Chapter 1
Thermodynamic Game and The Kac Limit in Quantum Lattices

J.-B. Bru, W. de Siqueira Pedra, K. Rodrigues Alves

Abstract A mathematically rigorous computation of the pressure and equilibrium states of important short-range quantum models on lattices (like the Hubbard model) to show possible phase transitions is generally elusive, beyond perturbative arguments, even after decades of mathematical studies. By contrast, such a question can be solved for mean-field models. This is done by using some form of the Bogoliubov approximation, leading to the thermodynamic game introduced in [1]. Here we illustrate this abstract result on a specific, albeit still general, example. We then state recent results contributing a precise mathematical relation between mean-field and short-range models via the long-range limit that is known in the literature as the Kac limit. This paves the way for studying phase transitions, or at least important fingerprints of them like strong correlations at long distances, for models having interactions whose ranges are finite, but very large as compared to the lattice constant. It also sheds a new light on mean-field models. If both attractive and repulsive long-range forces are present then it turns out that the limit mean-field model is not necessarily what one traditionally guesses.

Contents

1.1 Introduction

Realistic effective interparticle interactions of quantum many-body systems are widely seen as being short-range, not mean-field. However, the rigorous mathematical analysis of phase diagrams of short-range models turns out to be extremely difficult, in general, with many important fundamental questions remaining open still nowadays. By contrast, mean-field models come from different approximations or Ansätze, and are thus less realistic, in a sense, but are technically advantageous, while capturing surprisingly well many real physical phenomena. In fact, the study of phase diagrams of mean-field models can be performed by using what we call the thermodynamic game, as well as the corresponding self-consistency equations, as proven in [1]. This is explained in Section 1.3.3.

Then, we discuss a precise mathematical relation between mean-field and short-range models, by using the long-range limit that is known in the literature as the Kac limit. This relation was recently established [2] in an abstract, model-independent, way. To be more pedagogical, however, we restrict our discussions to a specific, albeit still general, example. This gives us the opportunity to illustrate unconventional results of [1, 2], for a mean-field model with both repulsive and attractive mean-field interactions.

The paper is organized as follows: Section 1.2 presents the general mathematical framework, while Section 1.3 explains our pedagogical example. Finally, some historical observations on the Bogoliubov approximation, the thermodynamic game and the Kac limit are given in Section 1.4.

Remark 1 (Quantum spin systems)
Our pedagogical example refers to lattice fermion systems, which are, from a technical point of view, slightly more difficult than quantum spin systems, because of a non-commutativity issue at different lattice sites. However, all the results presented in [1, 2] hold true for quantum spin systems, via obvious modifications.

Remark 2 (Periodic quantum lattice systems)
Our study focuses on lattice fermion systems that are translation-invariant (in space). However, all the results presented in [1, 2] hold true for (space-)periodic lattice fermion systems, by appropriately redefining the spin set[1]. A similar argument holds true for quantum spin systems.

1.2 Algebraic Formulation of Lattice Fermion Systems

1.2.1 Background Lattice

Fix once and for all the dimension $d \in \mathbb{N}$ of the (cubic) lattice $\mathbb{Z}^d$. In order to define the thermodynamic limit, we use the cubic boxes

$$\Lambda_\ell \doteq \{(x_1, \ldots, x_d) \in \mathbb{Z}^d : |x_1|, \ldots, |x_d| \leq \ell\} , \qquad \ell \in \mathbb{N} , \tag{1.1}$$

as a so-called van Hove sequence.

1.2.2 The CAR C^*-Algebra

$\mathcal{U}$ denotes the universal unital C^*-algebra generated by elements $\{a_{x,\mathrm{s}}\}_{x\in\mathbb{Z}^d,\mathrm{s}\in\mathrm{S}}$ satisfying the canonical anti-commutation relations (CAR):

$$\begin{cases} a_{x,\mathrm{s}}a_{y,\mathrm{t}} + a_{y,\mathrm{t}}a_{x,\mathrm{s}} = 0 \\ a_{x,\mathrm{s}}^*a_{y,\mathrm{t}} + a_{y,\mathrm{t}}a_{x,\mathrm{s}}^* = \delta_{x,y}\delta_{\mathrm{s},\mathrm{t}}\mathbf{1} \end{cases} , \qquad x, y \in \mathbb{Z}^d,\ \mathrm{s}, \mathrm{t} \in \mathrm{S} , \tag{1.2}$$

where $\mathbf{1}$ stands for the unit of the algebra, $\delta_{\cdot,\cdot}$ is the Kronecker delta and S is some finite set (representing an orthonormal basis of spin modes), which is fixed once and for all to be $\mathrm{S} \doteq \{\uparrow, \downarrow\}$, in light of the example discussed below. Observe that the CAR C^*-algebra $\mathcal{U}$ is separable. In fact, the unital C^*-algebra $\mathcal{U}$ is a so-called approximately finite-dimensional (AF) C^*-algebra, i.e., it is generated by an increasing family of finite-dimensional C^*-subalgebras.

[1] In fact, one can see the lattice points in a (space) period as a single point in an equivalent lattice on which particles have an enlarged spin set.

1.2.3 States of Lattice Fermion Systems

States on the C^*-algebra $\mathcal{U}$ are, by definition, linear functionals $\rho : \mathcal{U} \to \mathbb{C}$ which are positive, i.e., for all elements $A \in \mathcal{U}$, $\rho(|A|^2) \geq 0$, and normalized, i.e., $\rho(\mathfrak{1}) = 1$. Equivalently, the linear functional ρ is a state iff $\rho(\mathfrak{1}) = 1$ and $\|\rho\|_{\mathcal{U}^*} = 1$. See, e.g., [3, Section 2.3]. The set of all states on $\mathcal{U}$ is denoted by

$$E \doteq \bigcap_{A \in \mathcal{U}} \{\rho \in \mathcal{U}^* : \rho(\mathfrak{1}) = 1,\ \rho(|A|^2) \geq 0\} . \tag{1.3}$$

Here, for any $A \in \mathcal{U}$, $|A|^2$ denotes the positive product $A^*A \in \mathcal{U}$. The convex set E is compact with respect to the weak* topology. Additionally, since $\mathcal{U}$ is separable, it is metrizable. See, e.g., [4].

1.2.4 Translation-Invariant States

Lattice translations refer to the group homomorphism $x \mapsto \alpha_x$ from $(\mathbb{Z}^d, +)$ to the group of $*$-automorphisms of the CAR C^*-algebra $\mathcal{U}$ of the (infinite) lattice $\mathbb{Z}^d$, which is uniquely defined by the condition

$$\alpha_x(a_{y,\mathrm{s}}) = a_{y+x,\mathrm{s}} , \quad y \in \mathbb{Z}^d,\ \mathrm{s} \in \mathrm{S} . \tag{1.4}$$

Via this group homomorphism we define the translation invariance of states: The state $\rho \in E$ is said to be translation-invariant iff it satisfies $\rho \circ \alpha_x = \rho$ for all $x \in \mathbb{Z}^d$. The space of translation-invariant states on $\mathcal{U}$ is the convex set

$$E_1 \doteq \bigcap_{x \in \mathbb{Z}^d,\, A \in \mathcal{U}} \{\rho \in \mathcal{U}^* : \rho(\mathfrak{1}) = 1,\ \rho(|A|^2) \geq 0,\ \rho = \rho \circ \alpha_x\} , \tag{1.5}$$

which is again metrizable and compact with respect to the weak* topology. Even if the structure of the set of translation-invariant states is not really explicitly used below, it is still instructive to say a few words about E_1 since some of its properties are pivotal for all the theory described in the rest of the paper.

For instance, thanks to the Krein-Milman theorem [4, Theorem 3.23], E_1 is the weak*-closure of the convex hull of the (non-empty) set $\mathcal{E}(E_1)$ of its extreme points, which turns out to be a weak*-dense (G_δ) subset [1, Corollary 4.6]:

$$E_1 = \overline{\mathrm{co}}\,(\mathcal{E}(E_1)) = \overline{\mathcal{E}(E_1)} , \tag{1.6}$$

where $\overline{\mathrm{co}}(K)$ denotes the weak*-closed convex hull of a set K. This fact is well-known and is also true for quantum spin systems on lattices [3, Example 4.3.26 and discussions p. 464]. Meanwhile, since E_1 is metrizable (because $\mathcal{U}$ is separable), Choquet's theorem applies: By [1, Theorem 1.9], for any $\rho \in E_1$, there is a unique probability measure[2] μ_ρ on E_1 such that $\mu_\rho(\mathcal{E}(E_1)) = 1$ and

$$\rho(A) = \int_{E_1} \hat{\rho}(A)\, \mu_\rho(\mathrm{d}\hat{\rho}) , \qquad A \in \mathcal{U} . \tag{1.7}$$

In particular, E_1 is a Choquet simplex. In fact, up to an affine homeomorphism, E_1 is the so-called Poulsen simplex [1, Theorem 1.12]. (Note in passing that μ_ρ is an orthogonal measure, thanks to [5, Theorem 5.1].)

The unique decomposition of a translation-invariant state $\rho \in E_1$ in terms of extreme translation-invariant states $\hat{\rho} \in \mathcal{E}(E_1)$ is called the *ergodic* decomposition of ρ because of the following fact: Define the space-averages of any element $A \in \mathcal{U}$ by

$$A_\ell \doteq \frac{1}{|\Lambda_\ell|} \sum_{x \in \Lambda_\ell} \alpha_x(A) , \qquad \ell \in \mathbb{N}_0 . \tag{1.8}$$

Then, by definition, a translation-invariant state $\hat{\rho} \in E_1$ is said to be *ergodic* if

$$\lim_{\ell \to \infty} \hat{\rho}(|A_\ell|^2) = \lim_{\ell \to \infty} \frac{1}{|\Lambda_\ell|^2} \sum_{x,y \in \Lambda_\ell} \rho(\alpha_y(A^*)\, \alpha_x(A)) = |\hat{\rho}(A)|^2 , \qquad A \in \mathcal{U} . \tag{1.9}$$

[2] For E is a metrizable compact space, any finite Borel measure is regular and tight. Thus, here, probability measures are just the same as normalized Borel measures.

By [1, Theorem 1.16], any extreme translation-invariant state is ergodic and vice-versa. In other words, the (weak*-dense) set of extreme translation-invariant states is equal to

$$\mathcal{E}(E_1) = \{\hat{\rho} \in E_1 : \hat{\rho} \text{ is ergodic}\} = \bigcap_{A \in \mathcal{U}} \left\{ \hat{\rho} \in E_1 : \lim_{\ell \to \infty} \hat{\rho}(|A_\ell|^2) = |\hat{\rho}(A)|^2 \right\} . \tag{1.10}$$

This last observation is not explicitly used in this paper, but it highlights the use of the space-averaging functionals $\Delta_\pm : E_1 \to \mathbb{R}$ defined by Equations (1.18)–(1.19), which are key objects for the study of thermodynamic properties of mean-field models.

1.3 From Short-Range to Mean-Field Models

1.3.1 The Short-Range Model

For two parameters $\gamma_-, \gamma_+ \in (0,1)$ and $\ell \in \mathbb{N}$, we define the translation-invariant local Hamiltonians

$$H_{\Lambda_\ell}(\gamma_-, \gamma_+) \doteq T_{\Lambda_\ell} - H_{\Lambda_\ell,-} + H_{\Lambda_\ell,+} , \tag{1.11}$$

where

$$\begin{aligned} T_{\Lambda_\ell} &\doteq \sum_{x,y \in \Lambda_\ell,\ \mathrm{s} \in \{\uparrow,\downarrow\}} \varepsilon(x-y)\, a_{x,\mathrm{s}}^* a_{y,\mathrm{s}} , \\ H_{\Lambda_\ell,-} &\doteq \sum_{x,y \in \Lambda_\ell} \gamma_-^d f_-(\gamma_-(x-y))\, a_{y,\uparrow}^* a_{y,\downarrow}^* a_{x,\downarrow} a_{x,\uparrow} , \\ H_{\Lambda_\ell,+} &\doteq \sum_{x,y \in \Lambda_\ell, \mathrm{s},\mathrm{t} \in \{\uparrow,\downarrow\}} \gamma_+^d f_+(\gamma_+(x-y))\, a_{y,\mathrm{t}}^* a_{y,\mathrm{t}} a_{x,\mathrm{s}}^* a_{x,\mathrm{s}} . \end{aligned}$$

Here, ε is some reflection-symmetric real-valued function on $\mathbb{Z}^d$, i.e.,

$$\varepsilon(-z) = \varepsilon(z) \in \mathbb{R} , \qquad z \in \mathbb{Z}^d .$$

It represents the kinetic part of the model. Usually, $\varepsilon(x) = v(|x|)$ for some function $v : \mathbb{R}_0^+ \to \mathbb{R}$. Instead, one could have used the condition

$$\varepsilon(-z) = \overline{\varepsilon(z)} \in \mathbb{C} , \qquad z \in \mathbb{Z}^d,$$

on the one-particle hopping strength with minor changes. This slightly more general situation allows, for instance, for an external magnetic potential in the model. For simplicity we stick to the real case and assume additionally that ε is finitely supported. This encompasses the case of the discrete Laplacian, which corresponds to the choice

$$\varepsilon(z) = \begin{cases} 0 & \text{for } |z| > 1 \\ -1 & \text{for } |z| = 1 \\ 2d & \text{for } z = 0 \end{cases} , \qquad z \in \mathbb{Z}^d .$$

The reflection-symmetric real-valued function f_+ is a (non-zero) pair potential encoding interparticle forces, whose range is tuned by the parameter $\gamma_+ \in (0,1)$. The (non-zero) reflection-symmetric real-valued function f_- encodes the hopping strength of Cooper pairs. This model thus implements a BCS interaction whose range is tuned by the parameter $\gamma_- \in (0,1)$. Similar to the one-particle hopping strength ε, with minor modifications in order to include external magnetic forces, one could have used a complex-valued function f_- satisfying

$$f_-(-z) = \overline{f_-(z)} \in \mathbb{C} , \qquad z \in \mathbb{Z}^d .$$

Again for simplicity we stick to the real case. As is usual in theoretical physics, f_-, f_+ are assumed to be fast decaying and positive definite, i.e., the Fourier transforms $\hat{f}_-, \hat{f}_+$ of f_-, f_+ are positive functions on $\mathbb{R}^d$. This choice for f_+ is reminiscent of a superstability condition, which is essential in the bosonic case [6, Section 2.2 and Appendix G]. For simplicity, we assume that $f_-, f_+ \in C_0^{2d}(\mathbb{R}^d, \mathbb{R})$, that is, they are both compactly supported and sufficiently regular. For technical reasons, we also assume that the Fourier transform of the Cooper pair hopping strength is scaling monotone, in the sense that

$$\hat{f}_-(\gamma^{-1}k) \leq \hat{f}_-(k) \ , \qquad k \in \mathbb{R}^d \ , \ \gamma\,(0,1) \ .$$

The definition of the Fourier transform of an integrable function $f : \mathbb{R}^d \to \mathbb{C}$ we use here is

$$\hat{f}\,(k) \doteq \int_{\mathbb{R}^d} f\,(x)\,\mathrm{e}^{-ik\cdot x}\mathrm{d}^d x \ , \qquad k \in \mathbb{R}^d \ . \tag{1.12}$$

By [1, Theorem 2.12], the pressure of the model can be represented in the thermodynamic limit as a variational problem over translation-invariant states: First, for arbitrary parameters $\gamma_-, \gamma_+ \in (0,1)$, the energy density functional

$$\mathfrak{e}(\gamma_-,\gamma_+) : E_1 \to \mathbb{R}$$

is defined by

$$\mathfrak{e}(\gamma_-,\gamma_+)\,(\rho) \doteq \lim_{\ell\to\infty} \frac{1}{|\Lambda_\ell|}\rho\left(H_{\Lambda_\ell}\left(\gamma_-,\gamma_+\right)\right) < \infty$$

for any translation-invariant state $\rho \in E_1$. This limit exists, thanks to [1, Definition 1.31 and Lemma 1.32]. It can be split into three parts:

$$\mathfrak{e}(\gamma_-,\gamma_+) = \underbrace{\mathfrak{e}_+}_{\text{repulsive interaction term } (+)} - \underbrace{\mathfrak{e}_-}_{\text{attractive interaction term } (-)} + \underbrace{\mathfrak{e}_0}_{\text{kinetic term}} \ , \tag{1.13}$$

where, for any translation-invariant state $\rho \in E_1$,

$$\mathfrak{e}_\pm\,(\rho) \doteq \lim_{\ell\to\infty} \frac{1}{|\Lambda_\ell|}\rho\left(H_{\Lambda_\ell,\pm}\right) < \infty \qquad \text{and} \qquad \mathfrak{e}_0\,(\rho) \doteq \lim_{\ell\to\infty} \frac{1}{|\Lambda_\ell|}\rho\left(T_{\Lambda_\ell}\right) < \infty. \tag{1.14}$$

See again [1, Definition 1.31 and Lemma 1.32].

Then, for any inverse temperature $\beta \in (0,\infty)$ and $\gamma_-, \gamma_+ \in (0,1)$, the free energy density functional $\mathfrak{f}_\beta(\gamma_-,\gamma_+) : E_1 \to \mathbb{R}$ is defined by

$$\mathfrak{f}_\beta(\gamma_-,\gamma_+) \doteq \mathfrak{e}(\gamma_-,\gamma_+) - \beta^{-1}\mathfrak{s} \ , \tag{1.15}$$

where $\mathfrak{s} : E_1 \to \mathbb{R}_0^+$ is the entropy density functional defined as the thermodynamic limit of the von Neumann entropy $S_\ell(\rho)$ per unit volume:

$$\mathfrak{s}\,(\rho) \doteq \lim_{\ell\to\infty} \frac{1}{|\Lambda_\ell|} S_\ell(\rho) \ , \qquad \rho \in E_1 \ .$$

For more details, see [1, Definition 1.28 and Lemma 1.29]. With these definitions, by [1, Theorem 2.12], the thermodynamic limit of the (grand-canonical) pressure equals

$$P_\beta\,(\gamma_-,\gamma_+) \doteq \lim_{\ell\to\infty} \frac{1}{\beta|\Lambda_\ell|} \ln \mathrm{Tr}(\mathrm{e}^{-\beta H_{\Lambda_\ell}(\gamma_-,\gamma_+)}) = -\inf \mathfrak{f}_\beta(\gamma_-,\gamma_+)\,(E_1) < \infty \tag{1.16}$$

for any inverse temperature $\beta \in (0,\infty)$ and parameters $\gamma_-, \gamma_+ \in (0,1)$. See also [7]. The free energy density functional $\mathfrak{f}_\beta(\gamma_-,\gamma_+)$ is weak*-lower semicontinuous [1, Lemmata 1.29 and 1.32] and the (space homogeneous) infinite volume equilibrium states of the short-range model are defined as being the minimizers of this functional. They form thus the set

$$\Omega_\beta\,(\gamma_-,\gamma_+) \doteq \{\omega \in E_1 : \mathfrak{f}_\beta(\gamma_-,\gamma_+)\,(\omega) = -P_\beta\,(\gamma_-,\gamma_+)\}$$

for any $\beta \in (0,\infty)$ and parameters $\gamma_-, \gamma_+ \in (0,1)$. By [1, Lemma 2.16], it is a (non-empty) convex weak*-compact subset of the space E_1 of translation-invariant states. (It is even a face of E_1.)

Last but not least, $\Omega_\beta(\gamma_-,\gamma_+)$ can be directly related to the limit of Gibbs states associated with the local Hamiltonians $H_{\Lambda_\ell}(\gamma_-,\gamma_+)$, $\ell \in \mathbb{N}$: The set of all states on $\mathcal{U}$ being weak*-compact, the sequence of Gibbs states of the local Hamiltonians, seen as periodic states on $\mathcal{U}$, has weak*-convergent subsequences. However, it is not clear that such limits always belong to the set E_1 of translation-invariant states. In fact, if a weak*-convergent sequence of Gibbs states has a translation-invariant state ω as its limit, then the state ω must belong to $\Omega_\beta(\gamma_-,\gamma_+)$. This condition can be ensured by imposing periodic boundary conditions, as explained in [1, Chapter 3]. In particular, in this case, the weak*-accumulation points of Gibbs states belong to $\Omega_\beta(\gamma_-,\gamma_+)$. See [1, Theorem 3.13].

1.3.2 The Mean-Field Model

The Kac, or long-range, limit refers to the limits $\gamma_\pm \to 0^+$ of the short-range model that is already in infinite volume, i.e., $\gamma_\pm \to 0^+$ after the thermodynamic limit $\ell \to \infty$. For small parameters $\gamma_\pm \ll 1$, the short-range model defined in finite volume by (1.11) has interparticle $(+)$ and BCS $(-)$ interactions whose ranges ($\mathcal{O}(\gamma_\pm^{-1})$) are very large as compared to lattice constant (here, $\mathcal{O}(1)$), but the interaction strength is small as $\gamma_\pm^d$, in such a way that the first Born approximation[3] to the scattering lengths of the interparticle and BCS potentials remain constant, as is usual. One therefore expects to have some effective mean-field, or long-range, model in the limits $\gamma_\pm \to 0^+$.

The effective local Hamiltonians for the long-range limit of the above short-range model should be

$$H_{\Lambda_\ell}^\sharp(\eta_-,\eta_+) \doteq T_{\Lambda_\ell} + \underbrace{\frac{\eta_+}{|\Lambda_\ell|}\sum_{x,y\in\Lambda_\ell,\mathrm{s},\mathrm{t}\in\{\uparrow,\downarrow\}} a_{y,\mathrm{t}}^* a_{y,\mathrm{t}} a_{x,\mathrm{s}}^* a_{x,\mathrm{s}}}_{\text{mean-field repulsion }(+)} - \underbrace{\frac{\eta_-}{|\Lambda_\ell|}\sum_{x,y\in\Lambda_\ell} a_{y,\uparrow}^* a_{y,\downarrow}^* a_{x,\downarrow} a_{x,\uparrow}}_{\text{mean-field attraction }(-)} \tag{1.17}$$

for all natural numbers $\ell \in \mathbb{N}$ and some positive parameters $\eta_-,\eta_+ \in \mathbb{R}_0^+$. Compare these local Hamiltonians with (1.11). They refer to a mean-field model and we can use the setting of [1]:

The space-averaging functionals $\Delta_\pm : E_1 \to \mathbb{R}$ associated with the mean-field repulsions $(+)$ and attractions $(-)$ are equal to

$$\Delta_\pm(\rho) \doteq \lim_{\ell\to\infty} \frac{1}{|\Lambda_\ell|^2} \sum_{x,y\in\Lambda_\ell} \rho\left(\alpha_y\left(A_\pm^*\right)\alpha_x\left(A_\pm\right)\right) \in \left[|\rho(A_\pm)|^2, \|A_\pm\|_{\mathcal{U}}^2\right] , \tag{1.18}$$

for any translation-invariant state $\rho \in E_1$, where

$$A_- \doteq a_{0,\downarrow}a_{0,\uparrow} \qquad \text{and} \qquad A_+ \doteq a_{0,\uparrow}^* a_{0,\uparrow} + a_{0,\downarrow}^* a_{0,\downarrow} . \tag{1.19}$$

Compare this functional with Equations (1.8)–(1.10) and recall the ergodic property of extreme points of the convex weak*-compact space E_1 of translation-invariant states. Note that the space-averaging functionals $\Delta_\pm$ are well-defined, thanks to [1, Lemma 4.10].

For any inverse temperature $\beta \in (0,\infty)$ and $\eta_-,\eta_+ \in \mathbb{R}_0^+$, the free energy density functional $\mathfrak{f}_\beta^\sharp(\eta_-,\eta_+) : E_1 \to \mathbb{R}$ is then defined by

$$\mathfrak{f}_\beta^\sharp(\eta_-,\eta_+) \doteq \eta_+\Delta_+ - \eta_-\Delta_- + \mathfrak{e}_0 - \beta^{-1}\mathfrak{s} . \tag{1.20}$$

Compare this definition with Equations (1.13)–(1.15), the corresponding one for the short-range model. Similar to the short-range case, by [1, Theorem 2.12], the thermodynamic limit of the (grand-canonical) pressure of the mean-field model equals

$$P_\beta^\sharp(\eta_-,\eta_+) \doteq \lim_{\ell\to\infty} \frac{1}{\beta|\Lambda_\ell|} \ln \mathrm{Tr}(\mathrm{e}^{-\beta H_{\Lambda_\ell}^\sharp(\eta_-,\eta_+)}) = -\inf \mathfrak{f}_\beta^\sharp(\eta_-,\eta_+)(E_1) < \infty \tag{1.21}$$

for any inverse temperature $\beta \in (0,\infty)$ and parameters $\eta_-,\eta_+ \in \mathbb{R}_0^+$.

The free energy density functional $\mathfrak{f}_\beta^\sharp(\eta_-,\eta_+)$ is generally **not** weak*-lower semicontinuous:

$$\mathfrak{f}_\beta^\sharp(\eta_-,\eta_+) = \underbrace{\eta_+\Delta_+}_{\text{upper semicont.}} + \underbrace{\left(-\eta_-\Delta_- + \mathfrak{e}_0 - \beta^{-1}\mathfrak{s}\right)}_{\text{lower semicont.}} .$$

See [1, Theorem 1.18, Lemmata 1.29 and 1.32]. The free energy density functional $\mathfrak{f}_\beta^\sharp(\eta_-,\eta_+)$ on the convex weak*-compact space E_1 of translation-invariant states has thus a **topological** drawback. In particular, in contrast with the short-range model, this functional does not necessarily have minimizers.

[3] I.e., $\int_{\mathbb{R}^d} \gamma_\pm^d f_\pm(\gamma_\pm x)\,\mathrm{d}x = \int_{\mathbb{R}^d} f_\pm(x)\,\mathrm{d}x \doteq \hat{f}_\pm(0)$.

Observe that the situation is much simpler in absence of mean-field repulsions ($\eta_+ = 0$), since in this special case, exactly as for the short-range model, one can define equilibrium states as the minimizers of $\mathfrak{f}^\sharp_\beta(\eta_-, 0)$, this functional being weak*-lower semicontinuous. In order to define the equilibrium states of the mean-field model for general parameters $\eta_-, \eta_+ \in \mathbb{R}^+_0$ we consider approximating minimizers of $\mathfrak{f}^\sharp_\beta(\eta_-, \eta_+)$ instead of strict ones:

$$\Omega^\sharp_\beta(\eta_-, \eta_+) \doteq \Big\{\omega \in E_1 : \exists (\rho_n)_{n\in\mathbb{N}} \subseteq E_1 \text{ weak}^* \text{ converging to } \omega \\ \text{so that } \lim_{n\to\infty} \mathfrak{f}^\sharp_\beta(\eta_-, \eta_+)(\rho_n) = -P^\sharp_\beta(\eta_-, \eta_+)\Big\}$$

for $\beta \in (0, \infty)$ and $\eta_-, \eta_+ \in \mathbb{R}^+_0$. Recall that the space E_1 of translation-invariant states is weak*-compact and metrizable and thus, any sequence $(\rho_n)_{n\in\mathbb{N}}$ of approximating minimizers converges along subsequences. By [1, Lemma 2.16], $\Omega^\sharp_\beta(\eta_-, \eta_+)$ is a (non-empty) convex weak*-compact subspace of E_1.

Again, $\Omega^\sharp_\beta(\eta_-, \eta_+)$ can be directly related to the limit of Gibbs states associated with the local Hamiltonians $H^\sharp_{\Lambda_\ell}(\eta_-, \eta_+)$, $\ell \in \mathbb{N}$: If a weak*-convergent subsequence of Gibbs states has a translation-invariant state ω as its limit, then the state ω must belong to $\Omega^\sharp_\beta(\eta_-, \eta_+)$. Exactly as in the short-range case, this condition can be ensured by imposing periodic boundary conditions, as explained in [1, Chapter 3]. In particular, in this case, the weak*-accumulation points of Gibbs states belong to $\Omega^\sharp_\beta(\eta_-, \eta_+)$. See again [1, Theorem 3.13].

1.3.3 Thermodynamic Game

A mathematically rigorous computation of the pressure and equilibrium states of the short-range model to show possible phase transitions is elusive, beyond perturbative arguments, even after decades of mathematical studies. By contrast, such a question can be solved for the corresponding mean-field model. In fact, in the mean-field case, one can use a suitable version of Bogoliubov's approximation method, in order to compute pressures and correlation functions (equilibrium states). It led in the eighties to the *approximating Hamiltonian method* [16–18], which involves in general a variational problem made of a supremum and an infimum to compute the infinite volume pressure. This method has been revisited for lattice fermions and strongly extended (even on the level of the pressure) in [1], where the view point of game theory is used by interpreting the mean-field attractions and repulsions of the mean-field model as players of a zero-sum game. This leads to the *thermodynamic game*, introduced in [1, Section 2.7]:

To define the payoff function of the thermodynamic game we introduce so-called approximating (or effective) local short-range Hamiltonians for the mean-field model:

$$\tilde{H}_{\Lambda_\ell}(\eta_-, \eta_+, c_-, c_+) \doteq T_{\Lambda_\ell} + \eta_+^{1/2}(\bar{c}_+ + c_+) \sum_{x\in\Lambda_\ell, \mathrm{s}\in\{\uparrow,\downarrow\}} a^*_{x,\mathrm{s}} a_{x,\mathrm{s}} \\ + \eta_-^{1/2} \sum_{x\in\Lambda_\ell} \left(\bar{c}_- a^*_{x,\uparrow} a^*_{x,\downarrow} + c_- a_{x,\downarrow} a_{x,\uparrow}\right) \tag{1.22}$$

for two complex numbers $c_-, c_+ \in \mathbb{C}$, natural numbers $\ell \in \mathbb{N}$ and some positive parameters $\eta_-, \eta_+ \in \mathbb{R}^+_0$. Notice that the approximating model is only indirectly related to the original short-range model from which the mean-field model is obtained via the long-range limit. Given an inverse temperature $\beta \in (0, \infty)$, we define $\tilde{P}_\beta : \mathbb{C}^2 \times (\mathbb{R}^+_0)^2 \to \mathbb{R}$ to be the function defined by the infinite volume pressure

$$\tilde{P}_\beta(c_-, c_+, \eta_+, \eta_-) \doteq \lim_{\ell\to\infty} \frac{1}{\beta|\Lambda_\ell|} \ln \mathrm{Tr}(\mathrm{e}^{-\beta \tilde{H}_{\Lambda_\ell}(\eta_-, \eta_+, c_-, c_+)}) < \infty ,$$

for $c_-, c_+ \in \mathbb{C}$ and $\eta_-, \eta_+ \in \mathbb{R}^+_0$, which exists, thanks to [1, Theorem 2.12]. Then, for fixed $\beta \in (0, \infty)$ and $\eta_-, \eta_+ \in \mathbb{R}^+_0$, the approximating free energy density $h_\beta : \mathbb{C}^2 \to \mathbb{R}$ is defined by

$$h_\beta(c_-, c_+) \doteq -|c_+|^2 + |c_-|^2 - \tilde{P}_\beta(c_-, c_+, \eta_+, \eta_-) , \qquad c_-, c_+ \in \mathbb{C} .$$

Given an inverse temperature $\beta \in (0, \infty)$ and fixed parameters $\eta_-, \eta_+ \in \mathbb{R}^+_0$, this function is seen as the payoff function of a two-person zero-sum game, the thermodynamic game associated with the mean-field model:

i.) The two players are denoted by $(-)$ and $(+)$. In fact, we interpret the mean-field attractions and repulsions as two players that we respectively call the attractive and repulsive players.

ii.) The value $h_{\beta}(c_{-},c_{+}) \in \mathbb{R}$ is the loss of the player $(-)$ making the decision c_{-} and the gain of the second one making the decision c_{+}:

$(-)$ Without exchange of information, by minimizing

$$h_{\beta}^{\sharp}(c_{-}) \doteq \sup_{c_{+}\in\mathbb{C}} h_{\beta}(c_{-},c_{+}) \ ,$$

the player $(-)$ obtains her/his least maximum loss

$$\mathrm{F}_{\beta}^{\sharp}(\eta_{-},\eta_{+}) \doteq \inf_{c_{-}\in\mathbb{C}} h_{\beta}^{\sharp}(c_{-}) \ .$$

$(+)$ By maximizing

$$h_{\beta}^{\flat}(c_{+}) \doteq \inf_{c_{-}\in\mathbb{C}} h_{\beta}(c_{-},c_{+}) \ ,$$

the player $(+)$ obtains her/his greatest minimum gain

$$\mathrm{F}_{\beta}^{\flat}(\eta_{-},\eta_{+}) \doteq \sup_{c_{+}\in\mathbb{C}} h_{\beta}^{\flat}(c_{+}) \leq \mathrm{F}_{\beta}^{\sharp}(\eta_{-},\eta_{+}) \ .$$

$\mathrm{F}_{\beta}^{\flat}(\eta_{-},\eta_{+})$ and $\mathrm{F}_{\beta}^{\sharp}(\eta_{-},\eta_{+})$ are called the conservative values of the thermodynamic game, while

$$[\mathrm{F}_{\beta}^{\flat}(\eta_{-},\eta_{+}),\mathrm{F}_{\beta}^{\sharp}(\eta_{-},\eta_{+})] \tag{1.23}$$

is its duality interval.

iii.) The corresponding sets of conservative strategies are

$$\begin{aligned} \mathcal{C}_{\beta}^{\flat}(\eta_{-},\eta_{+}) &\doteq \left\{ d_{+}\in\mathbb{C} : \mathrm{F}_{\beta}^{\flat}(\eta_{-},\eta_{+}) = h_{\beta}^{\flat}(d_{+}) \right\} \ , \\ \mathcal{C}_{\beta}^{\sharp}(\eta_{-},\eta_{+}) &\doteq \left\{ d_{-}\in\mathbb{C} : \mathrm{F}_{\beta}^{\sharp}(\eta_{-},\eta_{+}) = h_{\beta}^{\sharp}(d_{-}) \right\} \ . \end{aligned} \tag{1.24}$$

In the particular case of a purely repulsive mean-field model, i.e., when $\eta_{-}=0$, $\mathcal{C}_{\beta}^{\sharp}=\mathbb{C}$, as h_{β} is independent of c_{-}. Similarly, if $\eta_{+}=0$ then $\mathcal{C}_{\beta}^{\flat}=\mathbb{C}$. In both cases, we have

$$\mathrm{F}_{\beta}^{\flat}(0,\eta_{+}) = \mathrm{F}_{\beta}^{\sharp}(0,\eta_{+}) \qquad \text{and} \qquad \mathrm{F}_{\beta}^{\flat}(\eta_{-},0) = \mathrm{F}_{\beta}^{\sharp}(\eta_{-},0) \tag{1.25}$$

for $\eta_{-},\eta_{+}\in\mathbb{R}_{0}^{+}$. By [1, Lemma 8.4], the sets of conservatives strategies have the following important properties:

($\flat$) $\mathcal{C}_{\beta}^{\flat}(\eta_{-},\eta_{+}) \subseteq \mathbb{C}$ has exactly one element d_{+} when $\eta_{+}\in(0,\infty)$.
($\sharp$) $\mathcal{C}_{\beta}^{\sharp}(\eta_{-},\eta_{+}) \subseteq \mathbb{C}$ is non-empty and bounded when $\eta_{-}\in(0,\infty)$.

The relevance of the thermodynamic game results from the fact that the conservative values $\mathrm{F}_{\beta}^{\flat}(\eta_{-},\eta_{+})$ and $\mathrm{F}_{\beta}^{\sharp}(\eta_{-},\eta_{+})$ of the game can be written as *variational problems over states*, corresponding, in turn, to pressures in some long-range limit. To state the precise assertions, we start by recalling [1] two free energy functionals associated with the mean-field model at a given inverse temperature $\beta\in(0,\infty)$:

- The conventional free energy density functional $\mathfrak{f}_{\beta}^{\sharp}(\eta_{-},\eta_{+}) : E_{1}\rightarrow\mathbb{R}$ is defined by

$$\mathfrak{f}_{\beta}^{\sharp}(\eta_{-},\eta_{+}) = \underbrace{\eta_{+}\Delta_{+}}_{\text{affine upper semicont.}} + \underbrace{\left(-\eta_{-}\Delta_{-}+\mathfrak{e}_{0}-\beta^{-1}\mathfrak{s}\right)}_{\text{affine lower semicont.}} .$$

- The non-conventional free energy density functional $\mathfrak{f}_{\beta}^{\flat}(\eta_{-},\eta_{+}) : E_{1}\rightarrow\mathbb{R}$ is defined by

$$\mathfrak{f}_{\beta}^{\flat}(\eta_{-},\eta_{+})(\rho) \doteq \underbrace{\eta_{+}\left|\rho(a_{0,\uparrow}^{*}a_{0,\uparrow}+a_{0,\downarrow}^{*}a_{0,\downarrow})\right|^{2}}_{\text{convex cont.}} + \underbrace{\left(-\eta_{-}\Delta_{-}(\rho)+\mathfrak{e}_{0}(\rho)-\beta^{-1}\mathfrak{s}(\rho)\right)}_{\text{affine lower semicont.}}$$

for any translation-invariant state $\rho \in E_1$.

Note that $\mathfrak{f}_\beta^\flat(\eta_-,\eta_+) \leq \mathfrak{f}_\beta^\sharp(\eta_-,\eta_+)$, by [1, Equation (1.11)]. Then, by [1, Theorems 2.12 and 2.36], the following assertion holds true:

Theorem 1 *Fix an arbitrary reflection-symmetric finitely supported real-valued function ε on $\mathbb{Z}^d$. For any inverse temperature $\beta \in (0,\infty)$ and $\eta_-,\eta_+ \in \mathbb{R}_0^+$, the conservative values of the thermodynamic game equal:*

$$\mathrm{F}_\beta^\flat(\eta_-,\eta_+) \doteq \sup_{c_+\in\mathbb{C}} \inf_{c_-\in\mathbb{C}} h_\beta(c_-,c_+) = \inf \mathfrak{f}_\beta^\flat(\eta_-,\eta_+)(E_1) \doteq -P_\beta^\flat(\eta_-,\eta_+) \ ,$$
$$\mathrm{F}_\beta^\sharp(\eta_-,\eta_+) \doteq \inf_{c_-\in\mathbb{C}} \sup_{c_+\in\mathbb{C}} h_\beta(c_-,c_+) = \inf \mathfrak{f}_\beta^\sharp(\eta_-,\eta_+)(E_1) = -P_\beta^\sharp(\eta_-,\eta_+) \ .$$

We emphasize that $P_\beta^\flat$ is defined by the above infimum, in contrast with $P_\beta^\sharp$ which is, by definition, the thermodynamic limit of the pressure of the mean-field model, see (1.21).

Notice that the approximating Hamiltonians (1.22) are quadratic in the annihilation and creation operators. It can thus be diagonalized by a so-called Bogoliubov transformation and the pressures $P_\beta^\flat$ and $P_\beta^\sharp$ of Theorem 1, as well as the payoff function h_β of the thermodynamic game, can be analytically and numerically studied. Additionally, the sets $\Omega_\beta^\sharp(\eta_-,\eta_+)$ and

$$\Omega_\beta^\flat(\eta_-,\eta_+) \doteq \left\{\omega\in E_1 : \mathfrak{f}_\beta^\flat(\eta_-,\eta_+)(\omega) = \inf\ \mathfrak{f}_\beta^\flat(\eta_-,\eta_+)(E_1) \doteq -P_\beta^\flat(\eta_-,\eta_+)\right\}$$

of equilibrium states can be explicitly determined, thanks to [2, Theorem 4.3]. In fact, we obtain a (static) self-consistency condition for equilibrium states, which refers, in a sense, to Euler-Lagrange equations for the variational problems defining the conservative values of the thermodynamic game. In the physics literature on superconductors, these self-consistency conditions are named gap equations.

Note that the $\sup$ and the $\inf$ of Theorem 1 do not commute in general. See [1, p. 42]. A sufficient condition for the $\sup$ and $\inf$ to commute is given through Sion's minimax theorem [8] as follows:

Lemma 1 *Fix an arbitrary reflection-symmetric finitely supported function ε on $\mathbb{Z}^d$. Let $\beta,\eta_-,\eta_+ \in (0,\infty)$. If, for any fixed $c_+\in\mathbb{C}$, the function $h_\beta(\cdot,c_+)$ on $\mathbb{C}$ is quasi-convex, i.e., for all $r\in\mathbb{R}$, the level set*

$$\{c_-\in\mathbb{C} : h_\beta(c_-,c_+) \leq r\}$$

is convex, then $\mathrm{F}_\beta^\sharp(\eta_-,\eta_+) = \mathrm{F}_\beta^\flat(\eta_-,\eta_+)$.

Proof This lemma refers to [2, Lemma 4.2]. □

1.3.4 The Kac Limit

Even if the Kac limit is relatively standard in classical statistical mechanics [23], until 2022 there were *no* result in quantum mechanics after Lieb's result [20] from 1966 for quantum particles in the continuum. It refers here to the Kac, or long-range, limits $\gamma_\pm \to 0^+$ of the short-range model which we study in this subsection. Note that Lieb's result is quite restrictive, in which concerns the computation of correlation functions, in contrast with the results presented here which refer to the full equilibrium states, i.e., to all correlation functions.

First, using the cyclic representation of the C^*-algebra $\mathcal{U}$ induced by any arbitrary translation-invariant state as well as the spectral theorem, one can prove [2] that the energy densities (1.14) associated with the short-range attractions and repulsions converge pointwise to a mean-field one associated with the elements $A_\pm$ (1.19):

$$\lim_{\gamma_\pm\to0^+} \mathfrak{e}_\pm(\rho) \doteq \lim_{\gamma_\pm\to0^+} \lim_{\ell\to\infty} \frac{1}{|\Lambda_\ell|}\rho(H_{\Lambda_\ell,\pm}) = \hat{f}_\pm(0)\Delta_\pm(\rho) \doteq \lim_{\ell\to\infty} \frac{\hat{f}_\pm(0)}{|\Lambda_\ell|^2} \sum_{x,y\in\Lambda_\ell} \rho\left(\alpha_y\left(A_\pm^*\right)\alpha_x\left(A_\pm\right)\right) \tag{1.26}$$

for any translation-invariant state $\rho\in E_1$, where we have from (1.12) that

$$\hat{f}_\pm(0) \doteq \int_{\mathbb{R}^d} f(x)\,\mathrm{d}^d x \geq 0 \ .$$

Recall that f_-, f_+ are assumed to be positive definite, i.e., the Fourier transforms $\hat{f}_-, \hat{f}_+$ of f_-, f_+, respectively, are positive functions on $\mathbb{R}^d$. Comparing (1.13)–(1.16) and (1.20)–(1.21) in light of (1.26), one infers that the parameters $\eta_-, \eta_+ \in \mathbb{R}_0^+$ of the mean-field model to be taken as the limit $\gamma_\pm \to 0^+$ of the short-range one are

$$\eta_\pm = \hat{f}_\pm(0) \in \mathbb{R}_0^+ \ .$$

This is confirmed by [2, Theorem 5.15], which, in the example presented here, refers to the following statement:

Theorem 2 *Fix an arbitrary reflection-symmetric finitely supported real-valued function* ε *on* $\mathbb{Z}^d$*. Let* $f_-, f_+ \in C_0^{2d}\left(\mathbb{R}^d, \mathbb{R}\right)$ *be reflection-symmetric, positive definite functions on* $\mathbb{R}^d$ *with* $\hat{f}_-(\gamma^{-1}k) \leq \hat{f}_-(k)$ *for* $k \in \mathbb{R}^d$*. Fix an inverse temperature* $\beta \in (0, \infty)$*.*

i.) Convergence of infinite volume pressures:

$$\lim_{\gamma_+ \to 0^+} \lim_{\gamma_- \to 0^+} P_\beta(\gamma_-, \gamma_+) = P_\beta^\sharp\left(\hat{f}_-(0), \hat{f}_+(0)\right) \ .$$

ii.) Convergence of equilibrium states: For any $\gamma_+ \in (0,1)$*, take any weak** *accumulation point* ω_{γ_+} *of any net* $(\omega_{\gamma_-,\gamma_+})_{\gamma_- \in (0,1)} \subseteq \Omega_\beta(\gamma_-, \gamma_+)$ *as* $\gamma_- \to 0^+$*. Pick any weak** *accumulation point* ω *of the net* $(\omega_{\gamma_+})_{\gamma_+ \in (0,1)}$*, as* $\gamma_+ \to 0^+$*. Then,*

$$\omega_{\gamma_-,\gamma_+} \underset{weak^*,\gamma_- \to 0^+}{\rightarrow} \omega_{\gamma_+} \underset{weak^*,\gamma_+ \to 0^+}{\rightarrow} \omega \in \Omega_\beta^\sharp(\hat{f}_-(0), \hat{f}_+(0)) \ .$$

Note that any net of translation-invariant states like the equilibrium states are weak*-convergent, along subsequences, because E_1 is weak*-compact and metrizable.

This theorem demonstrates that the mean-field model is generally an idealization of the short-range one in the long-range limit. In addition, [2] provides explicit error estimates from which one can deduce approximated phase diagrams for the short-range model at sufficiently small parameters $\gamma_\pm \in (0,1)$.

Note that Theorem 2 uses a particular order for the limits: first $\gamma_- \to 0^+$ and then $\gamma_+ \to 0^+$. It means that the attractive range has to be much larger than the repulsive one. One can ask whether this is just a technical issue. As a matter of fact, it is generally **not** so:

Proposition 1 *Fix an arbitrary reflection-symmetric finitely supported real-valued function* ε *on* $\mathbb{Z}^d$*. Let* $f_-, f_+ \in C_0^{2d}\left(\mathbb{R}^d, \mathbb{R}\right)$ *be reflection-symmetric, positive definite functions on* $\mathbb{R}^d$ *with* $\hat{f}_-(\gamma^{-1}k) \leq \hat{f}_-(k)$ *for* $k \in \mathbb{R}^d$*. Fix* $\beta \in (0, \infty)$*. If* $(\gamma_{-,n})_{n\in\mathbb{N}}$ *and* $(\gamma_{+,n})_{n\in\mathbb{N}}$ *converges to zero, then*

$$P_\beta^\sharp\left(\hat{f}_-(0), \hat{f}_+(0)\right) \leq \liminf_{n\to\infty} P_\beta(\gamma_{+,n}, \gamma_{-,n}) \leq \limsup_{n\to\infty} P_\beta(\gamma_{+,n}, \gamma_{-,n}) \leq P_\beta^\flat\left(\hat{f}_-(0), \hat{f}_+(0)\right) \ .$$

Proof See [2, Proposition 5.14]. □

Recall that the $\sup$ and the $\inf$ in Theorem 1 do not commute in general. See [1, p. 42]. A sufficient condition for the $\sup$ and $\inf$ to commute is given by Lemma 1. In fact, we generally have

$$\mathrm{F}_\beta^\flat(\eta_-, \eta_+) \neq \mathrm{F}_\beta^\sharp(\eta_-, \eta_+) \ , \qquad \text{i.e.,} \qquad P_\beta^\sharp(\eta_-, \eta_+) \neq P_\beta^\flat(\eta_-, \eta_+) \ ,$$

(see Theorem 1). Hence, Proposition 1 suggests that, depending upon how the double limit $\gamma_\pm \to 0^+$ of the short-range model is taken, one can get an effective long-range system that is different from the one described by the *conventional* mean-field model, which is the thermodynamic limit the finite-volume system described from local Hamiltonians (1.17). Applying [2, Theorem 5.17] to the model presented here, one can indeed reach what we call the *non-conventional* mean-field system:

Theorem 3 *Fix an arbitrary reflection-symmetric finitely supported real-valued function* ε *on* $\mathbb{Z}^d$*. Let* $f_-, f_+ \in C_0^{2d}\left(\mathbb{R}^d, \mathbb{R}\right)$ *be reflection-symmetric, positive definite functions on* $\mathbb{R}^d$ *with* $\hat{f}_-(\gamma^{-1}k) \leq \hat{f}_-(k)$ *for* $k \in \mathbb{R}^d$*. Fix an inverse temperature* $\beta \in (0, \infty)$*.*

i.) Convergence of infinite-volume pressures:

$$\lim_{\gamma_- \to 0^+} \lim_{\gamma_+ \to 0^+} P_\beta(\gamma_-, \gamma_+) = P_\beta^\flat\left(\hat{f}_-(0), \hat{f}_+(0)\right) \ .$$

ii.) Convergence of equilibrium states: For any $\gamma_- \in (0,1)$*, take any weak* accumulation point* ω_{γ_-} *of any net* $(\omega_{\gamma_-,\gamma_+})_{\gamma_+\in(0,1)} \subseteq \Omega_\beta(\gamma_-,\gamma_+)$ *as* $\gamma_+ \to 0^+$*. Pick any weak* accumulation point* ω *of the net* $(\omega_{\gamma_-})_{\gamma_-\in(0,1)}$*, as* $\gamma_- \to 0^+$*. Then,*

$$\omega_{\gamma_-,\gamma_+} \underset{weak^*,\gamma_+\to 0^+}{\to} \omega_{\gamma_-} \underset{weak^*,\gamma_-\to 0^+}{\to} \omega \in \Omega_\beta^\flat(\hat{f}_-(0),\hat{f}_+(0)) \ .$$

Remark again that any net of translation-invariant states like the equilibrium states are weak*-convergent, along subsequences, because E_1 is weak*-compact and metrizable.

Recall that Lemma 1 gives a sufficient condition for the equality

$$\Omega_\beta^\sharp(\eta_-,\eta_+) = \Omega_\beta^\flat(\eta_-,\eta_+)$$

for any $\eta_-,\eta_+ \in \mathbb{R}_0^+$. However, there is no reason for this equality to hold true in general. In particular, Theorems 2 and 3 generally describe essentially different situations. One can indeed prove that the limit of Kac pressures can attain **all** the values in the interval

$$\mathrm{I}_\beta \doteq \left[P_\beta^\sharp(\hat{f}_-(0),\hat{f}_+(0)), P_\beta^\flat(\hat{f}_-(0),\hat{f}_+(0))\right] = -\left[\mathrm{F}_\beta^\flat(\hat{f}_-(0),\hat{f}_+(0)), \mathrm{F}_\beta^\sharp(\hat{f}_-(0),\hat{f}_+(0))\right] \ ,$$

which is minus the duality interval (1.23) of the thermodynamic game, by Theorem 1.

Theorem 4 *Fix an arbitrary reflection-symmetric finitely supported real-valued function* ε *on* $\mathbb{Z}^d$*. Let* $f_-, f_+ \in C_0^{2d}(\mathbb{R}^d,\mathbb{R})$ *be reflection-symmetric, positive definite functions on* $\mathbb{R}^d$ *with* $\hat{f}_-(\gamma^{-1}k) \leq \hat{f}_-(k)$ *for* $k \in \mathbb{R}^d$*. Fix an inverse temperature* $\beta \in (0,\infty)$*. For any* $\mathrm{p} \in \mathrm{I}_\beta$*, there are two sequences* $(\gamma_{+,n})_{n\in\mathbb{N}}$ *and* $(\gamma_{-,n})_{n\in\mathbb{N}}$ *of real numbers in the interval* $(0,1)$ *converging to zero, such that*

$$\lim_{n\to\infty} P_\beta(\gamma_{-,n},\gamma_{+,n}) = \mathrm{p} \ .$$

Proof. See [2, Theorem 5.19]. □

This theorem shows that the long-range limit $\gamma_\pm \to 0^+$ is possibly not what one would expect in a first guess. In fact, the results referring to the above examples are highly non-trivial: As expected, any Kac or long-range limit leads to mean-field pressures and equilibrium states. However, the limit mean-field model is **not necessarily** what one traditionally guesses when one mixes repulsive and attractive long-range components (in contrast with (1.25)). In fact, it strongly depends upon the hierarchy of ranges between attractive and repulsive interparticle forces. For instance, if the range of repulsive forces is much larger than the range of the attractive ones, then in the Kac limit for these forces one may get a mean-field model that is **unconventional**. See Theorems 3 and 4.

1.4 Historical Notes

A result like Theorem 1 justifies on the level of equilibrium correlation functions the replacement of specific operators appearing in the Hamiltonian of a given physical system by constants which are determined as solutions to some self-consistency equation or some associated variational problem. This refers to the Bogoliubov approximation, which was already used for (purely attractive mean-field) Fermi systems on lattices in 1957 to derive the celebrated Bardeen-Cooper-Schrieffer (BCS) theory for conventional type I superconductors [9–11]. The authors were of course inspired by Bogoliubov and his revolutionary paper [12]. A rigorous justification of this replacement was given on the level of ground states by Bogoliubov in 1960 [13]. Then a method for analyzing the Bogoliubov approximation in a systematic way – on the level of the pressure – like in Theorem 1 with both mean-field repulsions and attractions was introduced by Bogoliubov Jr. in 1966 [14, 15] and by Brankov, Kurbatov, Tonchev, Zagrebnov during the seventies and eighties [16–18]. As already mentioned, this method is known in the literature as the approximating Hamiltonian method and leads – on the class of Hamiltonians it applies – to a rigorous proof of the exactness of the Bogoliubov approximation on the level of the pressure, provided it is done in an appropriated manner. Note however that the results of [16–18] are much more restrictive than those of Theorem 1. The main innovation of [1] is the fact that the Bogoliubov approximation is not only proven for pressure-like quantities (for instance, the thermodynamic limit of the logarithm of canonical or grand-canonical partition functions), as in previous works, but also for equilibrium states, i.e., for **all** correlation functions. See discussions in [1, Section 2.10] for more details.

The analysis of long-range limits $\gamma_{\pm} \to 0^{+}$ follows a rather old sequence of studies on the Kac limit, basically starting from 1959 with Kac's work on classical one-dimensional spin systems. The first important result [19] in this period was provided by Penrose and Lebowitz in 1966, who proved the convergence of the free energy of a classical system towards the one of the van der Waals theory. Shortly after that, the results of this seminal paper were extended to quantum systems (Boltzmann, Bose, or Fermi statistics) by Lieb in [20]. In 1971, Penrose and Lebowitz went considerably further than [19] with the paper [21]. See also [22] for a review of all these results of classical statistical mechanics. These outcomes form the mainstays of the subsequent results on the Kac limit and we recommend the book [23], published in 2009, for a more recent review on the subject in classical statistical mechanics, including the so-called Lebowitz-Penrose theorem and a more exhaustive list of references.

Studies on the Kac limit are still performed nowadays in classical statistical mechanics, see, e.g., [24–26]. By contrast, to our knowledge [2] is the unique recent study on the subject for quantum systems and the sole important results before [2] are those of [20], which refer to quantum particles in the continuum, but may certainly be extended to lattice systems. The main innovation of [2] is the fact that the convergence in the Kac limit is not only done for pressure-like quantities, as in previous works, but also for equilibrium states, i.e., for **all** correlation functions. These results on states were made possible by the variational approach of [1] for equilibrium states of mean-field models. Additionally, also in contrast with previous results on Kac limits, this method allows for coexistence of both attractive and repulsive long-range forces. This important extension is related to the game theoretical characterization of equilibrium states of mean-field models (cf. thermodynamic game). This study thus paves the way for studying phase transitions[4], or at least important fingerprints of them like strong correlations at long distances, for models having interactions whose ranges are finite, but very large as compared to the lattice constant. It also sheds a new light on mean-field models by connecting them with short-range ones, in a mathematically precise manner. Such studies can be important for future theoretical developments in many-body theory, since long-range interactions are expected to imply effective classical background fields, in the spirit of the Higgs mechanism of quantum field theory. This is shown in [5, 27, 28] for the infinite volume dynamics of mean-field models.

Acknowledgments: This work is supported by CNPq (309723/2020-5), the COST Action CA18232 financed by the European Cooperation in Science and Technology (COST) and also by the Basque Government through the BERC 2022-2025 program and by the Ministry of Science and Innovation: BCAM Severo Ochoa accreditation and CEX2021-001142-S / MICIN / AEI / 10.13039/501100011033 and PID2020-112948GB-I00 funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe". We thank Domingos Marchetti for valuable discussions and hints.

References

1. J.-B. Bru and W. de Siqueira Pedra, Non-cooperative Equilibria of Fermi Systems With Long Range Interactions, *Memoirs of the AMS* **224**, no. 1052 (2013).
2. J.-B. Bru, W. de Siqueira Pedra and K. Rodrigues Alves, From Short-Range to Mean-Field Models in Quantum Lattices, submitted preprint (2022). See arXiv:2203.01021 [math-ph] (52 pages).
3. O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics, Vol. I, 2nd ed.* New York: Springer-Verlag, 1987.
4. W. Rudin, *Functional Analysis*. McGraw-Hill Science, 1991.
5. J.-B. Bru and W. de Siqueira Pedra, Quantum Dynamics Generated by Long-Range Interactions for Lattice-Fermion and Quantum Spins, *J. Math. Anal. Appl.* **493**(1) (2021) 124517 (pp 65).
6. V.A. Zagrebnov and J.-B. Bru, The Bogoliubov Model of Weakly Imperfect Bose Gas, *Phys. Rep.* **350** (2001) 291-434.
7. H. Araki and H. Moriya, Equilibrium Statistical Mechanics of Fermion Lattice Systems, *Rev. Math. Phys.* **15** (2003) 93-198.
8. H. Komiya, Elementary Proof For Sion's minimax theorem, *Kodai Math. J.* **11**(1) (1988) 5-7.
9. L. N. Cooper, Bound Electron Pairs in a Degenerate Fermi Gas, *Phys. Rev.* **104** (1956) 1189-1190.
10. J. Bardeen, L.N. Cooper and J.R. Schrieffer, Microscopic Theory of Superconductivity, *Phys. Rev.* **106** (1957) 162-164.
11. J. Bardeen, L.N. Cooper and J.R. Schrieffer, Theory of Superconductivity, *Phys. Rev.* **108** (1957) 1175-1204.
12. N.N. Bogoliubov, On the theory of superfluidity, *J. Phys. (USSR)* **11** (1947) 23-32.
13. N.N. Bogoliubov, On some problems of the theory of superconductivity, *Physica* **26** (1960) S1-S16.
14. N.N. Bogoliubov Jr., *A method for studying model Hamiltonians*, Oxford: Pergamon, 1977.
15. N.N. Bogoliubov Jr., On model dynamical systems in statistical mechanics, *Physica* **32** (1966) 933.

[4] Mean-field repulsions have generally a geometrical effect by possibly breaking the face structure of the set of (generalized) equilibrium states (see [1, Lemma 9.8]). When this appears, we have long-range order of correlations without necessarily a non-unique equilibrium state (i.e., first order phase transition). See [1, Section 2.9].

16. N.N. Bogoliubov Jr., J.G. Brankov, V.A. Zagrebnov, A.M. Kurbatov and N.S. Tonchev, Metod approksimiruyushchego gamil'toniana v statisticheskoi fizike[5], *Sofia: Izdat. Bulgar. Akad. Nauk*[6], 1981
17. N.N. Bogoliubov Jr., J.G. Brankov, V.A. Zagrebnov, A.M. Kurbatov and N.S. Tonchev, Some classes of exactly soluble models of problems in Quantum Statistical Mechanics: the method of the approximating Hamiltonian, *Russ. Math. Surv.* **39** (1984) 1-50.
18. J.G. Brankov, D.M. Danchev and N.S. Tonchev, *Theory of Critical Phenomena in Finite–size Systems: Scaling and Quantum Effects.* Singapore–New Jersey–London–Hong Kong: Word Scientific, 2000.
19. J. Lebowitz and O. Penrose, A Rigorous Treatment of the Van der Waals-Maxwell Theory of the Vapor-Liquid Transition, *J. Math. Phys.* **7** (1966) 98.
20. E. Lieb, Quantum-mechanical extension of the Lebowitz-Penrose theorem on the Van Der Waals theory, *J. Math. Phys.* **7**(6) (1966) 1016-1024.
21. O. Penrose I and J. L. Lebowitz, Rigorous Treatment of Metastable States in the van der Waals-Maxwell Theory, *J. Stat. Phys.* **3**(2) (1971) 211-236.
22. P. C. Hemmer and J. L. Lebowitz, Systems with Weak Long-Range Potentials, pp 107-203 in *Phase Transitions and Critical Phenomena (Volume 5b)*, by C. Domb, and M.S. Green (eds), Academic Press Inc, 1976.
23. E. Presutti, *Scaling Limits in Statistical Mechanics and Microstructures in Continuum Mechanics*, Berlin, Heidelberg: Springer, 2009.
24. S. Franz and F.L. Toninelli, Kac Limit for Finite-Range Spin Glasses, *Phys. Rev. Lett.* **92** (2004) 030602 (3 pages).
25. S. Franz and F.L. Toninelli, Finite-range spin glasses in the Kac limit: free energy and local observables, *J. Phys. A: Math. Gen.* **37** (2004) 7433-7446.
26. S. Franz, Spin glass models with Kac interactions, *Eur. Phys. J. B* **64** (2008) 557-561.
27. J.-B. Bru and W. de Siqueira Pedra, Classical Dynamics Generated by Long-Range Interactions for Lattice Fermions and Quantum Spins, *J. Math. Anal. Appl.* **493**(1) (2021) 124434 (pp 61).
28. J.-B. Bru and W. de Siqueira Pedra, Entanglement of Classical and Quantum Short-Range Dynamics in Mean-Field Systems, *Annals of Physics* **434** (2021) 168643 (pp 31).

Jean-Bernard Bru
1. Departamento de Matemáticas
Facultad de Ciencia y Tecnología
Universidad del País Vasco / Euskal Herriko Unibertsitatea, UPV/EHU
Apartado 644, 48080 Bilbao

2. EHU Quantum Center, University of the Basque Country UPV/EHU

3. BCAM - Basque Center for Applied Mathematics
Mazarredo, 14, 48009 Bilbao

4. IKERBASQUE, Basque Foundation for Science
48011, Bilbao

Walter de Siqueira Pedra
1. Departamento de Física Matemática,
Instituto de Física,
Universidade de São Paulo
Rua do Matão 1371
CEP 05508-090 São Paulo, SP Brasil

2. BCAM - Basque Center for Applied Mathematics (*As external scientific member*)
Mazarredo, 14, 48009 Bilbao

K. Rodrigues Alves
1. Departamento de Matemáticas
Facultad de Ciencia y Tecnología
Universidad del País Vasco / Euskal Herriko Unibertsitatea, UPV/EHU
Apartado 644, 48080 Bilbao

2. BCAM - Basque Center for Applied Mathematics
Mazarredo, 14, 48009 Bilbao

[5] The Approximating Hamiltonian Method in Statistical Physics.
[6] Publ. House Bulg. Acad. Sci.