The e-value and the Full Bayesian Significance Test: Logical Properties and Philosophical Consequences

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The e-value, ev$(H|X)$ – also named the *epistemic-value* of hypothesis H given observations X, or the evidence-value of observations X in favor (or in support) of hypothesis H – is a Bayesian statistical significance measure introduced in 1999 by Carlos Alberto de Bragança Pereira and Julio Michael Stern, together with the FBST – the Full Bayesian Significance Test, see [5]. The definitions of e-value and the FBST were further refined and generalized by subsequent works of several researchers at USP – the University of São Paulo, and UFSCar – the Federal University of São Carlos, in Brazil, including Wagner Borges, Luís Gustavo Esteves, Rafael Izbicki, Regina Madruga, Rafael Bassi Stern, and Sergio Wechsler, see [1–4, 6].

The e-value was specially designed to assess the *statistical significance* or the *logical truth value* of *sharp* or *precise* hypotheses in the context of Bayesian statistics. The e-value has desirable asymptotic, geometrical (invariance), and logical (compositional) properties that allow consistent and coherent evaluation and testing of sharp statistical hypotheses. Furthermore, in applied modeling, the FBST offers an easy to implement and powerful statistical test that is fully compliant with Bayesian principles of good inference, like the likelihood principle.

In the context of statistical test of hypotheses, a *compositional logic* is conveyed by an algebraic formalism that allows the evaluation of truth-functions of composite models and truth-values of composite hypotheses by algebraic operations on the corresponding truth functions of elementary models and truth values of elementary hypotheses. The e-value and the FBST have a rich, expressive and intuitive compositional logic, while traditional truth-values and accompanying tests offered by either frequentist (classical) statistics, like the p-value, or by Bayesian statistics, like Bayes factors, have important and well-known deficiencies in this regard, specially in cases involving sharp statistical hypotheses. Furthermore, *logically coherent* evaluations and testing of sets and subsets of statistical hypotheses should render sequences of inferential reasoning that do not generate internal contradictions or anti-intuitive results. As expected, the e-value and the FBST comply with well-established rules of logical coherence, even in the case of sharp hypotheses, while traditional

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alternatives often fail to do so. Asymptotically, $\text{sev}(H|X)$ – the standarized e-value – shares several properties of the p-value, the well-known significance measure of frequentist statistics; Nevertheless, $\text{sev}(H|X)$ retains many theoretical characteristics of the Bayesian framework. For several theoretical developments and practical applications of the e-value and the FBST, see [7], [12].

Section 2 reviews the Bayesian statistical framework; Section 3 defines the e-value; Sections 4, 5 and 6 explain the invariance, asymptotic and compositional properties of the e-value; Section 7 defines the GFBST - the Generalized Full Bayesian Significance Test and its logical properties; Sections 8 and 9 comment on computational implementation and give a detailed numerical example in model selection; Section 10 lists a representative assortment of articles from many practical applications of the e-value and the FBST already published in the scientific literature. Section 11 considers the philosophical consequences of the aforementioned developments by briefly commenting the *Objective cognitive constructivism* epistemological framework, that was specifically developed to accomodate the formal properties of the e-value and the FBST, and renders a naturalized approach to ontology and metaphysics. Section 12 presents some topics for further research at the interface between Logic and Statistics.

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