_n \delta_n * b_m \delta_m) = \phi(a_n \delta_n)\, \phi(b_m \delta_m).$$

For this end, we need to check that

$$i_{n+m}^{-1}(\theta^n(\theta^{-n}(a_n) b_m)) = i_n^{-1}(a_n)\, i_m^{-1}(b_m)$$

or that

$$\theta^n(\theta^{-n}(i_n(x_n))\, i_m(y_m)) = i_{n+m}(x_n y_m)$$

which follows from (4.19).

To check that ϕ preserves adjoints it is enough to show that, for any n and any $a \in B_n B_n^*$,

$$\phi((a\delta_n)^*) = \phi(a\delta_n)^*$$

which translates to

$$i_{-n}^{-1}(\theta^{-n}(a^*)) = i_n^{-1}(a)^*.$$

Writing $a = i_n(x_n)$, for $x_n \in B_n$, the above becomes

$$i_{-n}^{-1}(\theta^{-n}(i_n(x_n)^*)) = x_n^*$$

which is equivalent to the conclusion of (4.20).

It follows that ϕ extends to a *-homomorphism from $C^*(B_0, \Theta)$ to B which is surjective, because its image contains any B_n, and which is equivariant with respect to the dual action, since spectral subspaces are mapped accordingly.

It remains to show that ϕ is injective, but in view of (2.9) we only need to check injectivity on the fixed point subalgebra of $C^*(B_0, \Theta)$, a fact that follows from (3.11). ∎

5. Representations of Covariance Algebras

The parallel between our covariance algebras and crossed product algebras is reflected, also, in their representation theory which we now study in detail. In particular, it is possible to define an analog for the regular representation of crossed products which is our next goal.

Fix, for the remainder of this section, a C^*-algebra A and a partial automorphism $\Theta = (\theta, I, J)$ of A.

(5.1) Definition. Let π be a representation of A on the Hilbert space $\mathscr{H}$. The regular representation $\tilde{\pi}$ of $C^*(A, \Theta)$, associated to π, is that which is obtained by inducing π to L [18] via the conditional expectation E of (3.9) and then extending it to the enveloping C^*-algebra of L.

A result which could perhaps be thought of as a new manifestation of the amenability of the group of integers is the following (see [15]):

(5.2) Theorem. *If π is a faithful representation of A then $\tilde{\pi}$ is faithful on $C^*(A, \Theta)$.*

Proof. The inducing process [18] leads us to consider the Hilbert space $\tilde{\mathscr{H}}$ obtained by completing $L \otimes \mathscr{H}$ under the semi-norm given by the (sometimes degenerated) inner product

$$\langle y_1 \otimes \xi_1, y_2 \otimes \xi_2 \rangle = \langle \pi(E(y_2^* y_1))\, \xi_1, \xi_2 \rangle \quad \text{for } y_i \in L, \quad \xi_i \in \mathscr{H}, \quad i = 1, 2.$$

The induced representation itself is specified by

$$\tilde{\pi}(x)(y \otimes \xi) = (xy) \otimes \xi \qquad \text{for} \quad x, y \in L, \quad \xi \in \mathscr{H}.$$

Let L_n be the nth spectral subspace for the dual action restricted to L, that is, $L_n = D_n \delta_n$. Since, for $n \neq m$, $E(L_n^* L_m) = 0$ and because $\bigoplus_n L_n$ is dense in L, we have that the subspaces $\tilde{\mathscr{H}}_n$ of $\tilde{\mathscr{H}}$, obtained by closing the image of $L_n \otimes \mathscr{H}$ in $\tilde{\mathscr{H}}$, form an orthogonal decomposition of $\tilde{\mathscr{H}}$.

If $y = d_n \delta_n$ for $d_n \in D_n$, then it is clear that $\tilde{\pi}(y)$ sends $\tilde{\mathscr{H}}_m$ into $\tilde{\mathscr{H}}_{m+n}$ for every m. Thus, if y is the finite sum $y = \sum_{n=-N}^{N} d_n \delta_n$, it follows that

$$p_{m+n} \circ \tilde{\pi}(y)|_{\tilde{\mathscr{H}}_m} = \tilde{\pi}(P_n(y))|_{\tilde{\mathscr{H}}_m} \qquad \text{for} \quad n, m \in \mathbf{Z},$$

where p_n is the orthogonal projection of $\tilde{\mathscr{H}}$ onto $\tilde{\mathscr{H}}_n$ and P_n is, as in (2.4), the nth spectral projection. Since the finite sums are dense in $C^*(A, \Theta)$, a continuity argument yields the above formula for all $y \in C^*(A, \Theta)$.

From this it can easily be seen that if $\tilde{\pi}(y) = 0$ then $\tilde{\pi}(P_n(y)) = 0$ for any y in the covariance algebra. This reduces our task to the easier one of proving $\tilde{\pi}$ to be injective on each spectral subspace of $C^*(A, \Theta)$, which we have already shown to be essentially $D_n \delta_n$. So let $d_n \in D_n$ and suppose that $\tilde{\pi}(d_n \delta_n) = 0$. Note that for $a \in D_0 = A$ and $\xi \in \mathscr{H}$

$$0 = \tilde{\pi}(d_n \delta_n)(a\delta_0 \otimes \xi) = \theta^n(\theta^{-n}(d_n)a)\delta_n \otimes \xi.$$

Therefore, the right hand side represents the zero vector in $\tilde{\mathscr{H}}$. By definition, its norm in $\tilde{\mathscr{H}}$ is computed by the horrible looking expression

$$\langle \pi(E((\theta^n(\theta^{-n}(d_n)a)\delta_n)^* * (\theta^n(\theta^{-n}(d_n)a)\,\delta_n)))\xi, \xi \rangle^{1/2}$$

which, fortunately, can be simplified to $\|\pi(\theta^{-n}(d_n)a)\xi\|$.

Since ξ can be any vector of $\mathscr{H}$, it follows that $\pi(\theta^{-n}(d_n)a) = 0$. But π was supposed faithful. Letting a run through an approximate unit for A we get $\theta^{-n}(d_n) = 0$ and hence $d_n = 0$. ∎

Representations of crossed product algebras are known to correspond to covariant representations of the corresponding C^*-dynamical system [15]. With the necessary modifications, the same is true for covariance algebras of partial automorphisms as we shall now see.

(5.3) DEFINITION. A covariant representation of the pair (A, Θ) is a triple $(\mathscr{H}, \pi, u)$ where π is a *-representation of A on the Hilbert space $\mathscr{H}$ and $u \in B(\mathscr{H})$ is a partial isometry whose initial space is $\pi(I)\mathscr{H}$ (meaning closed linear span) and whose final space is $\pi(J)\,\mathscr{H}$. In addition it is required that, for $a \in I$,

$$\pi(\theta(a)) = u\pi(a)u^*.$$

The rules of the game are often so though when dealing with algebraic properties of partial isometries that the reader may be stricken by the naiveness of the above definition. In fact that was also the author's first impression. It is really quite surprising that this definition happens to do its job in a satisfactory way.

A covariant representation gives rise to a representation of $C^*(A, \Theta)$ as we shall see. Before that, let us introduce some notation.

(5.4) DEFINITION. If n is a negative integer and if v is a partial isometry on a Hilbert space, then v^n stands for v^{-n*} (note that this will not cause any confusion since, when v is invertible, $v^* = v^{-1}$ and the two possible interpretations of v^n coincide).

For $y \in L$ (cf. (3.6)) put

$$(\pi \times u)(y) = \sum_{n \in \mathbf{Z}} \pi(y(n))u^n.$$

(5.5) PROPOSITION. *$\pi \times u$ is a representation of L on $\mathscr{H}$ and therefore, extends to a representation of $C^*(A, \Theta)$.*

Proof. The proof of the multiplicativity of $\pi \times u$ consists in justifying all steps of the following calculation, for $a_n \in D_n$ and $a_m \in D_m$

$$\begin{aligned}\pi(a_n)\, u^n \pi(a_m) u^m &= u^n \pi(\theta^{-n}(a_n) a_m) u^m \\ &= \pi(\theta^n(\theta^{-n}(a_n) a_m)) u^n u^m = \pi(\theta^n(\theta^{-n}(a_n) a_m)) u^{n+m}.\end{aligned}$$

The first two steps follow from the formula

$$u^n \pi(\theta^{-n}(a_n)) = \pi(a_n) u^n \qquad \text{for} \quad a_n \in D_n,$$

which can be proved, using induction, separately for $n > 0$ and $n < 0$. The final step is a consequence of the formula,

$$\pi(a_n)\, u^n u^m = \pi(a_n) u^{n+m} \qquad \text{for} \quad a_n \in D_n,$$

which is obviously true when n and m have the same sign, but which must be proved by induction otherwise. The proof that $\pi \times u$ is invariant under the star operation is left to the reader. ▮

(5.6) THEOREM. *Let σ be a *-representation of $C^*(A, \Theta)$ on the Hilbert space $\mathscr{H}$. Then there exists a covariant representation $(\pi, \mathscr{H}, u)$ of the pair (A, Θ) such that $\sigma = \pi \times u$.*

Proof. Let $B = C^*(A, \Theta)$ and let λ, ρ, $\lambda^\dagger$, and $\rho^\dagger$ be as in (4.11). Denote by π the restriction of σ to A, identified with $A\delta_0$. For $x_i \in I$ and $\xi_i \in \mathscr{H}$, $i = 1, ..., n$, let

$$u\left(\sum_{i=1}^{n} \pi(x_i)\xi_i\right) = \sum_{i=1}^{n} \sigma(\lambda(x_i\delta_0))\xi_i,$$

and let us verify that this extends to a well defined isometry from $\pi(I)\mathscr{H}$ to $\pi(J)\mathscr{H}$. We have

$$\begin{aligned}\left\|\sum_i \sigma(\lambda(x_i\delta_0))\xi_i\right\|^2 &= \sum_{ij} \langle \sigma(\lambda(x_j\delta_0)^* * \lambda(x_i\delta_0))\, \xi_i, \xi_j \rangle \\ &= \sum_{ij} \langle \pi(x_j^* x_i)\, \xi_i, \xi_j \rangle = \left\|\sum_i \pi(x_i)\xi_i\right\|,\end{aligned}$$

so u is well defined and isometric, but we still have to check that the image of u is $\pi(J)\mathscr{H}$. That image equals

$$\sigma(\lambda(B_1^* B_1))\mathscr{H} = \sigma(B_1 B_1^* B_1)\mathscr{H} \subseteq \pi(B_1 B_1^*)\mathscr{H} = \pi(J)\mathscr{H},$$

while

$$\begin{aligned}\pi(J)\mathscr{H} = \pi(B_1 B_1^*)\mathscr{H} &= \pi(B_1 B_1^* B_1 B_1^*)\mathscr{H} \subseteq \sigma(B_1 B_1^* B_1)\mathscr{H}\\ &= \sigma(\lambda(B_1^* B_1))\mathscr{H} = \mathrm{Im}(u).\end{aligned}$$

We next need to show that $(\pi, \mathscr{H}, u)$ satisfies the covariance equation $\pi(\theta(a)) = u\pi(a)u^*$ for $a \in I$. Given $b \in I$ and $\xi \in \mathscr{H}$ we have

$$\begin{aligned}u\pi(a)\,\pi(b)\xi &= u\pi(ab)\xi = \sigma(\lambda(ab\delta_0))\xi\\ &= \sigma((\theta(a)\delta_0) * \lambda(b\delta_0))\xi = \pi(\theta(a))u(\pi(b)\xi),\end{aligned}$$

which says that $u\pi(a) = \pi(\theta(a))u$ on $\pi(I)\mathscr{H}$. But both $u\pi(a)$ and $\pi(\theta(a))u$ vanish on the orthogonal complement of $\pi(I)\mathscr{H}$ hence the equality above holds on the whole of $\mathscr{H}$. So

$$u\pi(a)u^* = \pi(\theta(a))uu^* = \pi(\theta(a)).$$

Recall from (4.6) that $\lambda((a_1\delta_1)^*) = a_1^*\delta_0$ for $a_1 \in D_1$. Given $b_{-1} \in D_{-1} = I$, of the form $b_{-1} = \theta^{-1}(c_1^* d_1)$ with c_1, d_1 in D_1 note that

$$(c_1\delta_1)^* * (d_1\delta_1) = \theta^{-1}(c_1^* d_1)\delta_0 = b_{-1}\delta_0.$$

So

$$\lambda(b_{-1}\delta_0) = \lambda((c_1\delta_1)^*) * (d_1\delta_1) = c_1^* d_1\delta_1 = \theta(b_{-1})\delta_1.$$

The above form of b_{-1} is sufficiently general to ensure that

$$\lambda(b_{-1}\delta_0) = \theta(b_{-1})\delta_1$$

for any $b_{-1} \in D_{-1}$. This equality is the basis of the proof that $\pi \times u = \sigma$. In fact, since the dual action is semi-saturated as proven in (4.7), it is enough to show that $\pi \times u$ and σ coincide on $B_0 \cup B_1$. This is obvious for B_0 so it remains to prove that

$$\sigma(b_1\delta_1) = \pi(b_1)u \qquad \text{for} \quad b_1 \in D_1.$$

Both sides of the equal sign above represent operators which vanish on the orthogonal complement of $\pi(I)\mathscr{H}$ so, in order to prove their equality, we may restrict ourselves to $\pi(I)\mathscr{H}$. Given $a_{-1} \in I$ and $\xi \in \mathscr{H}$ we have for any $b_1 \in D_1$

$$\begin{aligned}\pi(b_1)u\pi(a_{-1})\xi &= \sigma(b_1\delta_0)\,\sigma(\lambda(a_{-1}\delta_0))\xi = \sigma((b_1\delta_0) * (\theta(a_{-1})\delta_1))\xi \\ &= \sigma(\theta(\theta^{-1}(b_1)a_{-1})\delta_1)\xi = \sigma((b_1\delta_1) * (a_{-1}\delta_0))\xi \\ &= \sigma(b_1\delta_1)\pi(a_{-1})\xi\end{aligned}$$

which concludes our proof. ∎

6. The Toeplitz Algebra of a Partial Automorphism

Let A be a C^*-algebra and $\Theta = (\theta, I, J)$ be a partial automorphism of A. We denote by B, the covariance algebra $C^*(A, \Theta)$ and regard A as a subalgebra of B in the obvious way.

Fixing a faithful representation of B allows us to assume that $B \subseteq B(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. Using Theorem (5.6) we conclude that there is a partial isometry u in $B(\mathscr{H})$ such that, for $x \in I$ and $y \in J$, we have

(i) $uxu^* = \theta(x)$

(ii) $xu^*u = x$

(iii) $yuu^* = y$.

Moreover, the set of finite sums of the form $\sum_{n \in \mathbf{Z}} a_n u^n$ with $a_n \in D_n = \mathrm{Dom}(\theta^{-n})$ is dense in B (see (5.4)). The dual action of S^1 on B is given, on the above mentioned dense set, by

$$\alpha_z\left(\sum a_n u^n\right) = \sum z^n a_n u^n \qquad \text{for} \quad z \in S^1.$$

So the nth spectral subspace for α is given by $B_n = D_n u^n$ (see (3.11)).

In the following, $\mathbf{N}^*$ denotes the set of strictly positive integers, S denotes the unilateral shift on $l_2(\mathbf{N}^*)$, while $P = 1 - SS^*$ and $Q = SS^*$. Also let e_{ij} denote the standard matrix units in $B(l_2(\mathbf{N}^*))$ for $i, j \geqslant 1$. Note that $e_{ij} = S^{i-1}S^{*j-1} - S^iS^{*j}$.

(6.1) Definition. The Toeplitz algebra for the pair (A, Θ) is the sub-C^*-algebra $\mathscr{E} = \mathscr{E}(A, \Theta)$, of operators on $\mathscr{H} \otimes l_2(\mathbf{N}^*)$, generated by the set

$$\{b_n \otimes S^n : n \in \mathbf{Z},\, b_n \in B_n\}.$$

Since the dual action α is semi-saturated, we have by (4.7) and (4.8), that $B_n = (B_1)^n$ for $n \geqslant 1$. Our next result is an immediate consequence of this.

(6.2) PROPOSITION. *$\mathscr{E}$ is generated by $(B_0 \otimes 1) \cup (B_1 \otimes S)$.*

Proof. It is enough to note that, for $n \geqslant 1$,

$$(B_n \otimes S^n) = ((B_1)^n \otimes S^n) = (B_1 \otimes S)^n. \quad \blacksquare$$

(6.3) DEFINITION. Let Λ be the subset of $B(\mathscr{H} \otimes l_2(\mathbf{N}^*))$ given by

$$\Lambda = \bigoplus_{i, j \geqslant 1} (B_i B_j^* \otimes e_{ij}).$$

From now on we will use the symbol $\oplus$ to denote the *closure* of the sum of any family of independent subspaces of a Banach space. Also, following (2.2), we regard $B_i B_j^*$ as the closed linear span of the set of products xy where $x \in B_i$ and $y \in B_j^*$.

Because, for any j, $B_j^* B_j \subseteq B_0$ and also because $B_i B_0 \subset B_i$, we have that Λ is a sub-C^*-algebra of $B(\mathscr{H} \otimes l_2(\mathbf{N}^*))$.

(6.4) PROPOSITION. *Λ is an ideal in $\mathscr{E}$.*

Proof. Since $B_n = (B_1)^n$, for $n \geqslant 1$, the set of elements of the form $b_{n-1} b_1 b_m^*$, the indices indicating the spectral subspace of B each factor lies in, is total (their linear span is dense) in $B_n B_m^*$. So the identity

$$\begin{aligned} b_{n-1} b_1 b_m^* \otimes e_{nm} = {} & (b_{n-1} \otimes S^{n-1})(b_m b_1^* \otimes S^{m-1})^* \\ & - (b_{n-1} b_1 \otimes S^n)(b_m \otimes S^m)^* \end{aligned}$$

implies that Λ is contained in $\mathscr{E}$. The verification that Λ is actually an ideal is now straightforward. $\blacksquare$

As in [17], one should now show that the quotient $\mathscr{E}/\Lambda$ is isomorphic to B, providing the short exact sequence

$$0 \to \Lambda \to \mathscr{E} \to B \to 0.$$

Even though we could attempt to prove this right now, we will deduce it from a much stronger result (Lemma (6.7) below), which we will need for various other reasons.

For each $z \in S^1$ let Δ_z be the infinite matrix

$$\Delta_z = \begin{pmatrix} 1 & & & \\ & z & & \\ & & z^2 & \\ & & & \ddots \end{pmatrix}$$

regarded as a unitary operator on $\mathscr{H} \otimes l_2(\mathbf{N}^*)$. Note that $\Delta_z(b_n \otimes S^n)\Delta_z^{-1} = z^n(b_n \otimes S^n)$ so that $\Delta_z \mathscr{E} \Delta_z^{-1} = \mathscr{E}$ and so we can define an action γ of S^1 on $\mathscr{E}$ by

$$\gamma_z(x) = \Delta_z x \Delta_z^{-1} \qquad \text{for} \quad x \in \mathscr{E}, \quad z \in S^1.$$

Our next goal is to show that γ is a semi-saturated regular action, which will enable us to describe $\mathscr{E}$ as the covariance algebra for a certain partial automorphism, by Theorem (4.21).

(6.5) PROPOSITION. *Let $\mathscr{E}_n$ denote the nth spectral subspace relative to γ. Then*

(i) $\mathscr{E}_0 = A \otimes 1 + \bigoplus_{n=1}^{\infty} (D_n \otimes e_{nn})$

(ii) $\mathscr{E}_1 = B_1 \otimes S + \bigoplus_{n=1}^{\infty} (B_{n+1} B_n^* \otimes e_{n+1,n})$

(iii) $\mathscr{E}_1^* \mathscr{E}_1 = D_{-1} \otimes 1 + \bigoplus_{n=1}^{\infty} ((D_{-1} \cap D_n) \otimes e_{nn})$

(iv) $\mathscr{E}_1 \mathscr{E}_1^* = D_1 \otimes Q + \bigoplus_{n=2}^{\infty} (D_n \otimes e_{nn})$.

Proof. Let Y_0 and Y_1 denote the right hand side of (i) and (ii), respectively. Define $Y_n = (Y_1)^n$ for $n > 1$, and $Y_n = (Y_1^*)^{-n}$ for $n < 0$. It can be shown without much difficulty, that $Y = \bigoplus_{n \in \mathbf{Z}} Y_n$ is a sub-C^*-algebra of $\mathscr{E}$.

Should the reader decide to verify this by himself, we suggest he start by proving that $Y_1 Y_0$ and $Y_0 Y_1$ are contained in Y_1 and that $Y_1^* Y_1$ and $Y_1 Y_1^*$ are contained in Y_0.

We next note that $B_0 \otimes 1 \subseteq Y_0$ and that $B_1 \otimes S \subseteq Y_1$ so, by (6.2) we get $Y = \mathscr{E}$. Since it is obvious that $Y_n \subseteq \mathscr{E}_n$ we must have $Y_n = \mathscr{E}_n$ for all n and in particular for $n = 0$ and $n = 1$, proving (i) and (ii). The two remaining statements follow easily from (ii). ∎

(6.6) PROPOSITION. *γ is a semi-saturated regular action of S^1 on $\mathscr{E}$. The roles of the maps θ and λ mentioned in* (4.4) *are played, respectively, by $\theta_{\mathscr{E}}$ and $\lambda_{\mathscr{E}}$ defined by*

$$\theta_{\mathscr{E}} : x \in \mathscr{E}_1^* \mathscr{E}_1 \mapsto (u \otimes S)\, x (u \otimes S)^* \in \mathscr{E}_1 \mathscr{E}_1^*$$

$$\lambda_{\mathscr{E}} : x^* \in \mathscr{E}_1^* \mapsto (u \otimes S) x^* \in \mathscr{E}_1 \mathscr{E}_1^*.$$

Therefore $\mathscr{E}$ is isomorphic to the covariance algebra $C^*(\mathscr{E}_0, (\theta_{\mathscr{E}}, \mathscr{E}_1^* \mathscr{E}_1, \mathscr{E}_1 \mathscr{E}_1^*))$.

Proof. The proof of regularity consists in verifying that the above, in fact, describes well defined maps between the indicated sets and that they satisfy (4.4). These are somewhat routine tasks, which we leave for the reader. Finally, since it is obvious that $B_0 \otimes 1 \subseteq \mathscr{E}_0$ and that $B_1 \otimes S \subseteq \mathscr{E}_1$, γ is seen to be semi-saturated by (6.2). ∎

Our next lemma is an important tool in obtaining representations of the Toeplitz algebra (compare Lemma 7.1 in [1]).

(6.7) LEMMA. *Let $\mathcal{H}$ be a Hilbert space and π be a representation of A on $\mathcal{H}$. Suppose that V is a partial isometry operating on $\mathcal{H}$ such that for all $x \in I$*

(i) $V\pi(x) = \pi(\theta(x))V$

(ii) $V^*V\pi(x) = \pi(x)$.

Then there exists a representation $\pi_{\mathcal{E}}$ of $\mathcal{E}$ on $\mathcal{H}$ such that

$$\pi_{\mathcal{E}}(a_n u^n \otimes S^n) = \pi(a_n)V^n \qquad \textit{for} \quad n \in \mathbf{Z}, \quad a_n \in D_n.$$

If, in addition, for all $y \in J$

(iii) $VV^*\pi(y) = \pi(y)$

then there exists a representation π_B of B on $\mathcal{H}$ such that

$$\pi_B(a_n u^n) = \pi(a_n)V^n \qquad \textit{for} \quad n \in \mathbf{Z}, \quad a_n \in D_n.$$

Proof. The second assertion, by far the easier, can be proved exactly as in the proof of (5.5). Now assume that V satisfies (i) and (ii). Our strategy will be to define a representation π_0 of $\mathcal{E}_0$ on $\mathcal{H}$, such that the pair (π_0, V) satisfies (i), (ii), and (iii) with respect to the partial automorphism $(\theta_{\mathcal{E}}, \mathcal{E}_1^* \mathcal{E}_1, \mathcal{E}_1 \mathcal{E}_1^*)$ of $\mathcal{E}_0$. Once that is done, we may apply the part of the present lemma which we have already taken care of. For $n \in \mathbf{N}^*$ define

$$\pi_n : a_n \in D_n \mapsto V^{n-1}\pi(\theta^{-(n-1)}(a_n))\, V^{n-1*} - V^n \pi(\theta^{-n}(a_n))\, V^{n*} \in B(\mathcal{H}).$$

The reader may now verify that π_n is a representation of D_n on $\mathcal{H}$ and that

$$\pi_n(a_n)\pi_m(a_m) = 0$$

for $n \neq m$, $a_n \in D_n$, and $a_m \in D_m$. In doing so, the following identities are helpful

$$V^{n*}V^n\pi(a_{-n}) = \pi(a_{-n})\, V^{n*}V^n = \pi(a_{-n}) \qquad \text{for} \quad n \geqslant 1, \quad a_{-n} \in D_{-n}$$

$$\pi(a_n)\, V^n V^{n*} V^n = \pi(a_n)\, V^n \qquad \text{for} \quad n \geqslant 1, \quad a_n \in D_n.$$

Next, we may define a representation $\tilde{\pi}$ of $\mathcal{E}_0$ by

$$\tilde{\pi}\left(a \otimes 1 + \sum_{n=1}^{N} a_n \otimes e_{nn}\right) = \pi(a) + \sum_{n=1}^{N} \pi_n(a_n).$$

We claim that $(\tilde{\pi}, V)$ satisfy (i), (ii), and (iii) with respect to $(\theta_{\mathscr{E}}, \mathscr{E}_1^*\mathscr{E}_1, \mathscr{E}_1\mathscr{E}_1^*)$. In fact let $x = a_{-1}\otimes 1 + \sum_{n=1}^N h_n \otimes e_{nn} \in \mathscr{E}_1^*\mathscr{E}_1$ where $a_{-1}\in D_{-1}$ and $h_n \in D_{-1}\cap D_n$. We have

$$\begin{aligned}
\tilde{\pi}(\theta_{\mathscr{E}}(x))V &= \tilde{\pi}\left((u\otimes S)\left(a_{-1}\otimes 1 + \sum_{n=1}^N h_n\otimes e_{nn}\right)(u\otimes S)^*\right)V\\
&= \tilde{\pi}\left(\theta(a_{-1})\otimes Q + \sum_{n=1}^N \theta(h_n)\otimes e_{n+1,n+1}\right)V\\
&= \tilde{\pi}\left(\theta(a_{-1})\otimes 1 - \theta(a_{-1})\otimes e_{11} + \sum_{n=1}^N \theta(h_n)\otimes e_{n+1,n+1}\right)V\\
&= \Bigg(\pi(\theta(a_{-1})) - \pi(\theta(a_{-1})) + V\pi(a_{-1})V^*\\
&\quad + \sum_{n=1}^N V^n\pi(\theta^{-n+1}(h_n))V^{n*} - V^{n+1}\pi(\theta^{-n}(h_n))V^{n+1*}\Bigg)V\\
&= V\pi(a_{-1}) + \left(\sum_{n=1}^N V^nV^{n-1*}\pi(h_n)V^* - V^{n+1}V^{n*}\pi(h_n)V^*\right)V\\
&= V\pi(a_{-1}) + \sum_{n=1}^N V^nV^{n-1*}\pi(h_n) - V^{n+1}V^{n*}\pi(h_n)\\
&= V\pi(a_{-1}) + V\sum_{n=1}^N V^{n-1}\pi(\theta^{-(n-1)}(h_n))V^{n-1*}\\
&\quad - V^n\pi(\theta^{-n}(h_n))V^{n*}\\
&= V\tilde{\pi}\left(a_{-1}\otimes 1 + \sum_{n=1}^N h_n\otimes e_{nn}\right) = V\tilde{\pi}(x),
\end{aligned}$$

showing (i). To check (iii) let $y = a_1\otimes Q + \sum_{n=2}^N a_n\otimes e_{nn}$ be in $\mathscr{E}_1\mathscr{E}_1^*$. Using that $VV^*V = V$ we have

$$\begin{aligned}
VV^*\tilde{\pi}(y) &= VV^*\tilde{\pi}\left(a_1\otimes 1 - a_1\otimes e_{11} + \sum_{n=2}^N a_n\otimes e_{nn}\right)\\
&= VV^*\Bigg(V\pi(\theta^{-1}(a_1))V^* + \sum_{n=2}^N V^{n-1}\pi(\theta^{-(n-1)}(a_n))V^{n*}\\
&\quad - V^n\pi(\theta^{-n}(a_n))V^{n*}\Bigg) = \pi(y).
\end{aligned}$$

The verification of (ii) is left to the reader. This implies, by our previous work, that there exists a representation $\pi_{\mathscr{E}}$ of $\mathscr{E}$ on $\mathscr{H}$ such that

$$\pi_{\mathscr{E}}(y_n w^n) = \tilde{\pi}(y_n)V^n$$

for $y_n \in \mathscr{E}_n \mathscr{E}_n^*$, where w is the partial isometry implementing the partial automorphism $\theta_{\mathscr{E}}$. But, as it can be easily verified, $w = u \otimes S$. Therefore, if $a_1 \in D_1$ we have that $a_1 \otimes Q \in \mathscr{E}_1 \mathscr{E}_1^*$ and so

$$\pi_{\mathscr{E}}(a_1 u \otimes S) = \pi_{\mathscr{E}}((a_1 \otimes Q)(u \otimes S)) = \tilde{\pi}(a_1 \otimes Q) V$$
$$= \tilde{\pi}(a_1 \otimes 1 - a_1 \otimes e_{11}) V = V\pi(\theta^{-1}(a_1)) V^* V = \pi(a_1) V.$$

To conclude the proof one has, for $a_n, b_n \in D_n$ and $n > 1$,

$$a_n b_n u^n \otimes S^n = (a_n u^{n-1} \otimes S^{n-1})(\theta^{-(n-1)}(b_n) u \otimes S)$$

so, by induction

$$\pi_{\mathscr{E}}(a_n b_n u^n \otimes S^n) = \pi(a_n)\, V^{n-1} \pi(\theta^{-(n-1)}(b_n)) V = \pi(a_n b_n) V^n. \quad \blacksquare$$

(6.8) PROPOSITION. *There exists a short exact sequence*

$$0 \longrightarrow A \xrightarrow{i} \mathscr{E} \xrightarrow{\phi} B \longrightarrow 0,$$

where i is the inclusion and ϕ is given by

$$\phi(a_n u^n \otimes S^n) = a_n u^n \qquad \textit{for} \quad n \in \mathbf{Z}, \quad a_n \in D_n.$$

Proof. By (6.7) there indeed exists $\phi: \mathscr{E} \to B$ satisfying $\phi(a_n u^n \otimes S^n) = a_n u^n$. The identity used in our proof of (6.4) shows that ϕ vanishes on A so it defines a homomorphism $\tilde{\phi}: \mathscr{E}/A \to B$ which is obviously surjective. The action γ drops to $\mathscr{E}/A$, since A is an invariant ideal, and then $\tilde{\phi}$ becomes a covariant homomorphism (with the dual action on B).

For covariant homomorphisms under actions of S^1, injectivity is equivalent to injectivity on the fixed point subalgebra by (2.9). The latter being true for $\tilde{\phi}$, we conclude that $\tilde{\phi}$ is an isomorphism and the proof is thus completed. $\blacksquare$

7. THE GENERALIZED PIMSNER–VOICULESCU EXACT SEQUENCE

In this section we obtain a generalization of the Pimsner–Voiculescu exact sequence [17] to our context of crossed products by partial automorphisms.

An important tool in the sequel will be the fact that the inclusion of a full corner of a C^*-algebra induces an isomorphism on K-theory, in the presence of strictly positive elements ([2], see also [14, Proposition 2.1]). In fact a much stronger result holds: If B is a full corner in the C^*-algebra A and if A has a strictly positive element, then the inclusion $\iota: B \to A$ induces an invertible element in $KK(B, A)$. This is a trivial consequence of [2, Lemma 2.5]. In fact, if v is an isometry in $M(A \otimes K)$ such that $vv^* = p \otimes 1$, where p is the projection in $M(A \otimes K)$ for which $pAp = B$, then let $\phi: A \otimes K \to B \otimes K$ be the isomorphism given by $\phi(x) = vxv^*$. The composition $\phi(\imath \otimes \mathrm{id}_K)$, once composed with the inclusion of $B \otimes K$ into $M_2(B \otimes K)$, can be easily shown to be homotopic to the latter map. Likewise, $(\imath \otimes \mathrm{id}_K)\phi$ when composed with inclusion of $A \otimes K$ into $M_2(A \otimes K)$ is also homotopic to the latter map. In a word, ϕ provides an inverse for $\imath$ in $KK(B, A)$.

The model of KK-theory we adopt is the one introduced by Cuntz in [7]. See also the exposition in Chapter (5) of [10] which is where we borrow our notation from.

As we already mentioned, we make intensive use of the result above and hence we need to ensure that several algebras in our construction, including A, have strictly positive elements. It is not hard to see that, in the case of A, this implies that D_n must have a strictly positive element for each n. However, we cannot think of a reasonably graceful set of hypotheses which could provide for this much, other than, of course, assuming our algebras to be separable (see 1.4.3 and 3.10.5 in [15]).

As in the previous section, let A be a C^*-algebra with a fixed partial automorphism $\Theta = (\theta, I, J)$ and let $B = C^*(A, \Theta)$. In most of our results below, A will, therefore, be assumed separable.

In the following, all occurrences of i refer to the inclusion homomorphism that should be clear from the context. Also we denote by θ_*^{-1} the map induced at the level of K groups by the composition

$$J \xrightarrow{\theta^{-1}} I \xrightarrow{i} A.$$

The following is our second main result.

(7.1) THEOREM. *Let $\Theta = (\theta, I, J)$ be a partial automorphism of the separable C^*-algebra A. Then there exists an exact sequence of K-groups*

$$\begin{array}{ccccc} K_0(J) & \xrightarrow{i_* - \theta_*^{-1}} & K_0(A) & \xrightarrow{i_*} & K_0(C^*(A, \Theta)) \\ \uparrow & & & & \downarrow \\ K_1(C^*(A, \Theta)) & \xleftarrow{i_*} & K_1(A) & \xleftarrow{i_* - \theta_*^{-1}} & K_1(J) \end{array}$$

Our proof will be, roughly speaking, based on [17] in the sense that we derive our result from the usual exact sequence of K-Theory for the extension

$$0 \to A \xrightarrow{i} \mathscr{E} \xrightarrow{\phi} C^*(A, \Theta) \to 0.$$

The crucial part in accomplishing this program is proving that $K_*(A)$ and $K_*(\mathscr{E})$ are isomorphic, at which point we can no longer follow the original method of Pimsner and Voiculescu but, instead, we must resort to techniques from *KK*-theory (see Proposition 5.5 in [6] and Theorem 7.2 in [1]) which will, in fact, lead us to the stronger result that A and $\mathscr{E}$ are *KK*-equivalent to each other.

In the following we let

$$d: A \to \mathscr{E}$$

and

$$j: J \to A$$

be defined by $d(a) = a \otimes 1$ for a in A and $j(x) = x \otimes e_{11}$ for $x \in J$.

(7.2) PROPOSITION. *The diagram*

$$\begin{array}{ccc} K_*(A) & \xrightarrow{i_*} & K_*(\mathscr{E}) \\ {\scriptstyle j_*}\uparrow & & \uparrow{\scriptstyle d_*} \\ K_*(J) & \xrightarrow{i_* - \theta_*^{-1}} & K_*(A) \end{array}$$

is commutative. Moreover, if A is separable then j induces a KK-equivalence.

Proof. Let $\tilde{J}$ denote J with an added unit and let $y = 1 + ab$ be an invertible element in $\tilde{J}$ with a and b in J. The K_1-class of y being denoted by $[y]_1$, we have

$$d_*\theta_*^{-1}[y]_1 = [1 + u^*abu \otimes 1]_1 = [1 + (u^*a \otimes S^*)(bu \otimes S)]_1.$$

Recall from Lemma (1.1) in [14] that the K_1 classes of $1 + rs$ and $1 + sr$ coincide, whenever $1 + rs$ is invertible. So we have

$$\begin{aligned} d_*\theta_*^{-1}[y]_1 &= [1 + ba \otimes Q]_1 = [1 + (b \otimes 1)(a \otimes Q)]_1 \\ &= [1 + (a \otimes Q)(b \otimes 1)]_1 = [1 + ab \otimes Q]_1. \end{aligned}$$

Following our diagram in the counterclockwise direction from $K_1(J)$ we then obtain

$$\begin{aligned} d_*(i_* - \theta_*^{-1})[y]_1 &= [1 + ab \otimes 1]_1 - [1 + ab \otimes Q]_1 \\ &= [1 + ab \otimes e_{11}]_1 = i_* j_*[y]_1. \end{aligned}$$

Clearly, any invertible element in $\tilde{J}$ is homotopic to an invertible element of the form $1 + ab$ as above. Moreover, by tensoring everything with $M_n(\mathbf{C})$ we get the above equality for any invertible $y \in M_n(\tilde{J})$. The commutativity of our diagram is therefore proved for K_1. The proof for K_0 follows by taking suspensions.

To conclude, we need to show that j induces a KK-equivalence. Note that j is an isomorphism from J onto $J \otimes e_{11} = B_1 B_1^* \otimes e_{11}$. The latter is obviously a corner of Λ and because, for $i, j > 1$

$$(B_i B_1^* \otimes e_{i1})(B_1 B_1^* \otimes e_{11})(B_1 B_j^* \otimes e_{1j})$$
$$= B_i B_1^* B_1 B_1^* B_1 B_j^* \otimes e_{ij} = B_i B_j^* \otimes e_{ij}$$

one sees that $J \otimes e_{11}$ is not contained in any proper ideal of Λ. In two words, $J \otimes e_{11}$ is a full corner of Λ. If we now invoke the separability of A, and hence also of Λ, the result will follow from our comments in the introduction to the present section. ∎

It follows that j_* is an isomorphism between the indicated K-groups and so all we need, in order to prove (7.1), is to show that d_* is an isomorphism as well. Then we just have to write down the K-theory exact sequence for

$$0 \to \Lambda \xrightarrow{i} \mathscr{E} \xrightarrow{\phi} B \to 0,$$

and to replace the two segments $K_*(\Lambda) \xrightarrow{i_*} K_*(\mathscr{E})$ there, by $K_*(J) \xrightarrow{i_* - \theta_*^{-1}} K_*(A)$.

(7.3) Definition. Let Λ_0 be the subalgebra of $B(\mathscr{H} \otimes l_2(\mathbf{N}))$ given by

$$\Lambda_0 = \bigoplus_{i, j \geqslant 0} B_i B_j^* \otimes e_{ij}$$

(note that, as opposed to (6.3), the indices i and j here, start at zero). Also let

$$j_0 : A \to \Lambda_0$$

be given by $j_0(a) = a \otimes e_{00}$.

(7.4) Proposition. *If A is separable then j_0 is a KK-equivalence.*

Proof. As in (7.2) it is clear that the image of j, namely $A \otimes e_{00} = B_0 B_0^* \otimes e_{00}$, is a corner of Λ_0. To show that it is also full note that for $i, j \geqslant 0$

$$(B_i B_0^* \otimes e_{i0})(B_0 B_0^* \otimes e_{00})(B_0 B_j^* \otimes e_{0j}) = B_i B_j^* \otimes e_{ij}.$$

The result now follows as in (7.2). ∎

Aiming at a proof that d induces an invertible element in $KK(A, \mathscr{E})$, we now introduce an element of $KK(\mathscr{E}, A_0)$ which will be shown to provide the required inverse, given the KK-equivalence between A and A_0.

In order to avoid confusion, we let T be the unilateral shift on $l_2(\mathbf{N})$ (as opposed to S, which operates on $l_2(\mathbf{N}^*)$). Define

$$\phi, \bar{\phi}: \mathscr{E} \to B(\mathscr{H} \otimes l_2(\mathbf{N}))$$

by

$$\phi(b_n \otimes S^n) = b_n \otimes T^n$$

and

$$\bar{\phi}(b_n \otimes S^n) = b_n \otimes T^n T T^*,$$

for $n \in \mathbf{Z}$ and $b_n \in B_n$.

(7.5) PROPOSITION. *The images of ϕ and $\bar{\phi}$ are contained in the multiplier algebra $M(A_0)$. Moreover, for every $x \in \mathscr{E}$ we have $\phi(x) - \bar{\phi}(x) \in A_0$.*

Proof. We have, for $i, j \geqslant 0$, $n \in \mathbf{Z}$, and $b_n \in B_n$

$$(b_n \otimes T^n)(B_i B_j^* \otimes e_{ij}) = [n + i \geqslant 0]\, b_n B_i B_j^* \otimes e_{n+i,j},$$

where $[n + i \geqslant 0]$ indicates the obvious boolean function assuming the real values 1 and 0. Also

$$(B_i B_j^* \otimes e_{ij})(b_n \otimes T^n) = [j - n \geqslant 0]\, B_i B_j^* b_n \otimes e_{i,j-n}$$

showing that $b_n \otimes T^n \in M(A_0)$. Similarly the image of $\bar{\phi}$ is also seen to be contained in $M(A_0)$.

Note that for $n \geqslant 0$, $b_n \in B_n$, and $x = b_n \otimes S^n$

$$\phi(x) - \bar{\phi}(x) = b_n \otimes e_{n,0}$$

which belongs to A_0. Since the set of x's as above, generates $\mathscr{E}$, it follows that $\phi(x) - \bar{\phi}(x)$ is in A_0 for all $x \in \mathscr{E}$. ∎

At this point we start our main KK-theory computations. We refer the reader to Chapter (5) in [10] for the main facts about the KK-product as well as for notation.

According to (5.1.1) in [10], $q(\phi, \bar{\phi})$ defines a homomorphism from $q\mathscr{E}$ to A_0 and therefore its homotopy class gives an element, denoted $[q(\phi, \bar{\phi})]$, in $KK(\mathscr{E}, A_0)$.

(7.6) PROPOSITION. *The product* $[q(d, 0)] \cdot [q(\phi, \bar{\phi})]$, *in* $KK(A, A_0)$, *equals* $[q(j_0, 0)]$.

Proof. Still using the notation from [10] and thus denoting by $1_{\mathscr{E}}$ the unit in the ring $KK(\mathscr{E}, \mathscr{E})$, we have $d^*(1_{\mathscr{E}}) = [q(d, 0)]$. So, by Theorem (5.1.15) in [10]

$$\begin{aligned}
[q(d, 0)] \cdot [q(\phi, \bar{\phi})] &= d^*(1_{\mathscr{E}}) \cdot [q(\phi, \bar{\phi})] \\
&= d^*(1_{\mathscr{E}} \cdot [q(\phi, \bar{\phi})]) = d^*([q(\phi, \bar{\phi})]) \\
&= [q(\phi, \bar{\phi})\, q(d)] = [q(\phi d, \bar{\phi} d)] \\
&= [q(\bar{\phi} d + j_0, \bar{\phi} d)] = [q(j_0, 0)]. \quad \blacksquare
\end{aligned}$$

Since $[q(j_0, 0)]$ is invertible in $KK(A, A_0)$ by (7.4), if we assume A to be separable, we find that $[q(d, 0)]$ is right invertible and we now concentrate our efforts in proving that it is also left invertible.

(7.7) DEFINITION. Let Ω_0 be the subset of $B(\mathscr{H} \otimes l_2(\mathbf{N}^*) \otimes l_2(N))$ given by

$$\Omega_0 = \bigoplus_{i, j \geq 0} ((B_i \otimes 1)\, \mathscr{E}(B_j^* \otimes 1)) \otimes e_{ij}.$$

Similarly to what we said with respect to both A and A_0, we have that Ω_0 is a subalgebra of operators on $\mathscr{H} \otimes l_2(\mathbf{N}^*) \otimes l_2(\mathbf{N})$ while the map

$$k_0 : \mathscr{E} \to \Omega_0$$

defined by $k_0(x) = x \otimes e_{00}$, induces the invertible element $[q(k_0, 0)]$ in $KK(\mathscr{E}, \Omega_0)$, when A is separable.

Let d' be the homomorphism

$$d' : A_0 \to \Omega_0$$

defined by

$$d'(b_i b_j^* \otimes e_{ij}) = b_i b_j^* \otimes 1 \otimes e_{ij} \qquad \text{for} \quad i, j \geq 0, \quad b_i b_j^* \in B_i B_j^*.$$

Note that the image of d' in fact lies in Ω_0 since, by (2.7)

$$\begin{aligned}
B_i B_j^* \otimes 1 = B_i B_i^* B_i B_j^* B_j B_j^* \otimes 1 &= (B_i \otimes 1)(B_i^* B_i B_j^* B_j \otimes 1)(B_j^* \otimes 1) \\
&\subseteq (B_i \otimes 1)(B_0 \otimes 1)(B_j^* \otimes 1) \subseteq (B_i \otimes 1)\, \mathscr{E}(B_j^* \otimes 1).
\end{aligned}$$

We then have the commutative diagram

$$\begin{array}{ccc} \Lambda_0 & \xrightarrow{d'} & \Omega_0 \\ {\scriptstyle j_0}\uparrow & & \uparrow{\scriptstyle k_0} \\ A & \xrightarrow{d} & \mathscr{E} \end{array}$$

in which the vertical arrows are invertible in the corresponding KK-groups in the separable case. It is then obvious that the existence of a left inverse to d can be verified by exhibiting a left inverse for d'.

(7.8) Proposition. *The product* $[q(\phi, \bar{\phi})] \cdot [q(d', 0)]$, *in* $KK(\mathscr{E}, \Omega_0)$, *equals* $[q(k_0, 0)]$.

Proof. Using (5.1.15) in [10], once more, as well as the fact that $d'_*(1_{\Lambda_0}) = [q(d', 0)]$, we have

$$[q(\phi, \bar{\phi})] \cdot [q(d', 0)] = [q(\phi, \bar{\phi})] \cdot d'_*(1_{\Lambda_0}) = d'_*([q(\phi, \bar{\phi})] \cdot 1_{\Lambda_0})$$
$$= d'_*([q(\phi, \bar{\phi})]) = [q(\phi, \bar{\phi}) d'].$$

Note that, in the last term above, we have written d' where the definition (see [10]) of d'_* calls for $\mathrm{id}_K \otimes d'$. We are justified in doing so because $q(\phi, \bar{\phi})$ takes values in Λ_0 which should be thought of as a subalgebra of $K \otimes \Lambda_0$, as is customary in K-theory.

Observing that d' extends to a homomorphism from $B(\mathscr{H} \otimes l_2(\mathbf{N}))$ to $B(\mathscr{H} \otimes l_2(\mathbf{N}^*) \otimes l_2(\mathbf{N}))$ in an obvious way, one sees that $q(\phi, \bar{\phi}) d' = q(\psi, \bar{\psi})$ where ψ and $\bar{\psi}$ are the maps from $\mathscr{E}$ to the multiplier algebra $M(\Omega_0)$ given by

$$\psi(b_n \otimes S^n) = b_n \otimes 1 \otimes T^n$$

and

$$\bar{\psi}(b_n \otimes S^n) = b_n \otimes 1 \otimes T^n T T^*$$

for $n \in \mathbf{Z}$ and $b_n \in B_n$. Recalling that $Q = SS^*$ and $P = 1 - SS^*$, consider the operator W on $\mathscr{H} \otimes l_2(\mathbf{N}^*) \otimes l_2(\mathbf{N})$ given by

$$W = 1 \otimes P \otimes e_{00} + 1 \otimes S \otimes e_{01} + 1 \otimes S^* \otimes e_{10}$$
$$+ 1 \otimes 1 \otimes (1 - e_{00} - e_{11}).$$

Its matrix with respect to the canonic basis of $l_2(\mathbf{N})$ is given by

$$W=\begin{pmatrix} 1\otimes P & 1\otimes S & 0 & & \cdots \\ 1\otimes S^* & 0 & 0 & & \cdots \\ 0 & 0 & 1 & & \cdots \\ & & & 1 & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}.$$

Clearly, W is a self-adjoint unitary which, therefore can be connected to the identity operator through the path of unitaries

$$W_t=(1+W)/2+e^{\pi i t}(1-W)/2 \qquad \text{for} \quad 0\leqslant t\leqslant 1.$$

If we let $\lambda=\lambda(t)=(1-e^{\pi i t})/2$ and $\mu=\mu(t)=(1+e^{\pi i t})/2$ we can write W_t as

$$W_t=\begin{pmatrix} 1\otimes P+\mu\otimes(1-P) & \lambda\otimes S & 0 & & \cdots \\ \lambda\otimes S^* & \mu\otimes 1 & 0 & & \cdots \\ 0 & 0 & 1 & & \cdots \\ & & & 1 & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}.$$

Let $V_t=W_t(u\otimes 1\otimes T)$ or, in matrix form,

$$V_t=\begin{pmatrix} \lambda u\otimes S & 0 & & \cdots \\ \mu u\otimes 1 & 0 & & \cdots \\ 0 & u\otimes 1 & 0 & \cdots \\ & 0 & u\otimes 1 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Observe that, because W_t commutes with $A\otimes 1\otimes 1$, we have for all $x\in I$

(i) $V_t(x\otimes 1\otimes 1)=(\theta(x)\otimes 1\otimes 1)V_t$

(ii) $V_t^*V_t(x\otimes 1\otimes 1)=x\otimes 1\otimes 1.$

These are precisely the assumptions in (6.7) so there exists a representation ψ_t of $\mathscr{E}$ on $\mathscr{H}\otimes l_2(\mathbf{N}^*)\otimes l_2(\mathbf{N})$ such that for $n\in\mathbf{Z}$ and $a_n\in D_n$

$$\psi_t(a_n u^n\otimes S^n)=(a_n\otimes 1\otimes 1)V_t^n.$$

Since $V_0=u\otimes 1\otimes T$, it is clear that $\psi_0=\psi$. On the other hand it is easy to see that

$$\psi_1(a_n u^n\otimes S^n)=a_n u^n\otimes S^n\otimes e_{00}+a_n u^n\otimes 1\otimes T^n TT^* \qquad \text{for} \quad a_n\in D_n$$

which gives $\psi_1(x)=k_0(x)+\bar{\psi}(x)$ for all $x\in\mathscr{E}$. Therefore, the pair $\psi_1, \bar{\psi}$ defines a homomorphism

$$q(\psi_1, \bar{\psi})\colon q\mathscr{E}\to\Omega_0$$

which obviously coincides with $q(k_0, 0)$. To conclude, we need to show that $\psi_t(x)\in M(\Omega_0)$ and that $\psi_t(x)-\bar{\psi}(x)\in\Omega_0$ for all t and for all $x\in\mathscr{E}$. In fact, once this is done, we obtain a well defined family of homomorphisms

$$q(\psi_t, \bar{\psi})\colon q\mathscr{E}\to\Omega_0,$$

providing a homotopy from $q(\psi, \bar{\psi})$ to $q(k_0, 0)$ and showing that $[q(\psi, \bar{\psi})]$ equals $[q(k_0, 0)]$ in $KK(\mathscr{E}, \Omega_0)$.

The fact that $\psi_t(x)-\bar{\psi}(x)\in\Omega_0$ is obvious for x of the form $x=a\otimes 1$ with $a\in A$. If $x=a_1u\otimes S$, for $a_1\in D_1$, we have

$$\psi_t(x)-\bar{\psi}(x)=\psi_t(x)-\psi(x)+\psi(x)-\bar{\psi}(x)$$

while

$$\psi_t(x)-\psi(x)=(a_1\otimes 1\otimes 1)\,W_t(u\otimes 1\otimes T)-(a_1u\otimes 1\otimes T)$$
$$=(W_t-I)(a_1u\otimes 1\otimes T)=\begin{pmatrix} \lambda a_1u\otimes S & 0 & \cdots \\ -\lambda a_1u\otimes 1 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

which clearly belongs to Ω_0. This implies two important facts. First, since we already know that $\psi(\mathscr{E})\subseteq M(\Omega_0)$, we conclude that $\psi_t(x)\in M(\Omega_0)$ for $x\in(B_0\otimes 1)\cup(B_1\otimes S)$, which is a generating set. So $\psi_t(x)\in M(\Omega_0)$ for all $x\in\mathscr{E}$. Second, because Ω_0 is an ideal in $M(\Omega_0)$, our task of proving that $\psi_t(x)-\bar{\psi}(x)\in\Omega_0$ needs only to be verified on a generating set, which the above computation also yields. ∎

Proof of (7.1). Our last proposition shows that $[q(d', 0)]$ is left invertible since $[q(k_0, 0)]$ is invertible. We conclude that $[q(d, 0)]$ is invertible and hence that A and $\mathscr{E}$ are KK-equivalent to each other. In particular the map

$$d_*\colon K_*(A)\to K_*(\mathscr{E})$$

is an isomorphism. Our result now follows from combining the K-theory exact sequence for the Toeplitz extension

$$0\to A\xrightarrow{i}\mathscr{E}\xrightarrow{\phi}B\to 0$$

with the conclusion of (7.2). ∎

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