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Pick's Theorem and Convergence of Multiple Fourier Series

L. Brandolini, L. Colzani, S. Robins, and G. Travaglini

Abstract. We add another brick to the large building comprising proofs of Pick's theorem. Although our proof is not the most elementary, it is short and reveals a connection between Pick's theorem and the pointwise convergence of multiple Fourier series of piecewise smooth functions.

1. INTRODUCTION. A polygon in the Cartesian plane is simple if it has no holes and if its boundary does not intersect itself. It is called an integer polygon if all of its vertices have integer coordinates. Let P be a simple integer polygon, $|P|$ its area, I the number of integer points strictly inside P, and B the number of integer points on the boundary ∂P. Then

Theorem 1 (Pick).

$$|P| = I + \frac{1}{2}B - 1. \tag{1}$$

In spite of the elementary statement, this is not an ancient result. It was published by Georg Pick in 1899, and first popularized by Hugo Steinhaus in 1937 in the Polish edition of *Mathematical Snapshots*; see [12, Chapter 4] for an English edition. The theorem has many proofs and interesting features. Its statement can be explained to elementary school children, who could be asked to verify it on examples. On the other hand, it can be related to certain nontrivial topics in mathematics. See, e.g., [5] for a connection to Euler's formula for planar graphs, or [10] for a connection to Minkowski's theorem on integer points in convex bodies, or [3] for a complex-analytic proof. A sketch of an easy proof runs as follows.

Step 1. A simple integer polygon can be triangulated into integer primitive triangles, with no integer points other than the vertices.

Step 2. Both terms $|P|$ and $I + \frac{1}{2}B - 1$ in (1) are "additive" with respect to the above triangulation.

Step 3. A primitive triangle together with one of its reflections gives a parallelogram that tiles the plane under integer translations.

Step 4. This latter parallelogram has area 1, so that (1) holds true for primitive triangles.

The purpose of this article is to exhibit a direct connection between Pick's theorem and harmonic analysis. The Fourier-analytic proof we give here is rather short and self-contained, it does not rely on any of the above geometric steps, and it is an elementary consequence of a classical result on pointwise convergence of multiple Fourier series. Moreover, it suggests a point of departure for higher-dimensional investigations.

In what follows, our standard reference for the harmonic analysis on Euclidean spaces is [13]. We recall some notations and some well-known results. If f and φ are

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integrable functions on $\mathbb{R}^d$, then $\varphi * f$ denotes the convolution:

$$\varphi * f(x) = \int_{\mathbb{R}^d} \varphi(x-y) f(y)\, dy.$$

Moreover, $\widehat{f}$ denotes the Fourier transform:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i \xi\cdot x} dx.$$

It is easily verified that if f is integrable on $\mathbb{R}^d$, then $\sum_{n\in\mathbb{Z}^d} f(n+x)$ is a periodic function integrable on the torus $\mathbb{R}^d/\mathbb{Z}^d$ and its Fourier coefficients are the restriction of $\widehat{f}$ to the integer points in $\mathbb{Z}^d$,

$$\begin{aligned}
&\int_{\mathbb{R}^d/\mathbb{Z}^d} \left(\sum_{n\in\mathbb{Z}^d} f(n+x)\right) e^{-2\pi i m\cdot x} dx \\
&\quad = \int_{\bigcup_{n\in\mathbb{Z}^d}\{n+[0,1)^d\}} f(y)\, e^{-2\pi i m\cdot(y-n)} dy \\
&\quad = \int_{\mathbb{R}^d} f(y)\, e^{-2\pi i m\cdot y} dy = \widehat{f}(m).
\end{aligned}$$

Hence, formally, one has the Poisson summation formula

$$\sum_{n\in\mathbb{Z}^d} f(n+x) = \sum_{m\in\mathbb{Z}^d} \widehat{f}(m)\, e^{2\pi i m\cdot x}. \tag{2}$$

Our proof of Pick's theorem is based on the Poisson summation formula applied to the characteristic function of a polygon. But there is a problem. The above formula as written does not immediately apply to nonsmooth functions, such as characteristic functions. Without additional assumptions the series in both sides of this identity may not converge pointwise. Even when both sides converge, they may differ. For example, observe that when a function is modified on a set of measure zero, such as the boundary of a polygon, the left-hand side of the formula may change, while the right-hand side remains the same. On the other hand, a correct formula can be obtained assuming natural regularity conditions on the function f and using suitable summability methods for the Fourier series.

Next, we recall the elementary facts that are required in order to use Poisson summation correctly. Recall that if φ and f are square integrable, then the convolution is uniformly bounded:

$$\begin{aligned}
|\varphi * f(x)| &= \left|\int_{\mathbb{R}^d} \varphi(x-y) f(y)\, dy\right| \\
&\le \left(\int_{\mathbb{R}^d} |\varphi(x-y)|^2\, dy\right)^{1/2} \left(\int_{\mathbb{R}^d} |f(y)|^2\, dy\right)^{1/2},
\end{aligned}$$

using the Cauchy–Schwartz inequality. It follows from the latter inequality, and the fact that square integrable functions can be approximated by continuous functions with compact support, that $\varphi * f$ is uniformly continuous. The Fourier transform of the convolution is the product of the Fourier transforms of the factors, i.e., $\widehat{\varphi * f}(\xi) = \widehat{\varphi}(\xi)\,\widehat{f}(\xi)$. The Fourier transform commutes with rotations, and in particular the Fourier transform of a radial function is radial. Moreover, for every dilation $\varepsilon > 0$ the Fourier transform of $\varphi_\varepsilon(x) := \varepsilon^{-d}\varphi\left(\varepsilon^{-1}x\right)$ is $\widehat{\varphi}(\varepsilon\xi)$. The proofs of these statements involve elementary manipulations of integrals. Finally, if φ is smooth with compact support, then $\widehat{\varphi}$ is smooth and it has rapid decay at infinity. This follows by repeated integrations by parts in the integral that defines the Fourier transform.

A quick sketch of our proof of Pick's theorem now runs as follows. Take a smooth radial function φ with compact support and integral 1, and define

$$f(x) = \varphi_\varepsilon * \chi_P(x),$$

so that $\widehat{f}(\xi) = \widehat{\varphi}(\varepsilon\xi)\,\widehat{\chi}_P(\xi)$. The Poisson summation formula applies to this function f, so that applying (2) with $x = 0$, we have

$$\sum_{n\in\mathbb{Z}^2} \varphi_\varepsilon * \chi_P(n) = \sum_{m\in\mathbb{Z}^2} \widehat{\varphi}(\varepsilon m)\,\widehat{\chi}_P(m).$$

Observe that the series on the left is finite, because $\varphi_\varepsilon * \chi_P$ has compact support, and the series on the right is absolutely convergent, because $\widehat{\chi}_P$ is bounded and $\widehat{\varphi}$ has fast decay at infinity. Also observe that, if ε is small enough, $\varphi_\varepsilon * \chi_P(n)$ is the normalized measure of the angle at the point n:

$$\begin{aligned}\varphi_\varepsilon * \chi_P(n) &= \varepsilon^{-2}\int_{\mathbb{R}^2} \varphi(\varepsilon y)\chi_P(n-y)dy \\ &= \begin{cases} 0 & \text{if } x \notin P, \\ 1 & \text{if } x \text{ is in the interior of } P, \\ 1/2 & \text{if } x \text{ is in the interior of a side of } P, \\ \alpha/2\pi & \text{if } x \text{ is a vertex of } P, \text{ with interior angle } \alpha. \end{cases}\end{aligned}$$

See Figure 1.

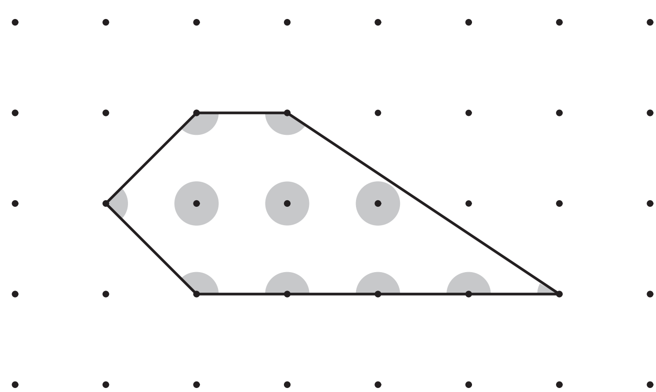

Figure 1. Values of $\varphi_\varepsilon * \chi_P(n)$ at the integer points.

From the formula for the sum of the interior angles of a polygon it follows that

$$\sum_{n\in\mathbb{Z}^2} \varphi_\varepsilon * \chi_P(n) = I + \frac{1}{2}B - 1.$$

One can compute explicitly the Fourier transform $\widehat{\chi}_P$ via the divergence theorem. In particular, we have

$$\widehat{\varphi}(0)\,\widehat{\chi}_P(0) = \widehat{\chi}_P(0) = |P|,$$

while for every $m \neq 0$,

$$\widehat{\varphi}(\varepsilon m)\,\widehat{\chi}_P(m) = -\widehat{\varphi}(-\varepsilon m)\,\widehat{\chi}_P(-m),$$

and one of the main points is that all of these terms cancel. It follows that

$$\sum_{m\in\mathbb{Z}^2} \widehat{\varphi}(\varepsilon m)\,\widehat{\chi}_P(m) = |P|.$$

Hence,

$$|P| = I + \frac{1}{2}B - 1.$$

This is only a sketch of the proof, but it is not difficult to fill in the details. The details are contained in what follows.

2. CONVERGENCE OF FOURIER EXPANSIONS.

The following variation on the classical Poisson summation formula is tailored for our problem.

Theorem 2. *Let φ and f be square integrable functions on $\mathbb{R}^d$ with compact support. Assume that $\int_{\mathbb{R}^d} \varphi(x)\,dx = 1$, and also assume that for every x,*

$$f(x) = \lim_{\varepsilon\to 0^+} \{\varphi_\varepsilon * f(x)\}. \tag{3}$$

Then, for every $\varepsilon > 0$,

$$\sum_{m\in\mathbb{Z}^d} \left|\widehat{\varphi}(\varepsilon m)\,\widehat{f}(m)\right| < +\infty.$$

Moreover, for every x,

$$\sum_{n\in\mathbb{Z}^d} f(n+x) = \lim_{\varepsilon\to 0^+} \left\{ \sum_{m\in\mathbb{Z}^d} \widehat{\varphi}(\varepsilon m)\,\widehat{f}(m)\,e^{2\pi i m\cdot x} \right\}. \tag{4}$$

If φ is smooth with compact support, then $\widehat{\varphi}$ has fast decay at infinity and the theorem reduces to the classical Poisson summation formula. See, e.g., [**13**, Chapter 7, Corollary 2.6] and [**13**, Chapter 2, Theorem 3.16] for similar results where φ is the Poisson kernel. Condition (3) is a regularity assumption on the function f, and it is satisfied at every point of continuity of the function. In particular, in dimension $d = 1$ and if φ is even, at a jump discontinuity of f the hypothesis is satisfied provided that

$$f(x) = \lim_{\varepsilon\to 0^+} \left\{ \frac{f(x+\varepsilon) + f(x-\varepsilon)}{2} \right\}.$$

Recall that the one-dimensional Fourier series of a piecewise smooth function at a jump discontinuity converges precisely to the above limit.

Proof. Under the assumption of the theorem, the convolution $\varphi_\varepsilon * f$ is bounded with compact support, the sum $\sum_{n\in\mathbb{Z}^d} \varphi_\varepsilon * f(n+x)$ is finite, and it gives a bounded function on the torus $\mathbb{R}^d/\mathbb{Z}^d = [0,1)^d$, with Fourier coefficients $\widehat{\varphi}(\varepsilon m)\,\widehat{f}(m)$,

$$\int_{\mathbb{R}^d/\mathbb{Z}^d} \left(\sum_{n\in\mathbb{Z}^d} \varphi_\varepsilon * f(n+x) \right) e^{-2\pi i m\cdot x} dx$$
$$= \int_{\mathbb{R}^d} \varphi_\varepsilon * f(y)\, e^{-2\pi i m\cdot y} dy = \widehat{\varphi}(\varepsilon m)\,\widehat{f}(m)\,.$$

Hence, the function

$$\sum_{n\in\mathbb{Z}^d} \varphi_\varepsilon * f(n+x)$$

has Fourier expansion

$$\sum_{m\in\mathbb{Z}^d} \widehat{\varphi}(\varepsilon m)\,\widehat{f}(m)\, e^{2\pi i m\cdot x}.$$

Recall that Fourier series may diverge at some points; hence it is not obvious that the above Fourier series converges and that it is equal pointwise to the function being expanded. Since the sum $\sum_{n\in\mathbb{Z}^d} \varphi_\varepsilon * f(n+x)$ is finite and has a bounded number of nonzero terms as $\varepsilon \to 0^+$, for every x the limit commutes with the sum:

$$\lim_{\varepsilon\to 0^+} \left\{ \sum_{n\in\mathbb{Z}^d} \varphi_\varepsilon * f(n+x) \right\} = \sum_{n\in\mathbb{Z}^d} f(n+x)\,.$$

Then it is enough to show that for every x,

$$\sum_{n\in\mathbb{Z}^d} \varphi_\varepsilon * f(n+x) = \sum_{m\in\mathbb{Z}^d} \widehat{\varphi}(\varepsilon m)\,\widehat{f}(m)\, e^{2\pi i m\cdot x}. \tag{5}$$

This follows from the fact that $\sum_{m\in\mathbb{Z}^d} \left|\widehat{\varphi}(\varepsilon m)\,\widehat{f}(m)\right|$ converges, which is a consequence of the following Plancherel–Polya type inequality. Let g be an integrable function with compact support and let ψ be a smooth compactly supported function with $\psi(x) = 1$ on the support of g. Since $g(x) = \psi(x)\,g(x)$, $\widehat{g}(\xi) = \widehat{\psi} * \widehat{g}(\xi)$, and $\widehat{\psi}$ is rapidly decreasing, we have

$$\sum_{m\in\mathbb{Z}^d} |\widehat{g}(m)| = \sum_{m\in\mathbb{Z}^d} \left| \int_{\mathbb{R}^d} \widehat{\psi}(m-\xi)\,\widehat{g}(\xi)\,d\xi \right|$$
$$\leqslant \sup_{\xi\in\mathbb{R}^d} \left\{ \sum_{m\in\mathbb{Z}^d} \left|\widehat{\psi}(m-\xi)\right| \right\} \int_{\mathbb{R}^d} |\widehat{g}(\xi)|\,d\xi \leqslant c \int_{\mathbb{R}^d} |\widehat{g}(\xi)|\,d\xi.$$

Observe that above the constant c depends on $\psi(x)$, hence on the support of g. See, e.g., [11, Chapter 3] for more general inequalities of this type.
Applying this inequality to the function $g(x) = \varphi_\varepsilon * f(x)$, we obtain

$$\sum_{m\in\mathbb{Z}^d} \left|\widehat{\varphi_\varepsilon * f}(m)\right| = \sum_{m\in\mathbb{Z}^d} \left|\widehat{\varphi}(\varepsilon m)\,\widehat{f}(m)\right| \leqslant c\int_{\mathbb{R}^d} \left|\widehat{\varphi}(\varepsilon\xi)\,\widehat{f}(\xi)\right| d\xi$$
$$\leqslant c\left(\int_{\mathbb{R}^d} |\widehat{\varphi}(\varepsilon\xi)|^2\, d\xi\right)^{1/2} \left(\int_{\mathbb{R}^d} \left|\widehat{f}(\xi)\right|^2 d\xi\right)^{1/2}$$
$$= c\varepsilon^{-d/2}\left(\int_{\mathbb{R}^d} |\varphi(x)|^2\, dx\right)^{1/2} \left(\int_{\mathbb{R}^d} |f(x)|^2\, dx\right)^{1/2}.$$

Observe that the factor $\varepsilon^{-d/2}$ does not contradict the existence of the limit as $\varepsilon \to 0^+$. The above estimate is just what we need to show the pointwise equality (5) for every fixed $\varepsilon > 0$, since we already observed that the limit of the left-hand side of (5) exists. ■

3. PICK'S THEOREM. Our proof of Pick's theorem below is a corollary of the version of the Poisson summation formula above, applied to characteristic functions of integer polygons. Such characteristic functions do not satisfy the assumption (3) of Theorem 2, but they can be regularized by modifying the values at the boundary. It is a classical argument to restate Pick's theorem in terms of normalized angles as follows. Define a regularization of the characteristic function of the polygon P:

$$\widetilde{\chi}_P(x) = \begin{cases} 0 & \text{if } x \notin P, \\ 1 & \text{if } x \text{ is in the interior of } P, \\ 1/2 & \text{if } x \text{ is in the interior of a side of } P, \\ \alpha/2\pi & \text{if } x \text{ is a vertex of } P, \text{ with interior angle } \alpha. \end{cases}$$

Assuming that P has N vertices, since the sum of the inner angles is $\pi(N-2)$, we have

$$\sum_{k\in\mathbb{Z}^2} \widetilde{\chi}_P(k) = \sum_{\text{interior points of } P} 1 + \sum_{\text{interior points of sides of } P} 1/2 + \sum_{\text{vertices of } P} \alpha/2\pi$$
$$= I + \frac{1}{2}(B-N) + \frac{1}{2}(N-2) = I + \frac{1}{2}B - 1.$$

Hence, Pick's theorem is reduced to the following.

Theorem 3. *If P is a simple integer polygon, then*

$$\sum_{n\in\mathbb{Z}^2} \widetilde{\chi}_P(n) = |P|.$$

Proof. Let φ be a square integrable radial function with compact support and integral 1; for example let $\varphi(x) = 4\pi^{-1}\chi_{\{|x|<1/2\}}(x)$. For $\varepsilon > 0$ small enough and for every $n \in \mathbb{Z}^2$ it can be easily shown that

$$\varphi_\varepsilon * \widetilde{\chi}_P(n) = \widetilde{\chi}_P(n).$$

Then $\widetilde{\chi}_P$ satisfies the assumption (3) of Theorem 2 and

$$\sum_{n\in\mathbb{Z}^2} \widetilde{\chi}_P(n) = \lim_{\varepsilon\to 0^+}\left\{\sum_{m\in\mathbb{Z}^2} \widehat{\varphi}(\varepsilon m)\,\widehat{\widetilde{\chi}}_P(m)\right\}. \tag{6}$$

Observe that in this identity the limit can be omitted as soon as ε is small enough, and also observe that $\widehat{\widetilde{\chi}}_P(m) = \widehat{\chi}_P(m)$, since $\widetilde{\chi}(x) = \chi(x)$ except on a set of measure zero. Let P have vertices $\{P_j\}$ and sides $\{P_j + t(P_{j+1} - P_j) : 0 \le t \le 1\}$ with outward unit normals $\{n_j\}$. Then, with the notation $P_{N+1} = P_1$, if $m \ne 0$ the divergence theorem yields

$$\begin{aligned}\widehat{\chi}_P(m) &= \int_P e^{-2\pi i m\cdot x}dx = \int_P \operatorname{div}\left(\frac{-m}{2\pi i\,|m|^2}e^{-2\pi i m\cdot x}\right)dx \\ &= \frac{-1}{2\pi i}\sum_{j=1}^{N}\frac{n_j\cdot m}{|m|^2}\left|P_{j+1}-P_j\right|\int_0^1 e^{-2\pi i m\cdot(P_j+t(P_{j+1}-P_j))}dt.\end{aligned}$$

The one-dimensional integrals can be computed explicitly, and when P_j and m belong to $\mathbb{Z}^2$, then

$$\int_0^1 e^{-2\pi i m\cdot(P_j+t(P_{j+1}-P_j))}dt = \begin{cases} 0 & \text{if } m\cdot(P_{j+1}-P_j) \ne 0,\\ 1 & \text{if } m\cdot(P_{j+1}-P_j) = 0.\end{cases}$$

Recalling that $\widehat{\varphi}(0) = 1$ and $\widehat{\chi}_P(0) = |P|$, we obtain

$$\begin{aligned}\sum_{m\in\mathbb{Z}^2}\widehat{\varphi}(\varepsilon m)\,\widehat{\chi}_P(m) &= \widehat{\chi}_P(0) + \sum_{m\in\mathbb{Z}^2\setminus\{0\}}\widehat{\varphi}(\varepsilon m)\,\widehat{\chi}_P(m) \\ &= |P| - \frac{1}{2\pi i}\sum_{j=1}^{N}\left|P_{j+1}-P_j\right|\left(\sum_{m\ne 0,\ m\cdot(P_{j+1}-P_j)=0}\widehat{\varphi}(\varepsilon m)\frac{m\cdot n_j}{|m|^2}\right).\end{aligned} \tag{7}$$

Finally, the sums inside the parentheses vanish, because, under the assumption that $\widehat{\varphi}(\varepsilon m)$ is radial, $\widehat{\varphi}(\varepsilon m)\,|m|^{-2}\,m\cdot n_j$ is an odd function of m. Hence

$$\sum_{m\in\mathbb{Z}^2}\widehat{\varphi}(\varepsilon m)\,\widehat{\chi}_P(m) = |P|.$$

■

Observe that in the proof of the above theorem the assumption that the polygon is simple can be weakened. In particular, the formulation of Pick's theorem in terms of normalized interior angles also holds for integer polygons with holes.

4. FURTHER REMARKS. Pick's theorem, in the naive form that we know it, fails in dimension $d \ge 3$. Indeed, as observed by J. E. Reeve, the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,N)$, has volume $N/6$, contains four integer points on the boundary, and has no integer points inside. Hence there is no simple relation between the volume and the integer points for general three-dimensional integer polytopes.

Fascinating relations do appear, however, when an integer polyhedron is dilated by an integer factor. By Ehrhart's theorem from the 1950s, the number of integer points in a dilated integer polyhedron P is a polynomial function of the integer dilation parameter, with leading coefficient equal to the volume of P. The reader may consult, for example, the books [1] and [2] for an account of Ehrhart's main theorems. The Poisson summation formula was used in [4] to analyze the Ehrhart polynomial of an integer polytope in $\mathbb{R}^d$.

The above-defined *regularized discrete volume* $\sum_{n\in\mathbb{Z}^d} \widetilde{\chi}_P(n)$ can be easily defined in every dimension, but in general it is no longer equal to the Euclidean volume $|P|$. However, as we see from equations (6) and (7), it is still true that

$$\sum_{n\in\mathbb{Z}^d} \widetilde{\chi}_P(n) = |P| \quad \text{if and only if} \quad \sum_{0\neq m\in\mathbb{Z}^d} \widehat{\varphi}(\varepsilon m)\,\widehat{\chi}_P(m) = 0 \tag{8}$$

for all sufficiently small $\varepsilon > 0$, and for every choice of φ as before. That is, if an integer polytope P satisfies (8), then by definition its continuous Euclidean volume is equal to its regularized discrete volume. We can call such integer polytopes *concrete polytopes*, following the tradition of [7], who used the first three letters of "continuous" and the last five letters of "discrete" to consider objects that can be described by both continuous methods and by discrete methods.

An interesting open problem is to characterize the concrete polytopes in $\mathbb{R}^n$; that is, what are the integer polytopes that enjoy the relation $\sum_{n\in\mathbb{Z}^d} \widetilde{\chi}_P(n) = |P|$?

As already shown by Barvinok [1], integer zonotopes are concrete polytopes, as well as integer symmetric polytopes whose facets are also symmetric. A more general family of concrete polytopes is given by multiple tilers. Indeed, an easy application of the Poisson summation formula (see [8], [9, p. 137]) tells us that the integer polytope P multi-tiles $\mathbb{R}^d$ with the lattice of integer translations, if and only if $\widehat{\chi}_P(m) = 0$ for every $m \in \mathbb{Z}\backslash\{0\}$, so that identity (8) is trivially satisfied. On the other hand Garber and Pak have recently shown that there exist concrete lattice polytopes in $\mathbb{R}^3$ which do not multi-tile $\mathbb{R}^3$ (see [6]).

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