

Remarks on the spectrum of a non-local Dirichlet problem

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ABSTRACT

In this paper, we analyse the spectrum of non-local Dirichlet problems with non-singular kernels in bounded open sets. The novelty is twofold. First we study the continuity of eigenvalues with respect to domain perturbation via Lebesgue measure. Next, under additional smooth conditions on the kernel and domain, we prove differentiability of simple eigenvalues computing their first derivative discussing extremum problems for eigenvalues.

1. Introduction

In this note, we discuss the spectrum set of a non-local equation with non-singular kernels and Dirichlet conditions in bounded open sets $\Omega \subset \mathbb{R}^N$. We consider the non-local eigenvalue problem

$$\begin{cases} (J * u)(x) - u(x) = -\lambda u(x), & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1.1)$$

where $J * u$ stands for the usual convolution

$$(J * u)(x) = \int_{\mathbb{R}^N} J(x-y)u(y)dy$$

with a kernel J . Throughout this article the function J satisfies the hypotheses

(H) $J \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ is a non-negative function, spherically symmetric and radially decreasing with $J(0) > 0$ and $\int_{\mathbb{R}^N} J(x) dx = 1$.

Our main goal is to study the continuity of the spectrum set with respect to the variation of the domain Ω . Next, assuming J and Ω are $\mathcal{C}^1$ -regular, we also show differentiability of simple eigenvalues computing an expression for their first derivative allowing Ω to vary in the set of open sets which are $\mathcal{C}^1$ -diffeomorphic.

Note that analysing the spectral properties of (1.1) is equivalent to study the spectrum of the linear operator $\mathcal{B}_\Omega : W_\Omega \mapsto W_\Omega$ where $W_\Omega = \{u \in L^2(\mathbb{R}^N) : u(x) \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega\}$ and

$$\mathcal{B}_\Omega u(x) \equiv u(x) - \int_{\Omega} J(x-y)u(y)dy, \quad x \in \Omega. \quad (1.2)$$

Moreover, one has that the operator $\mathcal{B}_\Omega$ is the sum of the identity on the Hilbert space W_Ω minus the compact and self-adjoint operator $\hat{\mathcal{J}}_\Omega : W_\Omega \mapsto W_\Omega$ which is defined by

$$\hat{\mathcal{J}}_\Omega = E_\Omega \circ \mathcal{J}_\Omega \circ R_\Omega, \quad (1.3)$$

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where $\mathcal{J}_\Omega : L^2(\Omega) \mapsto L^2(\Omega)$ is the linear operator defined by the convolution

$$\mathcal{J}_\Omega u(x) = (J * u)(x), \quad x \in \Omega, \quad (1.4)$$

$R_\Omega : W_\Omega \mapsto L^2(\Omega)$ is the restriction to Ω and $E_\Omega : L^2(\Omega) \mapsto W_\Omega$ is the standard extension by zero

$$E_\Omega u(x) = \begin{cases} u(x), & \text{if } x \in \Omega \\ 0, & \text{otherwise.} \end{cases}$$

We will see that there exists a precise relationship between the spectrum of the operators $\mathcal{B}_\Omega$ and $\mathcal{J}_\Omega$. Indeed, the continuity properties for the eigenvalues of $\mathcal{B}_\Omega$ will be obtained by an accurate analysis of the spectrum of $\mathcal{J}_\Omega$ via perturbation theory for linear operators developed in [21]. The convergence of the eigenvalues is obtained assuming that the Lebesgue measure of the symmetric difference of open sets goes to zero.

Along the whole paper, we say that a family of measurable sets $\{\Omega_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N$ converges in measure to $\Omega \subset \mathbb{R}^N$ as $n \rightarrow \infty$, if the symmetric difference $|\Omega_n \setminus \Omega| + |\Omega \setminus \Omega_n| \rightarrow 0$. Note that $|\mathcal{O}|$ denotes the Lebesgue measure of any measurable set $\mathcal{O} \subset \mathbb{R}^N$. Here, we mention one of the main results in this direction.

THEOREM 1.1. *Let D be a bounded set in $\mathbb{R}^N$ and $\Omega_n \subset D$ be a family of open bounded sets with $\mu_k(\Omega_n)$ denoting the k th eigenvalue of the operator $\mathcal{J}_{\Omega_n}$. Assume $\Omega_n \rightarrow \Omega$ in measure as $n \rightarrow \infty$ for some $\Omega \subset D$.*

Then, there exist positive constants C and δ , depending only on the domain Ω , such that, if $\mu_k(\Omega)$ is the k th eigenvalue of the operator $\mathcal{J}_\Omega$, then

$$|\mu_k(\Omega_n) - \mu_k(\Omega)| \leq C \|J\|_{L^\infty(\mathbb{R}^N)} [|\Omega_n \setminus \Omega| + |\Omega \setminus \Omega_n|]^{1/2}$$

whenever $|\Omega_n \setminus \Omega| + |\Omega \setminus \Omega_n| < \delta$. In particular,

$$|\mu_k(\Omega_n) - \mu_k(\Omega)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Moreover, if $\lambda_k(\Omega_n)$ is the k th eigenvalue of the operator $\mathcal{B}_{\Omega_n}$, we have

$$|\lambda_k(\Omega_n) - \lambda_k(\Omega)| \leq C \|J\|_{L^\infty(\mathbb{R}^N)} [|\Omega_n \setminus \Omega| + |\Omega \setminus \Omega_n|]^{1/2}$$

as $|\Omega_n \setminus \Omega| + |\Omega \setminus \Omega_n| < \delta$ with

$$|\lambda_k(\Omega_n) - \lambda_k(\Omega)| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

where $\lambda_k(\Omega)$ is the k th eigenvalue of the operator $\mathcal{B}_\Omega$.

Next, we follow the approach introduced in [18] to perturb Ω in order to take derivatives of simple eigenvalues with respect to the domain. More precisely, if $\Omega \subset \mathbb{R}^N$ is a $\mathcal{C}^1$ -regular open bounded set, and $h : \Omega \mapsto \mathbb{R}^N$ is a $\mathcal{C}^1$ -diffeomorphism to its image, we define the composition map

$$h^* v(x) = (v \circ h)(x), \quad x \in \Omega,$$

for any v set on $h(\Omega)$. $h^* : L^2(h(\Omega)) \mapsto L^2(\Omega)$ is an isomorphism with $(h^*)^{-1} = (h^{-1})^*$.

For such imbedding h and bounded region Ω , one can introduce the non-local Dirichlet operator $\mathcal{B}_{h(\Omega)}$ on the perturbed open set $h(\Omega)$ by

$$(\mathcal{B}_{h(\Omega)} v)(y) = v(y) - \int_{h(\Omega)} J(y-w) v(w) dw, \quad y \in h(\Omega), \quad (1.5)$$

with $\mathcal{B}_{h(\Omega)} : W_{h(\Omega)} \mapsto W_{h(\Omega)}$. On the other hand, we can use h^* to set $h^*\mathcal{B}_{h(\Omega)}h^{*-1} : W_\Omega \mapsto W_\Omega$ by

$$h^*\mathcal{B}_{h(\Omega)}h^{*-1}u(x) = \int_{h(\Omega)} J(h(x) - w)(u \circ h^{-1})(w)dw, \quad \forall x \in \Omega. \quad (1.6)$$

It is known that expressions (1.5) and (1.6) are the customary manner to describe motion or deformation of regions. Form (1.5) is called the Lagrangian description, and (1.6) the Eulerian one. The former is written in a fixed coordinate system while the Lagrangian does not. Also,

$$h^*\mathcal{B}_{h(\Omega)}h^{*-1}u(x) = v(y) - \int_{h(\Omega)} J(y - w)v(w)dw = (\mathcal{B}_{h(\Omega)}v)(y)$$

if we take $y = h(x)$ and $v(y) = (u \circ h^{-1})(y) = h^{*-1}u(y)$ for $y \in h(\Omega)$.

In this way, we perturb our eigenvalue problem (1.1). We take imbeddings $h : \Omega \mapsto \mathbb{R}^N$ varying in the set of diffeomorphisms $\text{Diff}^1(\Omega)$ studying the eigenvalues of the operators (1.5) and (1.6) which are the same. We have the following result concerning the derivative of simple eigenvalues.

THEOREM 1.2. *Let λ_0 be a simple eigenvalue for $\mathcal{B}_\Omega$ with corresponding normalized eigenfunction u_0 and $J \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R})$ satisfying (H). Then, there exists a neighbourhood $\mathcal{V}$ of the inclusion $i_\Omega \in \text{Diff}^1(\Omega)$, and $\mathcal{C}^1$ -functions (u_h, λ_h) from $\mathcal{V}$ into $L^2(\Omega) \times \mathbb{R}$ which satisfy $h^*\mathcal{B}_{h(\Omega)}h^{*-1}u_h(x) = \lambda_h u_h(x)$, $x \in \Omega$, with $u_h \in \mathcal{C}^1(\Omega)$. Also, λ_h is a simple eigenvalue, $(\lambda_{i_\Omega}, u_{i_\Omega}) = (\lambda_0, u_0)$, and the domain derivative is given by*

$$\frac{\partial \lambda}{\partial h}(i_\Omega) \cdot V = -(1 - \lambda_0) \int_{\partial \Omega} u_0^2 V \cdot N_\Omega dS \quad \text{for all } V \in \mathcal{C}^1(\Omega, \mathbb{R}^N), \quad (1.7)$$

where $\partial \Omega$ denotes the boundary of Ω and N_Ω its normal vector.

At this point, it is worth noticing that we are improving here results from [16] where the domain perturbation to the first eigenvalue of (1.1) was considered and formula (1.7) was first obtained. There, the authors have used the variational formulation of the first eigenvalue and the positivity of the corresponding eigenfunction which holds just in this particular case. Our result is more general since it holds for any simple eigenvalue also showing smooth persistence.

We mention some authors as [1, 15, 19] which associate J under conditions (H) to a radial probability density calling equation (1.1) a non-local analogous to the Dirichlet boundary conditions problem to the Laplacian. Indeed, several continuous models for species and human mobility have been proposed using such non-local approach, in order to look for more realistic dispersion equations [3, 8, 11]. Recall that hostile surroundings are modeled by the Dirichlet condition as in (1.1).

As one can see, for instance, in [7, 8, 13, 16, 20], that such kind of non-local models with non-singular kernels exhibit different properties when compared to their local and non-local analogs which are associated to singular kernels. The local and non-local analogs for (1.1) are given by unbounded operators with compact resolvent which guarantees the regularizing effect for the solutions. Hence, besides the applied models with non-singular kernels, the mathematical interest is mainly due to the fact that, in general, there is no regularizing effect and therefore no general compactness tools are available making their study different.

Finally, let us note that Theorem 1.1 is not true for standard local operators like the Laplacian. In the classical paper [12], the authors consider the Laplacian with Dirichlet boundary condition in a bounded domain from where a big number of periodic small balls (the holes) is removed. They consider $\Omega^\epsilon = \Omega \setminus \cup_i B_{r^\epsilon}(x_i)$ where $B_{r^\epsilon}(x_i)$ is a ball centred in $x_i \in \Omega$ of the form $x_i \in 2\epsilon\mathbb{Z}^N$ with radius $0 < r^\epsilon < \epsilon < 1$ and $\epsilon \rightarrow 0$. It is shown that there is a critical size of the holes (that is, a critical order of r^ϵ in ϵ) such that the resolvent operator

of the Dirichlet–Laplacian is not continuous at $\epsilon = 0$. Assuming $N \geq 3$, for instance, we have that the critical size of the holes is given by $a^\epsilon \sim \epsilon^{\frac{N}{N-2}}$. Hence, if we take $r^\epsilon \geq a^\epsilon$, then the continuity of the spectral set does not hold. We also mention [9] where the authors discuss capacity constraints to guarantee certain continuity of the spectra.

The paper is organized as follows: in Section 2, we show some preliminary results concerning to the spectrum of $\mathcal{J}_\Omega$ and $\mathcal{B}_\Omega$ also discussing isoperimetric inequalities for $\mathcal{B}_\Omega$. Such inequalities are an analogue of Rayleigh–Faber–Krahn and Hong–Krahn–Szegő inequalities and have been recently obtained for $\mathcal{J}_\Omega$ in [24]. For a recent review on isoperimetric inequalities, we refer to [6].

In Section 3, we study the continuity of eigenvalues with respect to Ω . We also take into account recent results concerning the convergence of eigenvalues posed in oscillating and perforated domains. Finally, in Section 4, we obtain the stability of a simple eigenvalue with respect to the variation of smooth domains performed by imbeddings, proving Theorem 1.2.

2. Basic facts and preliminary results

Let us first discuss the operator $\mathcal{J}_\Omega : L^2(\Omega) \mapsto L^2(\Omega)$ given by the convolution (1.4). Note $\mathcal{J}_\Omega$ is bounded, compact and self-adjoint satisfying

$$\|\mathcal{J}_\Omega\|_{L^2(\Omega)} \leq |\Omega| \|J\|_{L^\infty(\mathbb{R}^N)}.$$

Such a proof is straightforward and can be found, for instance, in [13, 23]. In the sequel, we mention other properties with respect to its spectral set which are also consequence of classical results from functional analysis.

REMARK 2.1. Since $\mathcal{J}_\Omega$ is compact and self-adjoint, one may obtain, for instance, from [21, Chapter V, Theorem 2.10], that the spectrum $\sigma(\mathcal{J}_\Omega)$ consists of at most a countable number of real eigenvalues with finite multiplicities, possible excepting zero. Let us enumerate their eigenvalues in decreasing order of magnitude

$$|\mu_1| \geq |\mu_2| \geq \dots$$

If $P_1, P_2, \dots$ are the associated eigenprojections of $\mathcal{J}_\Omega$, then P_i are orthogonal and self-adjoint with finite dimensional range. Also, we have the spectral representation

$$\mathcal{J}_\Omega = \sum_{i \geq 0} \mu_i P_i$$

in the sense of convergence in norm with projections forming a complete orthogonal family together with the orthogonal projection P_0 on the null space of $\mathcal{J}_\Omega$.

REMARK 2.2. From [21, Chapter V, Theorem 2.10], we have that $0 \in \sigma(\mathcal{J}_\Omega)$. Also, if there exists an infinite sequence of distinct eigenvalues μ_i , then $\mu_i \rightarrow 0$ as $i \rightarrow +\infty$, and then, zero belongs to the essential spectrum $\sigma_{ess}(\mathcal{J}_\Omega)$. On the other hand, if the set of eigenvalues is finite, its null space is not trivial, indeed, it is an infinite dimensional subspace of $L^2(\Omega)$.

REMARK 2.3. We note that $|\mu_1|$ is equal to the spectral radius of $\mathcal{J}_\Omega$ which coincides with its norm

$$|\mu_1| = \lim_{n \rightarrow +\infty} \|\mathcal{J}_\Omega^n\|^{1/n} = \|\mathcal{J}_\Omega\|.$$

Moreover, it is known from [23, 24], that the first eigenvalue μ_1 is positive, simple, whose corresponding eigenfunction u_1 can be chosen strictly positive in Ω .

Since the eigenvalues μ_i have finite multiplicity, we can set them in a decreasing order of magnitude also taking account their multiplicity. Hence, we denote by $u_1, u_2, \dots$ the corresponding eigenfunctions for each eigenvalue μ_i setting

$$\mathcal{J}_\Omega u_i(x) = \mu_i(\Omega) u_i(x).$$

Now, let us denote the range of $\mathcal{J}_\Omega$ by $R(\mathcal{J}_\Omega)$. Since $\mathcal{J}_\Omega$ is self-adjoint, $R(\mathcal{J}_\Omega)$ is orthogonal to the kernel of $\mathcal{J}_\Omega$, $\ker(\mathcal{J}_\Omega)$, setting a useful decomposition for $L^2(\Omega)$. From Remark 2.1, one gets

$$L^2(\Omega) = R(\mathcal{J}_\Omega) \oplus \ker(\mathcal{J}_\Omega).$$

We still have the following result concerning $R(\mathcal{J}_\Omega)$.

LEMMA 2.1. *Assume $R(\mathcal{J}_\Omega)$ is finite dimensional.*

Then, there exist a set of normalized eigenfunctions $\{u_1, \dots, u_m\} \subset L^2(\Omega)$, associated to non-zero eigenvalues $\mu_i(\Omega)$, such that

$$J(x - y) = \sum_{i=1}^m \mu_i(\Omega) u_i(x) u_i(y), \quad \text{a.e. } \Omega. \quad (2.8)$$

In particular, $J(x) = \sum_{i=1}^m \mu_i(\Omega) u_i(x) u_i(0)$ a.e. Ω , and $J(0)|\Omega| = \sum_{i=1}^m \mu_i(\Omega)$.

Proof. First, we recall that $L^2(\Omega)$ is the direct sum of $R(\mathcal{J}_\Omega)$ and $\ker(\mathcal{J}_\Omega)$. Thus, if $R(\mathcal{J}_\Omega)$ is finite dimensional, by 2.1 again, there exist $\{u_1, \dots, u_m\} \subset L^2(\Omega)$ given by orthogonal and normalized eigenfunctions of $\mathcal{J}_\Omega$, associated to non-zero eigenvalues $\mu_i(\Omega)$ such that

$$R(\mathcal{J}_\Omega) = [u_1, \dots, u_m].$$

Hence, we can take the orthogonal projections P_i as

$$P_i u(x) = \left(\int_{\Omega} u_i(y) u(y) dy \right) u_i(x), \quad x \in \Omega.$$

For all $u \in L^2(\Omega)$, we have

$$\mathcal{J}_\Omega u(x) = \int_{\Omega} J(x - y) u(y) dy = \sum_{i=1}^m \mu_i(\Omega) \left(\int_{\Omega} u_i(y) u(y) dy \right) u_i(x), \quad x \in \Omega.$$

Consequently,

$$0 = \int_{\Omega} \left(J(x - y) - \sum_{i=1}^m \mu_i(\Omega) u_i(x) u_i(y) \right) u(y) dy, \quad \forall u \in L^2(\Omega) \text{ and } \forall x \in \Omega,$$

completing the proof. $\square$

Now, let us consider the operator $\mathcal{B}_\Omega : W_\Omega \mapsto W_\Omega$ defined by (1.2). Since $\mathcal{B}_\Omega$ is a scalar combination of the identity and the self-adjoint operator $\hat{\mathcal{J}}_\Omega$, $\mathcal{B}_\Omega$ is also a bounded self-adjoint operator in $L^2(\Omega)$.

REMARK 2.4. We note that:

- (a) $\lambda(\Omega) \in \sigma(\mathcal{B}_\Omega)$ is an eigenvalue, if and only if, there exists $u \in L^2(\Omega)$, $u \neq 0$, with $u(x) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$, satisfying equation (1.1) for this same $\lambda(\Omega)$;
- (b) $u \in L^2(\Omega)$ is a fixed point of $\mathcal{B}_\Omega$, if and only if, u belongs to the null set of $\mathcal{J}_\Omega$;
- (c) $\lambda(\Omega) \in \sigma(\mathcal{B}_\Omega)$ is an eigenvalue, if and only if, $1 - \lambda(\Omega)$ is an eigenvalue of the compact operator $\mathcal{J}_\Omega$. Hence, the eigenvalues of $\mathcal{B}_\Omega$ are enumerated according to the eigenvalues of

$\mathcal{J}_\Omega$ setting $\lambda_k(\Omega) = 1 - \mu_k(\Omega)$ for $k \geq 1$. Also, $0 \in \sigma_{ess}(\mathcal{J}_\Omega)$, if and only if, $1 \in \sigma_{ess}(\mathcal{B}_\Omega)$, and zero is an eigenvalue of $\mathcal{J}_\Omega$, if and only if, $1 \in \sigma(\mathcal{J}_\Omega)$;

- (d) from Remark 2.3, we know that the first eigenvalue of $\mathcal{B}_\Omega$, which is given by $\lambda_1(\Omega) = 1 - \mu_1(\Omega)$, it is associated to a strictly positive eigenfunction which is also simple with

$$\lambda_1(\Omega) = 1 - \|\mathcal{J}_\Omega\| < 1;$$

- (e) Further, since we are assuming $\int_{\mathbb{R}^N} J(y) dy = 1$, we have

$$\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)(u(y) - u(x))^2 dy dx = \|u\|_{L^2(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)u(y)u(x) dy dx,$$

and then, we get from (d) that

$$\lambda_1(\Omega) = \inf_{u \neq 0 \text{ in } W_\Omega} \frac{\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)(u(y) - u(x))^2 dy dx}{\|u\|_{L^2(\mathbb{R}^N)}^2}. \quad (2.9)$$

For more details, see [1, 16].

Let us take u_1 , the first positive eigenfunction of $\mathcal{B}_\Omega$. It follows from (1.1) that

$$\begin{aligned} -\lambda_1(\Omega) \int_{\Omega} (u_1(x))^2 dx &= \int_{\Omega} u_1(x) \int_{\Omega} J(x-y)(u_1(y) - u_1(x)) dy \\ &= -\frac{1}{2} \int_{\Omega} \int_{\Omega} J(x-y)(u_1(y) - u_1(x))^2 dy dx \leq 0. \end{aligned}$$

Thus, $0 \leq \lambda_1(\Omega) < 1$ with $\lambda_1(\Omega) = 0$, if and only if, u_1 is a positive constant. Now, due to [1, Proposition 2.2], one can get that $J * u(x) - u(x) \equiv 0$ in Ω with $u(x) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$, if and only if, $u(x) \equiv 0$ in $\mathbb{R}^N$. Hence, we conclude that

$$0 < \lambda_1(\Omega) < 1 \quad \text{and} \quad 0 < \|\mathcal{J}_\Omega\| < 1 \quad (2.10)$$

for any bounded open set Ω .

Consequently, we obtain from (2.10) that $\mathcal{B}_\Omega$ is a perturbation of the identity being an invertible operator with continuous inverse given by $\mathcal{B}_\Omega^{-1}u = (I - \hat{\mathcal{J}}_\Omega)^{-1}u = \sum_{n=0}^{\infty} \hat{\mathcal{J}}_\Omega^n u$.

REMARK 2.5. Others informations and properties concerning the operators $\mathcal{J}_\Omega$ and $\mathcal{B}_\Omega$, and their spectrum set, can be seen, for instance, in [13, 20, 23] and references therein. Moreover, it is important to know that all the results discussed to this point remain valid substituting the radial condition on the function J with the even one, that is, assuming $J(-x) = J(x)$.

Finally, let us just mention some isoperimetric inequalities for the first and second eigenvalues of $\mathcal{B}_\Omega$. Due to the symmetric condition imposed on the kernel J , an analogue of Rayleigh–Faber–Krahn and Hong–Krahn–Szegö inequalities for $\mathcal{J}_\Omega$ have been shown in [24]. Hence, since Remark 2.4 gives a precise relationship between the spectrum of $\mathcal{J}_\Omega$ and $\mathcal{B}_\Omega$, we can easily extend the results from [24] to the Dirichlet problem (1.1).

Concerning the Rayleigh–Faber–Krahn inequality, we have the following result:

COROLLARY 2.1. *Let Ω^* denote an open ball with same measure as Ω . Then, under conditions (H), the ball Ω^* is a minimizer for the first eigenvalue of $\mathcal{B}_\Omega$, that is,*

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*).$$

Proof. It has been seen at [24, Theorem 2.1] that the first eigenvalue $\mu_1(\Omega)$ of $\mathcal{J}_\Omega$ achieves its maximum among open sets of given volume at the ball Ω^* . That is, $\mu_1(\Omega) \leq \mu_1(\Omega^*)$. Hence, we get the result from expression $\lambda_1(\Omega) = 1 - \mu_1(\Omega)$ given by Remark 2.4. $\square$

In Section 4, we give an example which shows that the first eigenvalue of (1.1) does not possess a maximizer among open bounded sets even with a fixed measure. Now we consider the minimizer of the second eigenvalue of $\mathcal{B}_\Omega$ among open sets of given volume. As we are going to see, the minimizer is no longer one ball, but the union of two identical balls whose mutual distance is going to infinity. It is an analogue of the Hong–Krahn–Szegő inequality [17] and it has been proven in [24, Theorem 2.3] for the compact operator $\mathcal{J}_\Omega$. First, we prove the existence of $\lambda_2(\Omega)$ (and $\mu_2(\Omega)$) for any $\Omega \subset \mathbb{R}^N$.

PROPOSITION 2.1. *Under conditions (H), we have $\dim(\mathcal{R}(\mathcal{J}_\Omega)) \geq 2$. In particular, there exists $\lambda_2(\Omega)$ for any bounded open domain $\Omega \subset \mathbb{R}^N$.*

Proof. Let us suppose that $\mathcal{J}_\Omega$ is a one-dimensional linear space. Then, by Lemma 2.1, taking $x = y$ in (2.8), we have that $J(0) = \mu_1(\Omega)(u_1(x))^2$ in Ω where $\mu_1(\Omega)$ is the first eigenvalue of $\mathcal{J}_\Omega$ with corresponding normalized eigenfunction $u_1 \in L^2(\Omega)$. Hence, we conclude that u_1 is a strictly positive constant which is a contradiction, since it satisfies (2.9) with $\lambda_1(\Omega) = 1 - \mu_1(\Omega) > 0$. Finally, as $\lambda_1(\Omega)$ is a simple eigenvalue, it follows that there exists at least another larger eigenvalue of $\mathcal{B}_\Omega$. $\square$

Now, let us optimize the second eigenvalue.

COROLLARY 2.2. *Under hypothesis (H), the minimum of the second eigenvalue of (1.1) among all bounded open sets with given volume is achieved by the disjoint union of two identical balls with mutual distance attaching to infinity.*

Proof. The result is a direct consequence of the expression $\lambda_2(\Omega) = 1 - \mu_2(\Omega)$ and [24, Theorem 2.3] where it has been proved that the maximum of $\mu_2(\Omega)$ is achieved in a disjoint union of identical balls with mutual distance going to infinity. $\square$

3. Continuity of eigenvalues

In this section, we discuss the continuity of the eigenvalues with respect to $\Omega \subset \mathbb{R}^N$. Note that this is not a trivial task since any change of Ω causes a change on the operator domain. In order to overcome this problem, we extend $\mathcal{J}_\Omega$ into a $L^2(D)$ for a larger bounded set $D \subset \mathbb{R}^N$.

Let us take $\Omega \subset D$. We define $\tilde{\mathcal{J}}_\Omega : L^2(D) \mapsto L^2(D)$ by

$$\tilde{\mathcal{J}}_\Omega u(x) = \begin{cases} \mathcal{J}_\Omega u(x) & x \in \Omega \\ 0 & x \in D \setminus \Omega \end{cases}.$$

Note that $\tilde{\mathcal{J}}_\Omega u(x) = \mathcal{J}_\Omega u(x)$ for all $x \in \Omega$, and then, $\tilde{\mathcal{J}}_\Omega$ is an extension of $\mathcal{J}_\Omega$ into $L^2(D)$. In fact, $\tilde{\mathcal{J}}_\Omega$ is somehow similar to the operator $\hat{\mathcal{J}}_\Omega$ introduced in (1.3) since $\tilde{\mathcal{J}}_\Omega = E_D \circ \mathcal{J}_\Omega \circ R_D$ where $E_D : L^2(\Omega) \mapsto L^2(D)$ is the extension by zero operator

$$E_D u(x) = \begin{cases} u(x), & \text{if } x \in \Omega \\ 0, & \text{otherwise} \end{cases} \quad (3.11)$$

and $R_D : L^2(D) \mapsto L^2(\Omega)$ is the restriction to Ω . Hence, since $\mathcal{J}_\Omega$ is compact and self-adjoint, it follows from [21, Theorem 4.8 Chapter 3] that $\tilde{\mathcal{J}}_\Omega$ is also a compact and self-adjoint operator acting on $L^2(D)$ with

$$\|\tilde{\mathcal{J}}_\Omega\|_{L^2(D)} \leq |\Omega| \|J\|_{L^\infty(\mathbb{R}^N)}.$$

Thus, we can argue as in Remark 2.1 getting from [21, Theorem 2.10 Chapter V] that $\sigma(\tilde{\mathcal{J}}_\Omega)$ consists of at most a countable number of real eigenvalues with finite multiplicities, possibly

excepting zero. We also enumerate their eigenvalues in decreasing order of magnitude

$$|\tilde{\mu}_1| \geq |\tilde{\mu}_2| \geq \dots$$

If $\tilde{P}_1, \tilde{P}_2, \dots$ are the associated eigenprojections, then $\tilde{P}_i$ are orthogonal and self-adjoint with finite dimensional range. Finally, we also get a spectral representation

$$\tilde{\mathcal{J}}_\Omega = \sum_{i \geq 0} \tilde{\mu}_i \tilde{P}_i$$

in the sense of convergence in norm with projections forming a complete orthogonal family together with the orthogonal projection $\tilde{P}_0$ on the null space of $\tilde{\mathcal{J}}_\Omega$.

In the sequel, we first get conditions, in order to guarantee the continuity of the operators $\tilde{\mathcal{J}}_\Omega$ with respect to Ω . Next, we note that the non-zero eigenvalues of $\tilde{\mathcal{J}}_\Omega$ and $\mathcal{J}_\Omega$ are equal. Here we study continuity via abstract results concerning perturbations for linear operators dealt in [21].

LEMMA 3.1. *Let Ω_1, Ω_2 be two bounded open sets in $D \subset \mathbb{R}^N$. Then, there exists $C > 0$ depending only on the measure of the set D such that*

$$\|\tilde{\mathcal{J}}_{\Omega_1} - \tilde{\mathcal{J}}_{\Omega_2}\|_D \leq C \|J\|_{L^\infty(\mathbb{R}^N)} [|\Omega_1 \setminus \Omega_2| + |\Omega_2 \setminus \Omega_1|]^{1/2}.$$

In particular, if $\Omega_n \subset D$ is a sequence of domains with $\Omega_n \rightarrow \Omega$ in measure for some $\Omega \subset D$ as $n \rightarrow \infty$, then

$$\|\tilde{\mathcal{J}}_{\Omega_n} - \tilde{\mathcal{J}}_\Omega\|_{L^2(D)} \rightarrow 0.$$

Proof. Note that

$$\tilde{\mathcal{J}}_{\Omega_1} u(x) - \tilde{\mathcal{J}}_{\Omega_2} u(x) = \begin{cases} \int_{\Omega_1} J(x-y)u(y)dy - \int_{\Omega_2} J(x-y)u(y)dy, & x \in \Omega_1 \cap \Omega_2, \\ \int_{\Omega_1} J(x-y)u(y)dy, & x \in \Omega_1 \setminus \Omega_2, \\ - \int_{\Omega_2} J(x-y)u(y)dy, & x \in \Omega_2 \setminus \Omega_1. \end{cases}$$

Hence, if $x \in \Omega_1 \cap \Omega_2$, we get

$$\begin{aligned} |\tilde{\mathcal{J}}_{\Omega_1} u(x) - \tilde{\mathcal{J}}_{\Omega_2} u(x)| &\leq \int_{\Omega_1 \setminus \Omega_2} J(x-y)|u(y)|dy + \int_{\Omega_2 \setminus \Omega_1} J(x-y)|u(y)|dy \\ &\leq \|J\|_{L^\infty(\mathbb{R}^N)} \|u\|_{L^2(D)} \left[|\Omega_1 \setminus \Omega_2|^{1/2} + |\Omega_2 \setminus \Omega_1|^{1/2} \right]. \end{aligned}$$

On the other hand, if $x \in (\Omega_1 \setminus \Omega_2) \cup (\Omega_2 \setminus \Omega_1)$,

$$|\tilde{\mathcal{J}}_{\Omega_1} u(x) - \tilde{\mathcal{J}}_{\Omega_2} u(x)| \leq \|J\|_{L^\infty(\mathbb{R}^N)} \|u\|_{L^2(D)} \left(|\Omega_1|^{1/2} + |\Omega_2|^{1/2} \right).$$

Consequently,

$$\begin{aligned} \int_D |\tilde{\mathcal{J}}_{\Omega_1} u(x) - \tilde{\mathcal{J}}_{\Omega_2} u(x)|^2 dx &\leq \int_{\Omega_1 \cup \Omega_2} |\tilde{\mathcal{J}}_{\Omega_1} u(x) - \tilde{\mathcal{J}}_{\Omega_2} u(x)|^2 dx \\ &\leq \|J\|_{L^\infty}^2 \|u\|_{L^2(D)}^2 \left(|\Omega_1 \setminus \Omega_2|^{1/2} + |\Omega_2 \setminus \Omega_1|^{1/2} \right)^2 |\Omega_1 \cap \Omega_2| \\ &\quad + \|J\|_{L^\infty}^2 \|u\|_{L^2(D)}^2 \left(|\Omega_1|^{1/2} + |\Omega_2|^{1/2} \right)^2 (|\Omega_1 \setminus \Omega_2| + |\Omega_2 \setminus \Omega_1|) \\ &\leq 2 \|J\|_{L^\infty}^2 \|u\|_{L^2(D)}^2 (|\Omega_1 \setminus \Omega_2| + |\Omega_2 \setminus \Omega_1|) (|\Omega_1 \cup \Omega_2| + |\Omega_1| + |\Omega_2|) \end{aligned}$$

proving the result. $\square$

Next, let us see that the sets of non-zero eigenvalues of $\tilde{\mathcal{J}}_\Omega$ and $\mathcal{J}_\Omega$ are equal.

LEMMA 3.2. *A non-zero value μ is an eigenvalue of the operator $\tilde{\mathcal{J}}_\Omega$, if and only if, it is a non-zero eigenvalue for $\mathcal{J}_\Omega$. Furthermore, we have that their multiplicity is preserved.*

Proof. We have that $\mu \neq 0$ is an eigenvalue of $\tilde{\mathcal{J}}_\Omega$, if and only if, there exists $\tilde{u} \neq 0$ in $L^2(D)$ with

$$\tilde{\mathcal{J}}_\Omega \tilde{u}(x) = \mu \tilde{u}(x), \quad x \in D \subset \mathbb{R}^N.$$

Thus, from definition of $\tilde{\mathcal{J}}_\Omega$, we get

$$\mathcal{J}_\Omega \tilde{u}(x) = \mu \tilde{u}(x), \quad x \in \Omega,$$

with $\tilde{u}(x) \equiv 0$ in $D \setminus \Omega$ since $\mu \neq 0$. Consequently, μ is also an eigenvalue of $\mathcal{J}_\Omega$ with corresponding eigenfunction $u(x) := \tilde{u}(x)$ for $x \in \Omega$. On the other hand, if $\mu \neq 0$ is an eigenvalue of $\mathcal{J}_\Omega$ with corresponding non-zero $u(x) \in L^2(\Omega)$, we have that the extension by zero of u into $L^2(D)$ is also an eigenfunction of $\tilde{\mathcal{J}}_\Omega$ associated to μ , completing the proof. $\square$

Now, let $s_T = \{\lambda_{p_1}, \dots, \lambda_{p_k}\}$ be a collection of finite eigenvalues of a compact and self-adjoint operator T and $P_{p_1}, \dots, P_{p_k}$ their associated orthogonal eigenprojections. We say that s_T is a finite system of eigenvalues with multiplicity $m \in \mathbb{N}$, if the range $R(P_{p_i})$ of P_{p_i} is finite and satisfies

$$\sum_{i=1}^k \dim(R(P_{p_i})) = m.$$

Note we can associate to s_T an orthogonal projection P_{s_T} given by $P_{s_T} = \sum_i P_{p_i}$. If in addition, all eigenvalues of s_T are simple, we call s_T a finite system of simple eigenvalues.

Our next result shows the persistence of a finite system of eigenvalues for $\tilde{\mathcal{J}}_\Omega$ when we perturb Ω . As we shall see, this is a direct consequence of the continuity of the operators with respect to Ω in norm and abstract results from perturbation theory of linear operators shown in [21].

LEMMA 3.3. *Let $s_{\tilde{\mathcal{J}}_\Omega} \subset \sigma(\tilde{\mathcal{J}}_\Omega)$ be a finite system of eigenvalues with multiplicity $m \in \mathbb{N}$ and $\mathcal{V} \subset \mathbb{R}$ a neighbourhood of $s_{\tilde{\mathcal{J}}_\Omega}$. Then, for all $\varepsilon > 0$, there exist $\delta > 0$ and a neighbourhood $\mathcal{V}_\varepsilon \subset \mathcal{V}$ of $s_{\tilde{\mathcal{J}}_\Omega}$ depending on $s_{\tilde{\mathcal{J}}_\Omega}$, $\mathcal{V}$ and $\tilde{\mathcal{J}}_\Omega$, such that, if $\tilde{\Omega} \subset D \subset \mathbb{R}^N$ satisfies*

$$|\Omega \setminus \tilde{\Omega}| + |\tilde{\Omega} \setminus \Omega| < \delta \tag{3.12}$$

then, $\tilde{\mathcal{J}}_{\tilde{\Omega}}$ also has a finite system of eigenvalues $s_{\tilde{\mathcal{J}}_{\tilde{\Omega}}}$ with multiplicity m and $s_{\tilde{\mathcal{J}}_{\tilde{\Omega}}} \subset \mathcal{V}_\varepsilon$. Furthermore, the orthogonal projections $P_{s_{\tilde{\mathcal{J}}_\Omega}}$ and $P_{s_{\tilde{\mathcal{J}}_{\tilde{\Omega}}}}$ associated to the finite systems $s_{\tilde{\mathcal{J}}_\Omega}$ and $s_{\tilde{\mathcal{J}}_{\tilde{\Omega}}}$ satisfy $\|P_{s_{\tilde{\mathcal{J}}_\Omega}} - P_{s_{\tilde{\mathcal{J}}_{\tilde{\Omega}}}}\|_{L^2(D)} < \varepsilon$.

Proof. Since $s_{\tilde{\mathcal{J}}_\Omega}$ is a finite collection of eigenvalues and $\mathcal{V}$ is a given neighbourhood, we can construct a finite collection of disjoint open disks B_i in $\mathbb{C}$ with radius $r_i > 0$ such that $s_{\tilde{\mathcal{J}}_\Omega} \subset (\cup_i \bar{B}_i) \cap \mathbb{R} \subset \mathcal{V}$ and $B_i \cap s_{\tilde{\mathcal{J}}_\Omega} = \tilde{\mu}_i(\Omega)$ for some eigenvalue $\tilde{\mu}_i(\Omega)$ of $\tilde{\mathcal{J}}_\Omega$. For each i , let us consider the circle Γ_i given by the boundary ∂B_i of B_i . Hence, for each i , we can separate $\sigma(\tilde{\mathcal{J}}_\Omega)$ in two natural parts $\sigma_{i,1}(\tilde{\mathcal{J}}_\Omega)$ and $\sigma_{i,2}(\tilde{\mathcal{J}}_\Omega)$ where $\sigma_{i,1}(\tilde{\mathcal{J}}_\Omega) = \sigma(\tilde{\mathcal{J}}_\Omega) \cap B_i$ and $\sigma_{i,2}(\tilde{\mathcal{J}}_\Omega) = \sigma(\tilde{\mathcal{J}}_\Omega) \cap \bar{B}_i^c$, and $L^2(\Omega) = M_{1,i} \oplus M_{i,2}$ where $M_{1,i}$ is the range of the orthogonal projection associated to $\tilde{\mu}_i(\Omega) \in B_i$, and $M_{i,2}$ is the enumerate union of all ranges given by the others eigenprojections and kernel of $\tilde{\mathcal{J}}_\Omega$.

It follows from Lemma 3.1, [21, Theorem 2.23, p. 206] and [21, Theorem 3.16, p. 212] that, for all $\varepsilon > 0$, there exist δ_i and $r_i > 0$ depending just on $\tilde{\mathcal{J}}_\Omega$ and Γ_i such that, if $\tilde{\Omega}$ satisfies

(3.12), then $\sigma(\tilde{\mathcal{J}}_{\tilde{\Omega}})$ can be likewise separated by Γ_i in two parts $\sigma_{i,1}(\tilde{\mathcal{J}}_{\tilde{\Omega}})$ and $\sigma_{i,2}(\tilde{\mathcal{J}}_{\tilde{\Omega}})$ with associated decomposition $L^2(\Omega) = \tilde{M}_{1,i} \oplus \tilde{M}_{i,2}$. $\tilde{M}_{1,i}$ and $\tilde{M}_{i,2}$ are, respectively, isomorphic with $M_{1,i}$ and $M_{i,2}$ and corresponding orthogonal projections ε -closed in operator norm. In particular, $\dim(\tilde{M}_{1,i}) = \dim(M_{1,i})$ and $\dim(\tilde{M}_{i,2}) = \dim(M_{i,2})$ and both $\sigma_{i,1}(\tilde{\mathcal{J}}_{\tilde{\Omega}})$ and $\sigma_{i,2}(\tilde{\mathcal{J}}_{\tilde{\Omega}})$ are non-empty if this is true for $\tilde{\mathcal{J}}_{\tilde{\Omega}}$. Since we are considering a finite collection of eigenvalues, the result follows taking $\delta = \min_i \{\delta_i\}$ and $\mathcal{V}_\varepsilon = (\cup_i B_i) \cap \mathcal{V}$. $\square$

As a direct consequence of Remark 2.4 and Lemmas 3.2 and 3.3, we obtain the continuity of a finite system of eigenvalues for the operators $\mathcal{J}_\Omega$ and $\mathcal{B}_\Omega$. We have the following result.

THEOREM 3.1. *Let $s_{\mathcal{J}_\Omega} \subset \sigma(\mathcal{J}_\Omega)$ be a finite system of eigenvalues with multiplicity $m \in \mathbb{N}$ and $\mathcal{V} \subset \mathbb{R}$ a neighbourhood of $s_{\mathcal{J}_\Omega}$.*

Then, for all $\varepsilon > 0$, there exist $\delta > 0$ and a neighbourhood $\mathcal{V}_\varepsilon \subset \mathcal{V}$ of $s_{\mathcal{J}_\Omega}$ depending on $s_{\mathcal{J}_\Omega}$, $\mathcal{V}$ and $\mathcal{J}_\Omega$ such that, if $\tilde{\Omega} \subset D \subset \mathbb{R}^N$ satisfies

$$|\Omega \setminus \tilde{\Omega}| + |\tilde{\Omega} \setminus \Omega| < \delta \quad (3.13)$$

then $\mathcal{J}_{\tilde{\Omega}}$ also has a finite system of eigenvalues $s_{\mathcal{J}_{\tilde{\Omega}}}$ with multiplicity m and $s_{\mathcal{J}_{\tilde{\Omega}}} \subset \mathcal{V}_\varepsilon$.

Furthermore, if $s_{\mathcal{B}_\Omega}$ is also a finite system of eigenvalues with multiplicity $m \in \mathbb{N}$ for the operator $\mathcal{B}_\Omega$, we have, under the same condition (3.13), the existence of a finite system of eigenvalues $s_{\mathcal{B}_{\tilde{\Omega}}} \subset \mathcal{V}_\varepsilon$ with multiplicity m .

REMARK 3.1. In the proof of Lemma 3.2, we obtain a relationship between the eigenprojections of the operators $\tilde{\mathcal{J}}_\Omega$ and $\mathcal{J}_\Omega$, and then, between the eigenprojections of the operators $\tilde{\mathcal{J}}_\Omega$ and $\mathcal{B}_\Omega$. Indeed, if $\tilde{P}_i$ and P_i are the eigenprojections of $\tilde{\mathcal{J}}_\Omega$ and $\mathcal{J}_\Omega$, respectively, we have that $P_i = R_D \circ \tilde{P}_i \circ E_D$ and $\tilde{P}_i = E_D \circ P_i \circ R_D$ for any $i \geq 1$ where R_D is the restriction operator to Ω and E_D is the extension by zero previously introduced in (3.11). Thus, by the action of the operator E_D , we obtain from Lemma 3.3 the continuity of the eigenspaces of $\mathcal{B}_\Omega$ associated to non-zero eigenvalues since they vanish outside of Ω .

We also note the persistence of a finite system of simple eigenvalues.

COROLLARY 3.1. *Let $s_{\mathcal{J}_\Omega} = \{\mu_1(\Omega), \dots, \mu_k(\Omega)\} \subset \sigma(\mathcal{J}_\Omega)$ be a finite system of simple eigenvalues with $s_{\mathcal{J}_\Omega} \subset \mathcal{V}$ for some open set $\mathcal{V} \subset \mathbb{R}$.*

Then, for all $\varepsilon > 0$, there exist $\delta > 0$ and a neighbourhood $\mathcal{V}_\varepsilon \subset \mathcal{V}$ of $s_{\mathcal{J}_\Omega}$ depending on $s_{\mathcal{J}_\Omega}$, $\mathcal{V}$ and $\mathcal{J}_\Omega$ such that, if $\tilde{\Omega} \subset D \subset \mathbb{R}^N$ satisfies (3.13), the operator $\mathcal{J}_{\tilde{\Omega}}$ also possesses a finite system of simple eigenvalue $s_{\mathcal{J}_{\tilde{\Omega}}} = \{\mu_1(\tilde{\Omega}), \dots, \mu_k(\tilde{\Omega})\} \subset \mathcal{V}_\varepsilon$.

Respectively, if $s_{\mathcal{B}_\Omega} = \{\lambda_1(\Omega), \dots, \lambda_k(\Omega)\} \subset \mathcal{V}$ is a finite system of simple eigenvalues for $\mathcal{B}_\Omega$, then there exists a finite system of simple eigenvalues $s_{\mathcal{B}_{\tilde{\Omega}}} = \{\lambda_1(\tilde{\Omega}), \dots, \lambda_k(\tilde{\Omega})\} \subset \mathcal{V}_\varepsilon$.

Proof. Let us apply Lemma 3.3 to each single system $\{\mu_i(\Omega)\} \subset s_{\mathcal{J}_\Omega}$. Since $\mu_i(\Omega)$ is simple, for each $i = 1, 2, \dots, k$, there exists $\delta_i > 0$ such that $\{\mu_i(\tilde{\Omega})\} \subset \sigma(\mathcal{J}_{\tilde{\Omega}})$ is also a simple eigenvalue whenever $\tilde{\Omega}$ satisfies (3.13) substituting δ with δ_i . Hence, as $s_{\mathcal{J}_\Omega}$ is a finite collection, the result follows if we take $\delta = \min\{\delta_1, \dots, \delta_k\}$ setting $s_{\mathcal{J}_{\tilde{\Omega}}}$ in a natural form. $\square$

Now, we are ready to obtain the convergence of single eigenvalues given by a sequence of bounded open sets.

LEMMA 3.4. *Let $\Omega_n \subset \mathbb{R}^N$ be a sequence of bounded open sets with*

$$|\Omega \setminus \Omega_n| + |\Omega_n \setminus \Omega| \rightarrow 0, \quad \text{as } n \rightarrow \infty$$

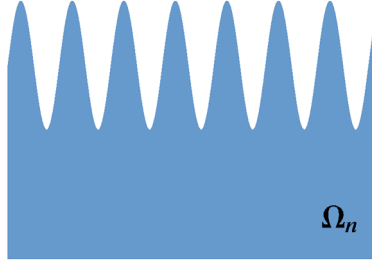


FIGURE 1 (colour online). A family of open sets with rough boundary.

for some bounded open set $\Omega \subset \mathbb{R}^N$. Then, if $\tilde{\mu}(\Omega)$ is an eigenvalue for $\tilde{\mathcal{J}}_\Omega$, there exists a family of eigenvalues $\tilde{\mu}(\Omega_n) \in \sigma(\tilde{\mathcal{J}}_{\Omega_n})$ such that

$$\tilde{\mu}(\Omega_n) \rightarrow \tilde{\mu}(\Omega), \quad \text{as } n \rightarrow \infty.$$

Proof. We just need to fix a small neighbourhood for the single eigenvalue $\tilde{\mu}(\Omega)$ applying Lemma 3.3 and Lemma 3.1. $\square$

Next, let us proof Theorem 1.1 which concerns the continuity of the eigenvalues providing an estimate of their rate of convergence. As one can see, Theorem 1.1 is a direct consequence of Lemma 3.2 and [17, Theorem 2.3.1].

Proof of Theorem 1.1. Using the spectral representation of $\tilde{\mathcal{J}}_\Omega$, we can have the following orthogonal decomposition for $L^2(\Omega)$

$$L^2(\Omega) = L_+^2(\Omega) \oplus L_0^2(\Omega) \oplus L_-^2(\Omega),$$

where $L_+^2(\Omega)$ and $L_-^2(\Omega)$ are defined by the eigenprojections associated to positive and negative eigenvalues, respectively, and $L_0^2(\Omega)$ denotes the null space of $\tilde{\mathcal{J}}_\Omega$. Note that such linear subspaces are $\tilde{\mathcal{J}}_\Omega$ invariant. Hence, since $\tilde{\mathcal{J}}_\Omega$ is a compact and self-adjoint operator, continuous with respect to $\Omega \subset \mathbb{R}^N$ by Lemma 3.3, we can use the min-max formula for their positive and negative eigenvalues. The result follows from Lemma 3.2, Remark 2.4 and [17, Theorem 2.3.1] applied to $\tilde{\mathcal{J}}_\Omega$. $\square$

Finally, let us consider two families of open sets discussing continuity of eigenvalues for the integral operators $\mathcal{J}_\Omega$ and $\mathcal{B}_\Omega$. First, we look at a family of open sets with rough boundary. Next, we analyse a periodically perforated domain. Below, we illustrate each family in Figures 1 and 2, respectively.

EXAMPLE 3.1 (Open sets with rough boundary). Let us consider the following family of domains

$$\Omega_n = \left\{ (x, y) \in \mathbb{R}^2 : x \in (0, 1) \text{ and } 0 < y < 1 + \frac{\sin(2\pi nx)}{n} \right\}.$$

The family Ω_n can be seen as a perturbation of the unit square $\Omega = (0, 1)^2$ and has been studied by many authors; see, for example, [2, 4 10] and references therein.

It is not difficult to see that

$$|\Omega \setminus \Omega_n| + |\Omega_n \setminus \Omega| = \frac{2n \int_0^{1/2n} \sin(2\pi nx) dx}{n} = \frac{2}{\pi n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

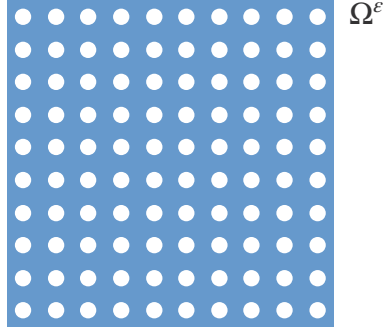


FIGURE 2 (colour online). A periodic perforated domain $\Omega^\epsilon = (0, 1)^2 \setminus A^\epsilon$.

Consequently, we may apply Theorem 1.1, Lemma 3.3, and Corollary 3.1 to this family of open sets evaluating the behaviour of their eigenvalues.

EXAMPLE 3.2 (Perforated domains). Let $Q \subset \mathbb{R}^N$ be the following cell

$$Q = (0, l_1) \times (0, l_2) \times \dots \times (0, l_N).$$

We perforate $\Omega \subset \mathbb{R}^N$ removing from it a set A^ϵ of periodically distributed holes set as follows: Take any open set $A \subset Q$ such that $Q \setminus A$ is a measurable set with $|Q \setminus A| \neq 0$. Denote by $\tau_\epsilon(A)$ all translated images of ϵA of the form $\epsilon(kl + A)$ for $k \in \mathbb{Z}^N$ and $kl = (k_1 l_1, \dots, k_N l_N)$. Now define $A^\epsilon = \Omega \cap \tau_\epsilon(A)$ introducing our perforated domain as

$$\Omega^\epsilon = \Omega \setminus A^\epsilon, \quad \epsilon > 0.$$

Note that, if the measure of the set A is non-zero, then $|\Omega \setminus \Omega^\epsilon| + |\Omega^\epsilon \setminus \Omega|$ does not converge to zero as $\epsilon \rightarrow 0$. Thus, Theorem 1.1 and Lemma 3.3, as well Corollary 3.1, cannot be applied to this family of open sets.

Indeed, it follows from [22, Lemma 3.1, Section 4.1] that the first eigenvalue $\lambda_1(\Omega^\epsilon)$ of the non-local Dirichlet operator $\mathcal{B}_{\Omega^\epsilon}$ converges to a value β_1 as $\epsilon \rightarrow 0$ which satisfies $\beta_1 \in (0, 1)$, and

$$\frac{\beta_1}{\mathcal{X}} \phi^*(x) = \mathcal{B}_\Omega \phi^*(x) + \frac{(1 - \mathcal{X})}{\mathcal{X}} \phi^*(x), \quad x \in \Omega, \quad (3.14)$$

for a strictly positive function $\phi^* \in L^2(\Omega)$, with $\phi^*(x) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$, and a positive constant $\mathcal{X}$

$$\mathcal{X} = \frac{|Q \setminus A|}{|Q|},$$

which is gotten by the limit of the characteristic function of the open sets Ω^ϵ as $\epsilon \rightarrow 0$.

We have:

COROLLARY 3.2 (Perforated domains). β_1 is the first eigenvalue of $\mathcal{B}_\Omega$, if and only if, $|A| = 0$, that is, when Ω is weakly perforated.

Proof. If β_1 is the first eigenvalue of $\mathcal{B}_\Omega$ and satisfies (3.14), taking, ϕ^* as a test function in equation (3.14), we get that

$$-\frac{(1 - \mathcal{X} - \beta_1)}{\mathcal{X}} \|\phi^*\|_{L^2(\Omega)}^2 = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x - y) (\phi^*(y) - \phi^*(x))^2 dy dx \geq \beta_1 \|\phi^*\|_{L^2(\Omega)}^2$$

and then, $\beta_1(1 - \mathcal{X}) \geq (1 - \mathcal{X})$. Since $\beta_1 \in (0, 1)$, we obtain $\mathcal{X} = 1$, which implies $|A| = 0$.

Reciprocally, if $|A| = 0$, then $|\Omega \setminus \Omega^\epsilon| + |\Omega^\epsilon \setminus \Omega| = 0$ for all $\epsilon > 0$, and then, we can apply Theorem 1.1 obtaining $\lambda_1(\Omega^\epsilon) \rightarrow \lambda_1(\Omega) = \beta_1$ as $\epsilon \rightarrow 0$, completing the proof. $\square$

REMARK 3.2. Finally, we would like to observe that other kinds of perforations could be considered and similar results could be obtained. In a general framework, the continuity of the eigenvalues will depend on the limit of the characteristic function χ_{Ω^ϵ} of the perforated domain Ω^ϵ . In fact, one can combine Theorem 1.1 and [22, Theorem 1.1] to show that the eigenvalues of $\mathcal{B}_{\Omega^\epsilon}$ are continuous, if and only if, $\chi_{\Omega^\epsilon} \rightharpoonup 1$ weakly* in $L^\infty(\Omega)$.

4. Domain derivative of simple eigenvalues

In this section, we perturb simple eigenvalues of operators $\mathcal{J}_\Omega$ and $\mathcal{B}_\Omega$ getting derivatives with respect to the domain Ω . We use the approach introduced in [18] perturbing a fixed domain Ω by diffeomorphisms. As a consequence, we extend the expression obtained to the domain derivative for the first eigenvalue in [16] for any simple one in the spectral set of $\mathcal{J}_\Omega$ and $\mathcal{B}_\Omega$.

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set $\mathcal{C}^1$ -regular. If $h : \Omega \mapsto \mathbb{R}^N$ is a $\mathcal{C}^1$ imbedding, that is, a diffeomorphism to its image, we set the composition map h^* (sometimes called pull-back) by

$$h^*v(x) = (v \circ h)(x), \quad x \in \Omega,$$

when v is any given function defined on $h(\Omega)$. It is not difficult to see $h^* : L^2(h(\Omega)) \mapsto L^2(\Omega)$ is an isomorphism with inverse $(h^*)^{-1} = (h^{-1})^*$.

For such imbedding h and a bounded region Ω , one has

$$(\mathcal{J}_{h(\Omega)}v)(y) = \int_{h(\Omega)} J(y-w)v(w)dw, \quad \forall y \in h(\Omega), \quad (4.15)$$

setting $\mathcal{J}_{h(\Omega)} : L^2(h(\Omega)) \mapsto L^2(h(\Omega))$. On the other hand, we can use the pull-back operator h^* to consider $h^*\mathcal{J}_{h(\Omega)}h^{*-1} : L^2(\Omega) \mapsto L^2(\Omega)$ given by

$$h^*\mathcal{J}_{h(\Omega)}h^{*-1}u(x) = \int_{h(\Omega)} J(h(x)-w)(u \circ h^{-1})(w)dw, \quad \forall x \in \Omega. \quad (4.16)$$

As we have already mentioned, expressions (4.15) and (4.16) are the customary way to describe motion or deformation of regions. (4.15) is called the Lagrangian description, and (4.16) the Eulerian one. The former is written in a fixed coordinate while the Lagrangian does not. It is easy to see

$$h^*\mathcal{J}_{h(\Omega)}h^{*-1}u(x) = \int_{h(\Omega)} J(y-w)v(w)dw = (\mathcal{J}_{h(\Omega)}v)(y) \quad (4.17)$$

if we take $y = h(x)$ and $v(y) = (u \circ h^{-1})(y) = h^{*-1}u(y)$ for $y \in h(\Omega)$.

Note $h^*\mathcal{J}_{h(\Omega)}h^{*-1}$ is a compact operator since h^* and h^{*-1} are isomorphisms and $\mathcal{J}_{h(\Omega)}$ is compact. On the other side, $h^*\mathcal{J}_{h(\Omega)}h^{*-1}$ is not a self-adjoint operator in $L^2(\Omega)$ for all h .

In fact, if we change the $L^2(\Omega)$ measure using the determinant of the Jacobian matrix Dh of h , we do obtain a self-adjoint operator. As J is even, by a change of variables, we have

$$\begin{aligned} \int_{\Omega} \varphi(x) h^*\mathcal{J}_{h(\Omega)}h^{*-1}u(x) |\det(Dh(x))| dx &= \int_{h(\Omega)} (\varphi \circ h^{-1})(y) \int_{h(\Omega)} J(y-w)(u \circ h^{-1})(w) dw dy \\ &= \int_{\Omega} \left(\int_{h(\Omega)} J(h(z)-y)(\varphi \circ h^{-1})(y) dy \right) u(z) |\det(Dh(z))| dz \\ &= \int_{\Omega} h^*\mathcal{J}_{h(\Omega)}h^{*-1}\varphi(z) u(z) |\det(Dh(z))| dz. \end{aligned}$$

Consequently, if we change the measure of $L^2(\Omega)$ taking

$$\hat{L}^2(\Omega) = \left\{ u : \Omega \mapsto \mathbb{R} : \int_{\Omega} u^2(x) |\det(Dh(x))| dx < \infty \right\}$$

we have that $h^* \mathcal{J}_{h(\Omega)} h^{*-1} : \hat{L}^2(\Omega) \mapsto \hat{L}^2(\Omega)$ is a compact self-adjoint operator in $\hat{L}^2(\Omega)$. As h is an imbedding, there exists $c > 0$ such that $|\det(Dh)| \geq c > 0$ in Ω , and then, $\hat{L}^2(\Omega)$ is well defined. Thus, we can conclude that $\sigma(h^* \mathcal{J}_{h(\Omega)} h^{*-1}) \subset \mathbb{R}$ for any imbedding $h : \Omega \mapsto \mathbb{R}$.

We have the following result:

PROPOSITION 4.1. *Let $h : \Omega \mapsto \mathbb{R}$ be an imbedding. Then, $\mu \in \mathbb{R}$ is an eigenvalue of $h^* \mathcal{J}_{h(\Omega)} h^{*-1}$, if and only if, is an eigenvalue for $\mathcal{J}_{h(\Omega)}$.*

Proof. Indeed, it follows from (4.17) that

$$h^* \mathcal{J}_{h(\Omega)} h^{*-1} u(x) = \mu u(x), \quad x \in \Omega,$$

if and only if,

$$\mathcal{J}_{h(\Omega)} v(y) = \mu v(y), \quad y \in h(\Omega),$$

for $v(y) = (u \circ h^{-1})(y)$ with $y \in h(\Omega)$. Also, since $h^{*-1} : L^2(\Omega) \mapsto L^2(h(\Omega))$ is an isomorphism, $u \neq 0$, if and only if, $v \neq 0$. $\square$

Now, let us study differentiability properties of simple eigenvalues $\mu_{h(\Omega)}$ of $\mathcal{J}_{h(\Omega)}$ with respect to h . For this, we denote by $\text{Diff}^1(\Omega) \subset \mathcal{C}^1(\Omega, \mathbb{R}^N)$ the set of $\mathcal{C}^1$ -functions $h : \Omega \mapsto \mathbb{R}$ which are imbeddings considering the map

$$F : \text{Diff}^1(\Omega) \times \mathbb{R} \times L^2(\Omega) \mapsto L^2(\Omega) \times \mathbb{R}$$

$$(h, \mu, u) \mapsto \left((h^* \mathcal{J}_{h(\Omega)} h^{*-1} - \mu)u, \int_{\Omega} u^2(x) |\det(Dh(x))| dx \right).$$

It is not difficult to see that $\text{Diff}^1(\Omega)$ is an open set of $\mathcal{C}^1(\Omega, \mathbb{R}^N)$ which denotes the space of $\mathcal{C}^1$ -functions from Ω into $\mathbb{R}^N$ whose derivatives extend continuously to the closure $\bar{\Omega}$ with the usual supremum norm. Hence, F can be seen as a map defined between Banach spaces.

Note, if $\mu_0 \in \mathbb{R}$ is an eigenvalue for $\mathcal{J}_{\Omega}$ for some $u_0 \in L^2(\Omega)$ with $\int_{\Omega} u_0^2(x) dx = 1$, then $F(i_{\Omega}, \mu_0, u_0) = (0, 1)$ where $i_{\Omega} \in \text{Diff}^1(\Omega)$ denotes the inclusion map of Ω into $\mathbb{R}^N$. On the other side, whenever $F(h, \mu, u) = (0, 1)$, we have from Proposition 4.1 that

$$\mathcal{J}_{h(\Omega)} v(y) = \mu v(y), \quad y \in h(\Omega), \quad \text{with} \quad \int_{h(\Omega)} v^2(y) dy = 1,$$

where $v(y) = (u \circ h^{-1})(y)$ for $y \in h(\Omega)$. In this way, we can use the map F to deal with eigenvalues and eigenfunctions of $\mathcal{J}_{h(\Omega)}$ and $h^* \mathcal{J}_{h(\Omega)} h^{*-1}$ perturbing the eigenvalue problem to the fixed domain Ω by diffeomorphisms h .

LEMMA 4.1. *Let μ_0 be a simple eigenvalue for $\mathcal{J}_{\Omega}$ with corresponding normalized eigenfunction u_0 and $J \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R})$ satisfying (H). Then, there exists a neighbourhood $\mathcal{V}$ of inclusion $i_{\Omega} \in \text{Diff}^1(\Omega)$, and $\mathcal{C}^1$ -functions u_h and μ_h from $\mathcal{V}$ into $L^2(\Omega)$ and $\mathbb{R}$, respectively, satisfying*

$$h^* \mathcal{J}_{h(\Omega)} h^{*-1} u_h(x) = \mu_h u_h(x), \quad x \in \Omega,$$

with $u_h \in \mathcal{C}^1(\Omega)$ for all $h \in \mathcal{V}$.

Moreover, μ_h is a simple eigenvalue with $(\mu_{i_\Omega}, u_{i_\Omega}) = (\mu_0, u_0)$ and domain derivative

$$\frac{\partial \mu}{\partial h}(i_\Omega) \cdot V = \mu_0 \int_{\partial\Omega} u_0^2 V \cdot N_\Omega dS \quad \forall V \in \mathcal{C}^1(\Omega, \mathbb{R}^N).$$

Proof. Under the additional condition $J \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R})$, we get from [14] that the map F is a $\mathcal{C}^1$ -function between Banach spaces (see also [18, Chapter 2]). In fact, F is linear with respect to the variables $\mu \in \mathbb{R}$ and $u \in L^2(\Omega)$. Also, it is of class $\mathcal{C}^1$ with respect to h , since expressions

$$\begin{aligned} h^* \mathcal{J}_{h(\Omega)} h^{*-1} u(x) &= \int_{h(\Omega)} J(h(x) - w)(u \circ h^{-1})(w) dw \\ &= \int_\Omega J(h(x) - h(z))u(z)|\det(Dh(z))|dz, \quad x \in \Omega, \end{aligned} \quad (4.18)$$

and $\int_\Omega u^2(x)|\det(Dh(x))|dx$ are set by compositions among smooth functions J , $\det$ and h which define $\mathcal{C}^1$ -maps in the variable $h \in \text{Diff}^1(\Omega)$.

Next, since μ_0 is a simple eigenvalue with $F(i_\Omega, \mu_0, u_0) = (0, 1)$, we are in condition to apply Implicit Function Theorem to F at $(i_\Omega, \mu_0, u_0) \in \text{Diff}^1(\Omega) \times \mathbb{R} \times L^2(\Omega)$. First, we see

$$\begin{aligned} \frac{\partial F}{\partial(\mu, u)}(i_\Omega, \mu_0, u_0) : \mathbb{R} \times L^2(\Omega) &\mapsto L^2(\Omega) \times \mathbb{R} \\ (\dot{\mu}, \dot{u}) &\mapsto \left((\mathcal{J}_\Omega - \mu_0)\dot{u} + \dot{\mu}u_0, 2 \int_\Omega u_0 \dot{u} dx \right) \end{aligned}$$

is an isomorphism. In fact, since μ_0 is a simple eigenvalue, its eigenfunction u_0 is orthogonal to the image of the operator $(\mathcal{J}_\Omega - \mu_0)$ satisfying $L^2(\Omega) = \text{R}(\mathcal{J}_\Omega - \mu_0) \oplus [u_0]$.

Thus, for any $f \in L^2(\Omega)$, there exists a unique $w \in \text{R}(\mathcal{J}_{h(\Omega)} - \mu_0)$ such that

$$(\mathcal{J}_\Omega - \mu_0)w = f - \dot{\mu}u_0 \quad \text{with} \quad \dot{\mu} = \int_\Omega f u_0$$

since for such $\dot{\mu}$, $f - \dot{\mu}u_0$ is orthogonal to u_0 in $L^2(\Omega)$ belonging to $\text{R}(\mathcal{J}_\Omega - \mu_0)$. Consequently, for all $(f, a) \in L^2(\Omega) \times \mathbb{R}$, we can take unique $\dot{u} = w + \frac{a}{2}u_0$ and $\dot{\mu} = \int_\Omega f u_0$ such that

$$\frac{\partial F}{\partial(\mu, u)}(i_\Omega, \mu_0, u_0)(\dot{\mu}, \dot{u}) = (f, a).$$

Therefore, by the Implicit Function Theorem, there exist $\mathcal{C}^1$ -functions $h \mapsto (\mu_h, u_h)$ such that $F(h, \mu_h, u_h) = (0, 1)$ whenever $\|h - i_\Omega\|_{\mathcal{C}^1(\Omega, \mathbb{R}^N)}$ is sufficiently small. Thus, we have a family of simple eigenvalues μ_h and corresponding eigenfunctions $v_h = (u_h \circ h^{-1})$ for $\mathcal{J}_{h(\Omega)}$ defined by any h in a neighbourhood of $i_\Omega \in \text{Diff}^1(\Omega)$ which is still differentiable with respect to h .

Finally, we compute the derivative of μ_h at $h = i_\Omega$. For this, it is enough to consider a curve of imbeddings $h(t, x) = x + tV(x)$ for a fixed $V \in \mathcal{C}^1(\Omega, \mathbb{R}^N)$ taking the Gateaux derivative at $t = 0$.

Note that

$$h(t)^* \mathcal{J}_{h(t, \Omega)} h(t)^{*-1} u_{h(t)}(x) = \mu_{h(t)} u_{h(t)}, \quad x \in \Omega,$$

and then,

$$\frac{\partial}{\partial t} \left(h(t)^* \mathcal{J}_{h(t, \Omega)} h(t)^{*-1} u_{h(t)}(x) \right) \Big|_{t=0} = \frac{\partial \mu_{i_\Omega}}{\partial t} u_0 + \mu_0 \frac{\partial u_{i_\Omega}}{\partial t} \quad \text{in } \Omega. \quad (4.19)$$

Thus, in order to complete our proof, we need to compute the derivative of the left side of (4.19). We proceed as in [18] using the anti-convective derivative D_t in the reference region Ω

$$D_t = \frac{\partial}{\partial t} - U(t, x) \cdot \frac{\partial}{\partial x} \quad \text{with} \quad U = \frac{\partial h^{-1}}{\partial x} \frac{\partial h}{\partial t}.$$

By [18, Lemma 2.1], we have

$$D_t \left(h(t)^* \mathcal{J}_{h(t), \Omega} h(t)^{* -1} u_{h(t)} \right) = h(t)^* \frac{\partial}{\partial t} \left(\mathcal{J}_{h(t), \Omega} h(t)^{* -1} u_{h(t)} \right) \quad \text{in } \Omega. \quad (4.20)$$

Now, set $v(t, y) = h(t)^{* -1} u_{h(t)}(y) = u_{h(t)}(h^{-1}(t, y))$, $y \in h(t, \Omega)$. Then, from (4.17), we get

$$\begin{aligned} \frac{\partial}{\partial t} \left(\mathcal{J}_{h(t), \Omega} h(t)^{* -1} u_{h(t)} \right) \Big|_{t=0} &= \frac{\partial}{\partial t} \left(\mathcal{J}_{h(t), \Omega} v \right) \Big|_{t=0} \\ &= \frac{\partial}{\partial t} \left(\int_{h(t), \Omega} J(y - w) v(t, w) dw \right) \Big|_{t=0} \quad \text{for } y \in h(t, \Omega). \end{aligned}$$

Due to [18, Theorem 1.11], we can compute domain derivatives for integrals obtaining

$$\begin{aligned} \frac{\partial}{\partial t} \left(\mathcal{J}_{h(t), \Omega} h(t)^{* -1} u_{h(t)} \right) \Big|_{t=0} &= \frac{\partial}{\partial t} \left(\mathcal{J}_{h(t), \Omega} v \right) \Big|_{t=0} \\ &= \int_{\Omega} J(x - w) (D_t u)(0, w) dw + \int_{\partial \Omega} J(x - z) u_0(z) (V \cdot N_{\Omega})(z) dS(z) \end{aligned}$$

where N_{Ω} is the unitary normal vector to $\partial \Omega$.

Note that the last integral on $\partial \Omega$ is well defined. Since J is $\mathcal{C}^1$, the eigenfunctions u_h and their derivatives can be continuously extended to the border $\partial \Omega$. Thus, $u_h \in \mathcal{C}^1(\Omega)$, and we can take the trace of u_h on $\partial \Omega$.

Consequently, from (4.19) and (4.20), we get

$$\begin{aligned} &\frac{\partial \mu_{i_{\Omega}}}{\partial t} u_0 + \mu_0 \frac{\partial u_{i_{\Omega}}}{\partial t} \\ &= \left[U(t, x) \cdot \frac{\partial}{\partial x} \left(h(t)^* \mathcal{J}_{h(t), \Omega} h(t)^{* -1} u_{h(t)} \right) + h(t)^* \frac{\partial}{\partial t} \left(\mathcal{J}_{h(t), \Omega} h(t)^{* -1} u_{h(t)} \right) \right] \Big|_{t=0} \\ &= V \cdot \frac{\partial}{\partial x} (\mathcal{J}_{\Omega} u_0) + \mathcal{J}_{\Omega} (D_t u_{h(t)}|_{t=0}) + \int_{\partial \Omega} J(\cdot - z) u_0(z) (V \cdot N_{\Omega})(z) dS(z) \quad \text{in } \Omega. \end{aligned}$$

Hence, multiplying by u_0 and integrating on Ω , we obtain

$$\begin{aligned} \frac{\partial \mu_{i_{\Omega}}}{\partial t} + \int_{\Omega} \mu_0 u_0 \frac{\partial u_{i_{\Omega}}}{\partial t} dx &= \int_{\Omega} \mu_0 u_0 \left(V \cdot \nabla u_0 + \frac{\partial u_{i_{\Omega}}}{\partial t} - V \cdot \nabla u_0 \right) dx \\ &\quad + \int_{\partial \Omega} \left(\int_{\Omega} J(x - z) u_0(x) dx \right) u_0(z) (V \cdot N_{\Omega})(z) dS(z), \end{aligned}$$

which implies

$$\frac{\partial \mu_{i_{\Omega}}}{\partial t} = \mu_0 \int_{\partial \Omega} u_0^2(z) (V \cdot N_{\Omega})(z) dS(z)$$

completing the proof. $\square$

Therefore, as a direct consequence of Lemma 4.1 and items (a) and (c) from Remark 2.4, we get Theorem 1.2 concerning the Dirichlet problem (1.1).

REMARK 4.1. From Corollary 2.1, we know $\lambda_1(\Omega^*)$ is simple, and a critical point to the map

$$h \in \text{Diff}^1(\Omega^*) \mapsto (\lambda_1(h(\Omega^*)), |h(\Omega^*)| = |\Omega^*|).$$

Hence, from Theorem 1.2

$$0 = \int_{\partial\Omega^*} u_1^2 V \cdot N_{\Omega^*} dS \quad \text{for all } V \in \mathcal{C}^1(\Omega^*, \mathbb{R}^N) \text{ such that } \int_{\partial\Omega^*} V \cdot N_{\Omega^*} = 0.$$

Therefore, the first eigenfunction u_1 associated to $\lambda_1(\Omega^*)$ satisfies the boundary condition $u_1(x) = c$ on $\partial\Omega$ for some constant $c \geq 0$.

REMARK 4.2. Finally, let us give an example which shows that in general, the first eigenvalue $\lambda_1(\Omega)$ of (1.1) does not possess a maximizer among open bounded sets with $|\Omega| = \text{constant}$.

For this, let $h : (0, 1)^2 \mapsto (0, a) \times (0, 1/a) \subset \mathbb{R}^2$ be the imbedding $h(x_1, x_2) = (ax_1, (1/a)x_2)$ for any $a > 0$. Note that $\det(Dh) = 1$ and $|h((0, 1)^2)| = 1$ for all a . Also, from (4.18) we have

$$h^* \mathcal{J}_{h((0,1)^2)} h^{*-1} u(x) = \int_{(0,1)^2} J(a(x_1 - y_1), (1/a)(x_2 - y_2)) u(y) dy, \quad \forall x \in (0, 1)^2.$$

Hence, since $J(x) \rightarrow 0$ as $|x| \rightarrow +\infty$ by hypothesis (H), we obtain that $h^* \mathcal{J}_{h((0,1)^2)} h^{*-1} u(x) \rightarrow 0$ as $a \rightarrow 0$, for all $x \in (0, 1)^2$ and $u \in L^2(\Omega)$. Therefore, one can get from Proposition 4.1 and Remark 2.3 that $\mu_1(h((0, 1)^2)) \rightarrow 0$ as $a \rightarrow 0$ implying that $\lambda_1(h((0, 1)^2)) \rightarrow 1$ as $a \rightarrow 0$. As $1 \in \sigma_{\text{ess}}(\mathcal{B}_\Omega)$ for any open set Ω , we conclude our assertion.

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