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Bounds for Quantile-Based Measures of Dependent Risks' Functions

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Abstract

This paper introduces two techniques for computing bounds of several quantile-based measures based on distortion functions. Marginal knowledge with the optional assumption of partial information about the structure of the dependence among the risks are assumed with the aim to compute bounds for risk measures of functions of dependent risks. Several examples in an insurance context are given. We use Embrechts *et al.* (2003) methodology to compute bounds for Value-at-Risk and the stochastic ordering approach, in the bidimensional and multidimensional cases.

MSC: IE43, IM11, IM12

Keywords: Bounds; Distortion functions; Risk measures; Stop-loss sums.

1 Introduction

The task of evaluating the riskiness of a portfolio is performed by defining an axiomatic measure which attempts to capture the main aspects of the risk inherent to the particular market. Due to the variety of different economic scenarios it is impossible to get a unique risk measure which can be applied effectively in every situation.

Value-at-Risk (VaR) is widely used in practice because of its simplicity. In accordance to several researchers and market people VaR is a risk measure that captures the essential features of a portfolio's risk, e.g. Jorion, (1997).

The VaR of a given risk X at level $p \in (0,1)$ is defined as

$$VaR_p(X) = inf\{x \in \Re, F(x) \geq p\} = F^{-1}(p),$$

where $F(x) = P(X \leq x)$ and $\Re = (-\infty, \infty)$. If the distribution function $F(x)$ is continuous, the VaR is a nondecreasing and left-continuous function of p. The VaR, specially for heavy-tailed returns, can be a very optimistic measure, in the sense that the probability of losing much more than VaR is not small given that the loss is greater than or equals its value. This can be understood if we accept the VaR as the answer to the question "How bad things can get?". It is easy to see that, in fact, VaR states that with probability p the loss will not exceed $VaR_p(X)$. Moreover, it is not a sub-additive nor it is a convex risk measure which may cause optimization problems.

These features of VaR can bring about practical problems as well as theoretical ones which motivates the introduction of several quantile-based risk measures. One of the most important of such measures is the Tail Value-at-Risk, TVaR, defined as

$$TVaR_p(X) = \frac{1}{1-p}\int_p^1 VaR_q(X)dq,\ p \in (0,1).$$

That is, the $TVaR_p(X)$ is the arithmetic mean of the quantiles, from p to one. This is the same thing as to say that the TVaR provides the expected loss incurred if it exceeds the value given by $VaR_p(X)$.

This feature makes TVaR much more conservative than VaR. The difference between them is that VaR provides an optimistic inferior bound to the loss incurred while TVaR gives the expected loss in the tail of the distribution. If X denotes the aggregated claims of a insurer's portfolio in a given period (a month, say) and K denotes aggregated provision for this portfolio, the difference $VaR_p(X) - K$ is the smallest additional capital amount required in order to prevent the insurer from technical insolvency with probability no more than *1-p*. If we fix the additional capital amount by $TVaR_p(X) - K$, then we can define "hard times" as the ones when X $\in [VaR_p(X), TVaR_p(X)]$. That is, in "hard times" the aggregated claims exceed $VaR_p(X)$, but the insurer does not spend the total available capital, according to Dhaene *et al.* (2006). Next we add some important properties of the TVaR, e.g. Rockafellar and Uryasev (2000, 2002): sub-additivity; monotonicity; homogeneity and translation invariance and it is a solution of an optimization problem.

Finally, it is important to point out that TVaR is not an appropriate risk measure in extreme situations, examples are low frequencies and high losses incurred. In this case can be used the Wang risk measure, WT, introduced by Wang (2000).

In fact, the TVaR and the VaR are a special case of a wider class of risk measures, defined as follows: For a given nondecreasing function $g : [0,1] \to [0,1]$, $g(0) = 0$ and $g(1) = 1$ called *distortion function* we can associate a quantile-based risk measure using the following expression:

$$D_g(X) = \int_0^\infty g(1 - F(t))dt = \int_0^1 F^{-1}(1-q)dg(q), \tag{1}$$

where the last relation on the right holds if g is increasing.

For instance, $VaR_p(X)$, $TVaR_p(X)$ and the Wang risk measure $WT_p(X)$ can be obtained by (1) using the distortion functions $g(x) = I(x > 1-p)$ (I(.) is the indicator function), $g(x) = min(\frac{x}{1-p}, 1)$ and $g(x) = \Phi[\Phi^{-1}(x)+\Phi^{-1}(p)]$, where Φ^{-1} is the inverse of Φ (Φ being the cumulative distribution of a standard normal variable), respectively.

Usually, one is interested in obtaining bounds for a given risk measure of a specified function $\Psi(.)$. Typical examples of such functions are (see Embrechts *et al.*, 2003):

- $\Psi(x_1, ..., x_n) = x_1 + ... + x_n$. In a insurance situation, one is often interested in the sum of random variables which can represent the aggregated claims of a portfolio in a given period;
- $\Psi(x_1, ..., x_n) = \sum_{i=1}^{n}(x_i - k)^+$, where $a^+ = max(a, 0)$, $k > 0$. This case corresponds to the functional underlying an excess-of-loss treaty in reinsurance for a loss greater than k. The $x_i's$ could be individual claims or reinsurance losses due to different lines of business;
- $\Psi(x_1, ..., x_n) = (\sum_{i=1}^{n} x_i - k)^+$, $k > 0$. This form has an interpretation in derivatives (e.g. Asian options or "stop-loss" reinsurance).

One can use different expressions for Ψ, depending on the situation. In practice, an insurance company might be interested in calculating the VaR of the sum of its portfolio's risks to avoid bankruptcy. Or the company might be interested in calculating the TVaR for the risks' sum in order to face the "hard times".

Computing risks measures for functions of dependent risks can be troublesome in the absence of independence (e.g. Denuit *et al.*, 2005 or Darkiewicz *et al.*, 2005), which is a realistic assumption. Frequently, one knows the marginal distributions of each risk, but only have partial or no information regarding their dependence structure. Our goal is to obtain bounds in such situation. Two approaches to do so will be discussed: in Section 2 the bounds are based on the methodology presented in Embrechts *et al.* (2003) to calculate the bounds for VaR, see Embrechts and Puccetti (2005) also. In Section 3 we provide bounds based on stochastic ordering approach. The pros and cons of each approach will be discussed.

2 Bounds for distribution functions

In this section we obtain bounds for risk measures associated with different distortion functions[1] as defined in the Introduction.

2.1 Bounds for VaR

Let $\mathbf{X} = (X_1, ..., X_n)$ be a random vector with known marginals $F_{X_i}(x_i)$, *i=1,...,n*. Let $\Psi : \Re^n \to \Re$ be an increasing and right-continuous in the last argument function for which we want to compute the risk measure and Ψ_{n-1} is the function Ψ with the first $n-1$ arguments held fixed. Define the copula by

$$C(u_1, ..., u_n) = P(U_1 \leq u_1, ..., U_n \leq u_n),$$

and the corresponding dual copula by

$$C^d(u_1, ..., u_n) = P(\{U_1 \leq u_1\} \cup ... \cup \{U_n \leq u_n\}),$$

[1]In Wang (1996) an interpretation of these distortion function is given in a insurance context.

where U_i is uniformly distributed over $[0,1]$, $i = 1, 2, ..., n$. We define the following quantities:

$$\tau_{C,\Psi}(s) = \sup_{x_1,...,x_{n-1}\in\Re} C(F_{X_1}(x_1), ..., F_{X_{n-1}}(x_{n-1}), F_{X_n}(\Psi_{n-1}^{-1}(s))),$$

$$\sigma_{C,\Psi}(s) = \int_{\{\Psi(x_1,...,x_n)\leq s\}} dC(F_{X_1}(x_1), ..., F_{X_n}(x_n))$$

and

$$\rho_{C,\Psi}(s) = \inf_{x_1,...,x_{n-1}\in\Re} C^d(F_{X_1}(x_1), ..., F_{X_{n-1}}(x_{n-1}), F_{X_n}(\Psi_{n-1}^{-1}(s))),$$

where $\Psi_{n-1}^{-1}(s) = sup\{x \in \Re | \Psi_{n-1}(x) \leq s\}$, $s \in [0, \infty]$, is the generalized inverse of $\Psi_{n-1}(x_1, ..., x_n)$. The expression $\sigma_{C,\Psi}(s)$ is, in fact, the distribution of Ψ.

Under the above notations the following statement is valid:

Theorem 1 (Embrechts *et al.*, 2003).[2] *If the copula C(.) associated to the random vector* $\mathbf{X}$ *satisfies* $C \geq C_0$ *and* $C^d \leq C_1^d$ *for a given pair of n-dimensional copulas* C_0 *and* C_1*, then*

$$\tau_{C_0,\Psi}(s) \leq \sigma_{C,\Psi}(s) \leq \rho_{C_1,\Psi}(s). \tag{2}$$

Besides, the limits are sharp, i.e., for each $s \in [0, \infty]$ *there exists a copula C(.) such that the lower bound is attained (and the same holds for the upper bound).*

Theorem 1 is used in Embrechts *et al.* (2003) to compute bounds of the VaR when the marginal distribution of each risk is known, but the risk manager does not know their joint distribution. Observe that in the conditions of Theorem 1 it is supposed that a partial information of the copula *C(.)* is known (C_0 and C_1^d are given). Here we will work under this hypothesis.

The partial information is a natural assumption if one, starting with marginals, chooses the right copula in order to describe the real dependence structure. If no information is known at all, which is very common, then in the two dimensional case we can use C_0 as Fréchet's lower bound and $C_1^d = C_0^d = min\{x_1 + x_2, 1\}$. For a multidimensional random vector this is not the case, since Fréchet's inferior limit is not always a valid distribution (see Nelsen, 1999). By applying (2) in the non-informative setting, we may fail to obtain "sharp" bounds. In Embrechts and Puccetti (2006) conditions to get a valid distribution from Fréchet's lower bound are given, and they are very restrictive. So, in a non-informative situation it is better to use, if possible, stochastic ordering (see Section 3) or a methodology presented in Embrechts and Puccetti (2006) which is complicated and it is hard to obtain numerical solutions and even harder to get analytical ones. Only a few expressions for the bounds are known in this case.

[2]The original proof of this theorem is wrong. The correction is given in Embrechts and Puccetti (2005).

If we are unsuccessful in applying any other methodology for calculating upper and lower bounds under absence of information, we can still use the inequalities (2). We will not be able to obtain sharp bounds, but at least some information about the lower and upper bounds will be available.

It is worth stressing that it is very hard to obtain analytical solutions to the bounds given in (2) and in most cases we will have to rely on numerical methods to compute those bounds. Such a methodology is presented in Williamson and Downs (1990). They developed a numerical method to compute the inverses $\rho^{-1}_{C_1,\Psi}(q)$ and $\tau^{-1}_{C_0,\Psi}(q)$ in the bidimensional setting, $q \in [0,1]$. A more generic approach is suggested by Embrechts *et al.* (2003), where the multidimensional case is treated. In the next sub-section we apply (2) for different types of distortion functions.

2.2 Bounds for other quantile-based measures

Let $TVaR_p(\Psi)$, $PT_p(\Psi)$ and $GT_p(\Psi)$ stand for the TVaR (obtained by distortion function $g(x) = \min(1, \frac{x}{1-p})$), the Power Transformation (when $g(x) = x^p$.) and Gini Transformation (substituting $g(x) = (1+p)x - px^2$), respectively. The Power Transformation, $PT_p(.)$, as a risk measure, incorporates the risk aversion toward uncertainty: the less is the value of p, the greater is the aversion. For Gini Transformation, the premium principle is the average absolute deviation from median. Denneberg (1990) generalizes the Gini principle. Accordingly, to determine the safety loading for insurance premiums, the average absolute deviation from the median is best suited. The parameter p gives the mark-up of the insurer in the tail of the distribution. Since $p \in (0,1)$, it will never exceed 100%.

Applying (2) we obtain the following bounds

$$\frac{1}{1-p}\int_p^1 \rho^{-1}_{C_1,\Psi}(q)dq \le TVaR_p(\Psi) \le \frac{1}{1-p}\int_p^1 \tau^{-1}_{C_0,\Psi}(q)dq,$$

$$\int_0^\infty [1-\rho_{C_1,\Psi}(t)]^p dt \le PT_p(\Psi) \le \int_0^\infty [1-\tau_{C_0,\Psi}(t)]^p dt$$

and

$$\int_0^\infty \{(1+p)[1-\rho_{C_1,\Psi}(t)] - p[1-\rho_{C_1,\Psi}(t]^2\}dt \le GT_p(\Psi) \le$$

$$\int_0^\infty \{(1+p)[1-\tau_{C_0,\Psi}(t)] - p[1-\tau_{C_0,\Psi}(t)]^2\}dt,$$

see Goncalves *et al.* (2005).

Example 1. Let X_1 and X_2 be two arbitrary risks with uniform distribution over $[0,1]$ and $\Psi(X_1,X_2) = X_1 + X_2$, then the above inequalities simplify to

$$\frac{1+p}{2} \le TVaR_p(X_1+X_2) \le \frac{3+p}{2},$$

$$\frac{1}{p+1} \le PT_p(X_1+X_2) \le \frac{p+2}{p+1} \tag{3}$$

and

$$\frac{3+p}{6} \leq GT_p(X_1+X_2) \leq \frac{9+p}{6},$$

correspondingly.

Bounds for other risk measures (discussed by Wang, 1996 and Dhaene *et al.*, 2006) can be found in Goncalves *et al.* (2005). In many cases it is necessary to use numerical methods to compute the bounds. This can be done by using the relation

$$D_g(X) = \int_0^1 F^{-1}(1-q)dg(q).$$

Applying the numerical method proposed in Williamson and Downs (1990) (to compute the inverses $\rho^{-1}_{C_1,\Psi}(q)$ and $\tau^{-1}_{C_0,\Psi}(q)$) and some numerical integration formula one is able to compute the following bounds [3]:

$$\int_0^1 \rho^{-1}_{C_1,\Psi}(1-q)pq^{p-1}dq \leq PT_p(\Psi) \leq \int_0^1 \tau^{-1}_{C_0,\Psi}(1-q)pq^{p-1}dq;$$

$$\int_0^1 \rho^{-1}_{C_1,\Psi}(1-q)[(1+p)-2pq]dq \leq GT_p(\Psi) \leq \int_0^1 \tau^{-1}_{C_0,\Psi}(1-q)[(1+p)-2pq]dq.$$

Unfortunately, there is no guarantee that all above bounds given here will be sharp. The reason is that we know that there is, theoretically, for each *"s"*, a copula *C(.)* attaining the upper (lower) bound. The integration is made over a set of values of *"s"* for the same copula *C(.)* and as a consequence there is no guarantee of bound's sharpness. As a matter of fact, later we will see that these bounds will fail to be sharp.

3 Bounds and stochastic ordering

In this section we obtain bounds for the quantile-based risk measures presented so far using *stochastic ordering* approach. First, we will restrict ourselves to the bivariate case.

3.1 The bidimensional case

Here we will obtain bounds for a function of two risks. At first we need several definitions and theorems related to stochastic ordering (see Müller and Stoyan, 2002).

- A function $f : \Re^2 \to \Re$ is said to be *supermodular* if

$$f(x_1+\epsilon, x_2+\delta) + f(x_1, x_2) \geq f(x_1, x_2+\delta) + f(x_1+\epsilon, x_2)$$

for all $x \in \Re^2$ and all $\epsilon, \delta > 0$;

[3] The bounds for TVaR as functions of the inverses $\rho^{-1}_{C_1,\Psi}(.)$ and $\tau^{-1}_{C_0,\Psi}(.)$ are given in the beginning of this subsection.

- Consider the random vectors $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$. We say that $\mathbf{X}$ is smaller than $\mathbf{Y}$ in *supermodular order* (written as $\mathbf{X} \leq_{sm} \mathbf{Y}$) if

$$Ef(\mathbf{X}) \leq Ef(\mathbf{Y})$$

  for all supermodular functions f such that the expectation exists;

- Let X e Y be two random variables. We say that X is smaller than Y in *stop-loss order* (denoted by $X \leq_{sl} Y$) if

$$Eh(X) \leq Eh(Y)$$

  for all increasing convex functions h such that the expectation exists.

The next two theorems are due to Müller (1997). The first one provides a relation between supermodular order and stop-loss order (it is adapted to the bivariate case) and the second theorem gives a relation of equivalence for supermodular order:

Theorem 2 (Müller, 1997). *Let $\mathbf{X}$ e $\mathbf{Y}$ be two bivariate random vectors such that $\mathbf{X} \leq_{sm} \mathbf{Y}$. Define $S = X_1 + X_2$ and $S' = Y_1 + Y_2$, then $S \leq_{sl} S'$.*

Theorem 3 (Müller, 1997). *Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ two random vectors with equal marginals. Then the following relations are equivalent:*

- $\mathbf{X} \leq_{sm} \mathbf{Y}$;
- $P(X_1 < s, X_2 < t) \leq P(Y_1 < s, Y_2 < s)$ *for all* $s,\ t \in \Re$;
- $P(X_1 > s, X_2 > t) \leq P(Y_1 > s, Y_2 > t)$ *for all* $s,\ t \in \Re$.

Observation. The second relation in Theorem 3 can serve as a definition of *superior orthant order* while the third one is known as *inferior orthant order.*

The next statement provides a relation between concave distortion function and stop-loss order. It plays an important role in obtaining the results we are looking for.

Theorem 4 (Wang *et al.*, 1997). *If the distortion function g is concave, then if $X \leq_{sl} Y$ one has $D_g(X) \leq D_g(Y)$.*

Under the hypothesis of Theorem 3 the next statement holds:

Proposition 1. *If a copula C satisfies $C_0 \leq C \leq C_1$, for all $(u, v) \in [0, 1]^2$ and a given pair of copulas C_0 and C_1, then for all concave distortion functions g,*

$$D_g(Z_1 + Z_2) \leq D_g(X_1 + X_2) \leq D_g(Y_1 + Y_2), \tag{4}$$

where D_g is given by (1) and the vectors $\mathbf{X} = (X_1, X_2)$, $\mathbf{Z} = (Z_1, Z_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ are represented by copulas C, C_0 and C_1, respectively. Besides, the bounds are sharps.

Proof. The inequality $C \leq C_1$ for all s and t implies that

$$P(X_1 < s, X_2 < t) \leq P(Y_1 < s, Y_2 < t).$$

From Theorem 3 it follows that $\mathbf{X} \leq_{sm} \mathbf{Y}$. By Theorem 2 one gets

$$X_1 + X_2 \leq_{sl} Y_1 + Y_2.$$

Finally, applying Theorem 4 we obtain the following relation

$$D_g(X_1 + X_2) \leq D_g(Y_1 + Y_2).$$

To prove that $D_g(X_1 + X_2) \geq D_g(Z_1 + Z_2)$ we proceed analogously, first showing that $Z \leq_{sm} X$. To ensure that the bounds are sharp, just take into account the cases $\mathbf{X} = \mathbf{Y}$ a.s and $\mathbf{X} = \mathbf{Z}$ a.s.

Let us note that Dhaene *et al.* (2006) prove a similar statement for the upper bounds of $D_g(X_1+X_2)$, with g concave, (the authors do not focus on the bidimensional case, however) when the copula C_1 is Fréchet's upper bound. In this case the use of supermodular order is not necessary, because if $\mathbf{Z} = (Z_1, Z_2)$ is a comonotonic random vector, then $Z_1 + Z_2 \geq_{sl} X_1 + X_2$, and the result follows directly from Theorem 4.

The next theorem states that increasing functions applied on the component X_i of $\mathbf{X}$ and Y_i of $\mathbf{Y}$, for $i = 1, \dots, n$, preserve supermodular order.

Theorem 5 (Müller, 1997). *Assume that* $\mathbf{X} \leq_{sm} \mathbf{Y}$ *and let* $\phi_1, \dots, \phi_n$ *be increasing functions in the support of* (X_i, Y_i), *for i=1,2,...,n. Then,*

$$(\phi_1(X_1), \dots, \phi_n(X_n)) \leq_{sm} (\phi_1(Y_1), \dots, \phi_n(X_n)).$$

Let us note that Theorem 5 gives a relation between stop-loss order and sum of the components of a random vector. We can achieve a wider result by applying this theorem (under the hypothesis of Theorem 3), as is shown in the following statement valid under the notations of Proposition 1.

Proposition 2. *Let* ϕ_1 *and* ϕ_2 *be increasing functions in the support of* X_1 *and* X_2 *respectively. If a copula* C *satisfies* $C_0 \leq C \leq C_1$, *for all* $(u, v) \in [0, 1]^2$, *and given pair of copulas* C_0 *and* C_1, *then for all concave distortion function* g,

$$D_g(\phi_1(Z_1) + \phi_2(Z_2)) \leq D_g(\phi_1(X_1) + \phi_2(X_2)) \leq D_g(\phi_1(Y_1) + \phi_2(Y_2)), \qquad (5)$$

and the bounds are sharp.

Proof. The proof is straightforward. Similarly to the proof of Proposition 1, one gets that $\mathbf{X} \leq_{sm} \mathbf{Y}$ and that $\mathbf{Z} \leq_{sm} \mathbf{X}$. By applying Theorem 5 it follows that $(\phi_1(Z_1), \phi_2(Z_2)) \leq_{sm} (\phi_1(X_1), \phi_2(X_2))$ and that $(\phi_1(X_1), \phi_2(X_2)) \leq_{sm} (\phi_1(Y_1), \phi_2(Y_2))$. Applying Theorem 2, we obtain that $\phi_1(Z_1) + \phi_2(Z_2) \leq_{sl} \phi_1(X_1) + \phi_2(X_2)$ and $\phi_1(X_1) + \phi_2(X_2) \leq_{sl} \phi_1(Y_1) + \phi_2(Y_2)$. Finally, using Theorem 3 it comes out that (5) is fulfilled.

To check sharpness of the bounds we proceed the same way we did in the proof of Proposition 1.

Lemma 1. *Proposition 2 is valid for the following functions:*

- $\Psi(x_1, x_2) = (x_1 - k)^+ + (x_2 - k)^+, \ k > 0;$
- $\Psi(x_1, x_2) = (x_1 + x_2 - k)^+, \ k > 0.$

Proof. For the first function the proof is obvious. For$(x_1 + x_2 - k)^+$, one just has to substitute $\phi_i(X_i) = X_i - \frac{k}{2}$, $i = 1, 2$. Then, $\phi_1(X_1) + \phi_2(X_2) \leq_{sl} \phi_1(Y_1) + \phi_2(Y_2)$. As a consequence, e.g. Müller and Stoyan (2002), for all $t \in \Re$

$$\int_t^\infty P(X_1 + X_2 - k > t)dt \leq \int_t^\infty P(Y_1 + Y_2 - k > t)dt. \tag{6}$$

Now we have to check if

$$\int_t^\infty P\{(X_1 + X_2 - k)^+ > t\}dt \leq \int_t^\infty P\{(Y_1 + Y_2 - k)^+ > t\}dt.$$

For $t < 0$ the above inequality is obvious. For $t \geq 0$, one has

$$P\{(X_1 + X_2 - k)^+ > t\} = P(X_1 + X_2 - k > t)$$

and

$$P\{(Y_1 + Y_2 - k)^+ > t\} = P(Y_1 + Y_2 - k > t).$$

Finally, applying (6) we conclude that $(x_1 + x_2 - k)^+ \leq_{sl} (y_1 + y_2 - k)^+$.

Note that the calculus of the bounds (5) is merely a computational problem since one knows the distribution of $\mathbf{Y}$ and $\mathbf{Z}$ by hypothesis. The assumption of partial knowledge of the copula C, i.e. information about C_0 and C_1 however, may not be realistic. Next we will discuss two extreme forms of dependence in order two obtain non-informative bounds.

For some uniform random variable U over [0, 1], define the following random vector

$$(Y_1, Y_2) = (F_{X_1}^{-1}(U), F_{X_2}^{-1}(U)).$$

This random vector has the same marginals as the random vector (X_1, X_2) and it is called *comonotonic*, see Embrechts *et al.* (2003).

Lemma 2 (Denneberg, 1994). *If X and Y are comonotonic random variables, then for $q \in [0, 1]$*

$$F_X^{-1}(q) + F_Y^{-1}(q) = F_{X+Y}^{-1}(q).$$

This lemma is useful to compute bounds for TVaR, since it is expressed easily as a function of the inverses of the distributions functions involved.

Define the following random vector

$$(Z_1, Z_2) = (F_{X_1}^{-1}(U), F_{X_2}^{-1}(1 - U)),$$

which has the same marginals as the random vector (X_1, X_2). It is is called *countermonotonic*, see Embrechts *et al.* (2003). Countermonotonicity cannot be extended to the multidimensional case. Unfortunately, we do not know an analogous relation as giving by Lemma 2 for the countermonotonic random variables.

When no information is available regarding the copula C at all, we can use Frechet's lower bound for C_0 and Fréchet's upper bound for C_1. And this can be done since the comonotonic and countermonotonic random vector satisfy the hypothesis of Theorem 3 and therefore, it is possible to apply Proposition 2. If the function Ψ is the sum of the risks, we can simplify the computation of the upper bound using Lemma 2 (being in the comonotonic case). Since the non-informative case is the most common situation, we will present some analytical examples.

Example 2. Suppose that X_1 and X_2 are two risks with uniform distribution over $[0, \theta]$ and $[0, \gamma]$ respectively and that $\Psi(X_1, X_2) = X_1 + X_2$, then we obtain the following bounds

$$\gamma + \frac{\theta - \gamma}{2}(1+p) \le TVaR_p(X_1 + X_2) \le \frac{\gamma + \theta}{2}(1+p),$$

$$\gamma + \frac{\theta - \gamma}{p+1} \le PT_p(X_1 + X_2) \le \frac{\theta + \gamma}{p+1}$$

and

$$\gamma + \frac{(\theta - \gamma)(3+p)}{6} \le GT_p(X_1 + X_2) \le \frac{(\gamma + \theta)(3+p)}{6}.$$

Setting $\theta = \gamma = 1$ we can see that the bounds (3) are not sharp. For instance, in the case of TVaR

$$1 \le TVaR_p(X_1 + X_2) \le (1+p),$$

while in Example 1 we found

$$\frac{1+p}{2} \le TVaR_p(X_1 + X_2) \le \frac{3+p}{2}.$$

Example 3. Suppose that X_1 and X_2 are exponentially distributed with parameters λ and β, correspondingly. In this case it is not possible to obtain the explicit expression of the lower bounds (it can be calculated numerically) but we can still obtain the upper ones:

$$TVaR_p(X_1 + X_2) \le \frac{\alpha + \beta}{\alpha\beta}(1 - ln(1-p)),$$

$$PT_p(X_1 + X_2) \le \frac{\alpha + \beta}{\alpha\beta p}$$

and

$$GT_p(X_1 + X_2) \le \frac{(\alpha + \beta)(2+p)}{2\alpha\beta}.$$

3.2 The multidimensional case

In the multidimensional case, the situations where we can apply stochastic orderings are more restrictive. Since Theorem 3 is no more valid for $n \geq 3$, we cannot offer a multivariate analog of Proposition 2. Nevertheless, it is still possible to obtain upper bounds for the non-informative scenario in some situations. Such a result will be useful if it could be applied for common functions of risk. Despite the fact that the findings of this section are not general, they can be applied quite often.

The definition of comonotonicity of the bivariate case can be extended directly to the multidimensional one, i.e., the vectors $\mathbf{X}$ and $\mathbf{Y}$ are comonotonic if

$$\mathbf{Y} = (Y_1, \ldots, Y_n) = (F_{X_1}^{-1}(U), \ldots, F_{X_n}^{-1}(U)) \; a.s., \tag{7}$$

where U is uniform over $[0,1]$, and the corresponding distribution attains the Frèchet upper bound. The properties of the comonotonicity here are similar to the bivariate case. Unfortunately, for multivariate countermonotonic random vectors we are unable to extend the corresponding properties valid for the bivariate case.

We will need the following theorem.

Theorem 6 (Kaas *et al.*, 2003). *For a n-variate vector of risks* $\mathbf{X}$, *define* $\mathbf{Y}$ *by (7). Then,*

$$Y_1 + Y_2 + \ldots + Y_n \geq_{sl} X_1 + X_2 + \ldots + X_n.$$

The next Lemma is useful since it shows how Theorem 6 can be expanded.

Lemma 3. *If* $\mathbf{Y} = (Y_1, Y_2, ..., Y_n)$ *can be represented by (7), then:*

- $\sum_{i=1}^{n}(X_i - k)^+ \leq_{sl} \sum_{i=1}^{n}(Y_i - k)^+, \; k > 0;$ (8)
- $(\sum_{i=1}^{n} X_i - k)^+ \leq_{sl} (\sum_{i=1}^{n} Y_i - k)^+, \; k > 0.$ (9)

Proof. To prove (8) we will show that $((Y_1 - k)^+, (Y_2 - k)^+, \ldots, (Y_n - k)^+)$ is a comonotonic random vector with the same distribution as

$$((X_1 - k)^+, (X_2 - k)^+, \ldots, (X_n - k)^+).$$

Note that for a given $1 \leq j \leq n$, we have that

$$(Y_j - k)^+ = \begin{cases} 0, & \text{if } F_{X_j}^{-1}(U) \leq k; \\ F_{X_j}^{-1}(U) - k, & \text{otherwise.} \end{cases}$$

Now, if we show that $(Y_j - k)^+ = F_{(X_j - k)^+}^{-1}(U)$ a.s., (8) will be a consequence of (7) and Theorem 6. Remind that $F_{X_j}^{-1}(U) = sup\{x \in \Re | P(X_j < x) \leq U\}$. Suppose that $F_{X_j}^{-1}(U) > k$. Since $F_{X_j}^{-1}(U) - k \geq 0$,

$$P(X - k \leq F_{X_j}^{-1}(U) - k) = P\{(X - k)^+ \leq F_{X_j}^{-1}(U) - k\}.$$

The last relation is true because the set where $X - k \leq F_{X_j}^{-1}(U) - k$ coincides with the set $(X-k)^+ \leq F_{X_j}^{-1}(U) - k$. Thus,

$$P\{(X_j - k)^+ \leq F_{X_j}^{-1}(U)\} = P\{X_j - k \leq F_{X_j}^{-1}(U)\} = P\{X_j \leq F_{X_j}^{-1}(U)\} \leq U.$$

Hence, $F_{(X_j-k)^+}^{-1}(U) = F_{X_j}^{-1}(U) - k$ a.s., which implies $F_{X_j}^{-1}(U) > k$, a.s.

Now, consider the case where $F_{X_j}^{-1}(U) \leq k$. In this situation

$$P\{(X_j - k)^+ \leq F_{X_j}^{-1}(U) - k\} = 1,$$

except when $F_{X_j}^{-1}(U) = k$. As a result, $F_{X_j}^{-1}(U) \leq k$ a.s., implies $F_{(X_j-k)^+}^{-1}(U) = 0$ a.s., which proves that $((Y_1 - k)^+, (Y_2 - k)^+, \ldots, (Y_n - k)^+)$ is a comonotonic random vector with the same distribution as $((X_1 - k)^+, (X_2 - k)^+, \ldots, (X_n - k)^+)$. Applying Theorem 6 we get $\sum_{i=1}^n (X_i - k)^+ \leq_{sl} \sum_{i=1}^n (Y_i - k)^+$.

It is easier to show (9). For each $d \in \Re$ we have that

$$\left((\sum_{i=1}^n x_i - k)^+ - d \right)^+ = \left(\sum_{i=1}^n x_i - (k+d) \right)^+ .$$

This relation and the proof of Theorem 6 (e.g. Kaas *et al.*, 2003) gives that the stop-loss premiums of the left side of this last expression are ordered. Since both sides of (9) have same expectation, the desired result follows.

We are ready to present the basic result in this subsection.

Proposition 3. *Let Y be a comonotonic random vector, g be a concave distortion function, and Ψ represented by $\sum_{i=1}^n Y_i$, $\sum_{i=1}^{i=n} (Y_i - k)^+$ or $(\sum_{i=1}^{i=n} Y_i - k)^+$. Then,*

$$D_g(\Psi(X_1, X_2 \ldots, X_n)) \leq D_g(\Psi(Y_1, Y_2, \ldots, Y_n))$$

and the inequality is sharp.

Proof. One just has to apply Theorems 4 and 6 and take into account Lemma 3.

The last statement shows that it is possible to obtain upper bounds in the non-informative case for the risk measures $TVaR$, PT and GT if Ψ is the corresponding sum. Otherwise, the methodology discussed in Section 2 should be used. In Dhaene *et al.* (2006) is presented a lower bound for the risk measure $D_g(\Psi)$ when g is concave and Ψ is the sum of the risks. Proposition 3 extends their result regarding the upper bounds for the *stop-loss* sums $\sum_{i=1}^{i=n} (Y_i - k)^+$ or $(\sum_{i=1}^{i=n} Y_i - k)^+$.

The following Lemma is a multivariate version of Lemma 2.

Lemma 4 (Denneberg, 1994). *If $X_1, X_2, \ldots, X_n$ are comonotonic random variable, then for $q \in [0, 1]$*

$$F_{X_1}^{-1}(q) + F_{X_2}^{-1}(q) + \ldots + F_{X_n}^{-1}(q) = F_{X_1+X_2+\ldots+X_n}^{-1}(q).$$

Next two examples illustrate the theory developed.

Example 4. Consider a n-dimensional vector of risks $\mathbf{X}$ such that the marginal distribution of X_i is uniform over $[0, \theta_i]$, for $i=1,2,\ldots,n$. Then, just extending the results of Example 2 and applying Lemma 4 for the TVaR, we get

$$TVaR_p(X_1 + \cdots + X_n) \leq \frac{\theta_1 + \theta_2 + \ldots + \theta_n}{2}(1+p),$$

$$PT_p(X_1 + \cdots + X_n) \leq \frac{\theta_1 + \theta_2 + \ldots + \theta_n}{p+1}$$

and

$$GT_p(X_1 + \cdots + X_n) \leq \frac{(\theta_1 + \theta_2 + \ldots + \theta_n)(3+p)}{6}.$$

Example 5. Consider n-dimensional vector of risks $\mathbf{X}$ such that the marginal distribution of X_i is exponential with parameter λ_i, for $i=1,2,\ldots,n$. Extending the results of Section 3.1 gives the following bounds:

$$TVaR_p(X_1 + \cdots + X_n) \leq \frac{\sum_{i=1}^n \prod_{j \neq i} \lambda_j}{\prod_i \lambda_i}[1 - ln(1-p)],$$

$$PT_p(X_1 + \cdots + X_n) \leq \frac{\sum_{i=1}^n \prod_{j \neq i} \lambda_j}{p \prod_i \lambda_i}$$

and

$$GT_p(X_1 + \cdots + X_n) \leq \frac{(2+p)\sum_{i=1}^n \prod_{j \neq i} \lambda_j}{2\prod_i \lambda_i}.$$

3.3 Application to Insurance

Suppose that $X_1, X_2, \ldots, X_n$ are risks of a insurer's portfolio, with marginal distributions $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$, correspondingly. We want to obtain upper bounds for the risk measures $TVaR$, PT, and GT when the function Ψ is a sum. By applying Proposition 3 with comonotonic copula C_1 we obtain the following bounds:

$$TVaR_p(X_1 + \cdots + X_n) \leq \frac{1}{1-p}\int_p^1 [F_{X_1}^{-1}(q) + F_{X_2}^{-1}(q) + \ldots + F_{X_m}^{-1}(q)]dq,$$

$$PT_p(X_1 + \cdots + X_n) \leq \int_0^\infty [H_{C_1,\Psi}(t)]^p dt,$$

and

$$GT_p(X_1 + \cdots + X_n) \leq \int_0^\infty [(1+p)\overline{H}_{C_1,\Psi}(t) - p\overline{H}_{C_1,\Psi}(t)^2]dt,$$

where

$$H_{C_1,\Psi}(t) = P\{\Psi(\mathbf{Y}) < t\} \text{ and } \overline{H}_{C_1,\Psi}(t) = 1 - H_{C_1,\Psi}(t)$$

with $\mathbf{Y}$ being comonotonic.

Below we show the upper bounds of these risk measures computed for a portfolio of five exponential risks when the parameters are $\lambda_i = \frac{1}{i}$ for $i = 1, 2, ..., 5$ and $p = 0.95$. The calculated values are as follows.

- The upper bound for the $TVaR_{0.95}(X_1 + \cdots + X_5)$ is 1095, which gives expected loss when the loss incurred is greater than the 0.95-quantile;
- The upper bound for $PT_{0.95}(X_1 + \cdots + X_5)$ is 288;
- The upper bound for $GT_{0.95}(X_1 + \cdots + X_5)$ is 404.

The difference between the calculated bounds is because of different premium principles they are assigned.

Let us note that the risks considered are nothing more than a insurance premium (pure premium for a insurance risk which includes the net expected loss plus a risk load charge, excluding expenses and commissions) for non-negative risks X_i. According to Wang *et al.* (1997) they have all the desirable properties one expect a risk measure should have. As we noted, the Power transformation, PT, can be used when one wants to incorporate the risk aversion of the insurers in the insurance price, and this aversion is characterized by the parameter p. The greater it is, the less is the risk aversion. In the example above, the insurer is not so risk averse. But if p were 0.1, then the upper bound would be 2740, naturally much higher than 288, calculated for $p = 0.95$.

Each of the measures cited above are given and discussed by Wang (1996). The insurer can choose the appropriated one depending on the economic situation in order to obtain the insurance premium. The hypothesis is that the markets are competitive, i.e., all the agents are price takers, which means that insurances prices cannot be influenced by a single agent.

4 Conclusions

We presented two techniques to compute bounds for the risk measures associated with distortion functions. The first one fail to be sharp. However, it is a wider approach in the sense that it is valid for a larger class of functions Ψ and it is not restricted to concave distortion functions. The second approach, based on stochastic ordering, provides sharp bounds and the computations are easier. On the other hand, the conditions of the corresponding theorems are stronger. Yet, they are very useful since cover a class of functions Ψ that is widely used in practice.

The technique that should be used will depend on the situation one has in hand. But it should be clear that the approach based on stochastic ordering ought to be used whenever it is possible.

Ordering of risks approach can be applied to adapt real situations described by complex models to more tractable and simpler models which are riskier and thus lead to conservative decisions. There are two reasons to prefer simpler models, at least. First, it is impossible to do calculations needed for the complex model, in general. Second, usually there is a lack of information about the parameters needed for its evaluation. So we can base our decision on the riskiest model consistent with the limited amount of information.

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