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Some inequalities and asymptotics for
a weighted alternate binomial sum

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Some inequalities and asymptotics for a weighted alternate binomial sum

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Abstract

We establish strict inequality bounds for the binomial sums $a_n = \sum_{i=0}^n \binom{n}{i} \frac{(-1)^i}{2i+1}$.
and prove the asymptotic result: $a_n \sim \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{2n+1}}$, $n \rightarrow \infty$.

1 Introduction

We are interested in studying the sums $\sum_{i=1}^n \binom{n}{i} (-1)^i \frac{1}{2i+1}$ for all $n \in \mathbb{N}$.

The sums are shown to be positive and strictly decreasing with n , having limit zero as n tends to infinity. The main results are bounds for these sums as well as their asymptotic behaviour.

2 Main results

Denote $a_n = \sum_{i=1}^n \binom{n}{i} \frac{(-1)^i}{2i+1}$.

Proposition 1 For all $n \in \mathbb{N}$, $\sum_{i=0}^n \binom{n}{i} \frac{(-1)^i}{2i+1} > 0$.

Proof: $\forall n \in \mathbb{N}$, $0 < \int_0^{\pi/2} \cos^{2n+1} x \, dx = \int_0^1 (1-u^2)^n \, du = \sum_{i=1}^n \binom{n}{i} (-1)^i \frac{u^{2i+1}}{2i+1} \Big|_0^1$.

□

Proposition 2 $\forall n \in \mathbb{N}, a_n > a_{n+1}$.

Proof: $\forall n \in \mathbb{N}, a_n = \int_0^{\pi/2} \cos^{2n+1} x \, dx > \int_0^{\pi/2} \cos^{2n+1} x \cos^2 x \, dx = a_{n+1}$. ■

Proposition 3 $\lim_{n \rightarrow \infty} a_n = 0$.

Proof: $\forall m > 1, \exists n \in \mathbb{N}, \left(\cos \left(\frac{1}{m} \right) \right)^{2n+1} < \frac{1}{m}$,

$$a_n = \int_0^{1/m} \cos^{2n+1} x \, dx + \int_{1/m}^{\pi/2} \cos^{2n+1} x \, dx < \frac{1}{m} + \frac{1}{m} \left(\frac{\pi}{2} - \frac{1}{m} \right) < \frac{\pi}{m}$$

and the result follows from Proposition 2. ■

Theorem 1 Let $0 < \mu < \sqrt{(2n+1) \ln 2}$. Then, for all μ and $n \in \mathbb{N}$

$$\begin{aligned} & \frac{1}{\sqrt{2n+1}} \frac{1}{\sqrt{\ln 2}} \int_0^\mu e^{-v^2} \, dv + \frac{1}{2n+2} \left[\left(\frac{\pi}{2} - z \right) \left(1 - \frac{1}{2} z^2 \right)^{2n+1} \right] \\ & < \sum_{i=0}^n \binom{n}{i} (-1)^i \frac{1}{2i+1} < \sqrt{\frac{6\pi}{11}} \frac{1}{\sqrt{2n+1}} + \frac{(\pi/2 - 1)^{2n+2}}{2n+2}, \end{aligned}$$

where $z = \mu / \sqrt{(2n+1) \ln 2}$.

Proof: From Taylor's formula

$$\cos x = 1 - \frac{x^2}{2!} + (\cos p) \frac{x^4}{4!} \leq 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

for some $p \in [0, x]$ and

$$\cos x \leq 1 - \frac{x^2}{2!} + \frac{x^2}{4!} = 1 - \frac{11}{24} x^2$$

for all $x \in [0, 1]$ with strict inequality on $(0, 1)$.

Remember that $\ln(1-u) = -\sum_{i=1}^{\infty} \frac{u^i}{i}$ for all $u \in [0, 1)$, so that $\forall x \in (0, 1)$

$$\cos^{2n+1} x < e^{(2n+1) \ln(1 - \frac{11}{24} x^2)} < e^{-(2n+1) \frac{11}{24} x^2}.$$

Summing the following inequalities

$$\begin{aligned} \int_0^1 \cos^{2n+1} x \, dx &< \int_0^1 e^{-(2n+1)\frac{11}{24}x^2} \, dx < \sqrt{\frac{24}{11(2n+1)}} \int_0^\infty e^{-v^2} \, dv = \sqrt{\frac{6\pi}{11(2n+1)}} \\ \int_1^{\pi/2} \cos^{2n+1} x \, dx &< \int_1^{\pi/2} \left(\frac{\pi}{2} - x\right)^{2n+1} \, dx = \frac{\left(\frac{\pi}{2} - 1\right)^{2n+2}}{2n+2}, \end{aligned}$$

we establish the upper bound.

Again, from Taylor's formula, for all $x \in (0, \pi/2)$, $\cos x > 1 - \frac{1}{2}x^2$ and we have for all $x \in (0, 1)$

$$\begin{aligned} \cos^{2n+1} x &> e^{(2n+1)\ln(1-\frac{1}{2}x^2)} = e^{-(2n+1)\sum_{i=1}^\infty \frac{(\frac{1}{2}x^2)^i}{i}} > e^{-(2n+1)\frac{(\sum_{i=1}^\infty (\frac{1}{2})^i)}{1}} x^2 \\ &= e^{-(2n+1)(\ln 2)x^2}. \end{aligned}$$

Since $z < 1$ we write

$$\begin{aligned} a_n &> \int_0^z e^{-(2n+1)(\ln 2)x^2} \, dx + \int_z^{\pi/2} \left(\left(1 - \frac{1}{2}z^2\right) + \frac{-(1 - \frac{1}{2}z^2)}{(\frac{\pi}{2} - z)}(x - z) \right)^{2n+1} \, dz \\ &= \frac{1}{\sqrt{(2n+1)\ln 2}} \int_0^\mu e^{-v^2} \, dv + \frac{1}{2n+2} \frac{(\frac{\pi}{2} - z)}{(1 - \frac{1}{2}z^2)} \left(1 - \frac{1}{2}z^2\right)^{2n+2} \end{aligned}$$

and the proof is complete. $\square$

For sequences x_n and y_n we write $x_n \sim y_n$ when $\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = 1$

Denote ℓ_n and u_n the lower and upper bounds given in Theorem 1. Now, letting $n \rightarrow \infty$ we have the asymptotic behaviours:

Clearly $u_n \sim \sqrt{\frac{6\pi}{11}} \frac{1}{\sqrt{2n+1}}$ and, choosing for example $\mu = \sqrt{(2n+1)\ln 2} - 1/n$, we have

$$\begin{aligned} \ell_n &= \frac{1}{\sqrt{2n+1}} \frac{1}{\sqrt{\ln 2}} \int_1^\mu e^{-v^2} \, dv + \frac{1}{2n+2} \left[\left(\frac{\pi}{2} - z\right) \left(1 - \frac{1}{2}z^2\right)^{2n+1} \right] \\ &\sim \frac{1}{\sqrt{2n+2}} \frac{1}{\sqrt{\ln 2}} \int_0^\infty e^{-v^2} \, dv = \frac{1}{2} \sqrt{\frac{\pi}{\ln 2}} \frac{1}{\sqrt{2n+1}}. \end{aligned}$$

Define b_n by $a_n = b_n \frac{1}{\sqrt{2n+1}}$. We have shown that

$$\frac{1}{2} \sqrt{\frac{\pi}{\ln 2}} \leq \liminf b_n \leq \limsup b_n \leq \sqrt{\frac{6\pi}{11}}.$$

We can improve these bounds as it is shown by the following theorems and propositions:

Theorem 2 $\forall n \in \mathbb{N}, \forall m > 1$,

$$\begin{aligned} a_n &< \sqrt{\frac{2}{1 - (1/12m^2)}} \frac{1}{\sqrt{2n+1}} \int_0^{\frac{1}{m} \sqrt{\frac{2n+1}{2} (1 - \frac{1}{12m^2})}} e^{-v^2} dv + \frac{\pi}{2} \left(\cos\left(\frac{1}{m}\right) \right)^{2n+1} \\ &< \sqrt{\frac{\pi}{2 - (1/6m^2)}} \frac{1}{\sqrt{2n+1}} + \frac{\pi}{2} \left(\cos\left(\frac{1}{m}\right) \right)^{2n+1}. \end{aligned}$$

Proof: For $0 < x < 1/m, m > 1$,

$$\begin{aligned} \cos x &< 1 - \frac{x^2}{2!} + \frac{x^2}{4!} \left(\frac{1}{m} \right)^2 = 1 - \frac{1}{2} \left(1 - \frac{1}{12m^2} \right) x^2, \\ a_n &< \int_0^{1/m} \cos^{2n+1} x \, dx + \int_{1/m}^{\pi/2} \left(\cos\left(\frac{1}{m}\right) \right)^{2n+1} dx \\ &< \int_0^{1/m} e^{-(2n+1)\frac{1}{2}(1 - \frac{1}{12m^2})x^2} dx + \frac{\pi}{2} \left(\cos\left(\frac{1}{m}\right) \right)^{2n+1} \\ &= \sqrt{\frac{2}{(2n+1)(1 - 1/12m^2)}} \int_0^{\frac{1}{m} \sqrt{\frac{(2n+1)}{2} (1 - \frac{1}{12m^2})}} e^{-v^2} dv + \frac{\pi}{2} \left(\cos\left(\frac{1}{m}\right) \right)^{2n+1} \end{aligned}$$

and the result follows from $\int_0^\infty e^{-v^2} dv = \frac{\sqrt{\pi}}{2}$. [1] □

Proposition 4 $\limsup b_n \leq \sqrt{\frac{\pi}{2}}$.

Proof: Letting $m = \sqrt[3]{2n+1}$ and $c_n = \frac{\pi}{2} \left(\cos\left(\frac{1}{m}\right) \right)^{2n+1} (2n+1)$ we have

$$\lim_{n \rightarrow \infty} \frac{1}{m} \sqrt{\frac{2n+1}{2} \left(1 - \frac{1}{12m^2} \right)} = +\infty$$

and

$$\lim_{n \rightarrow \infty} c_n = \lim_{y \rightarrow \infty} y^3 \left(\cos\left(\frac{1}{y}\right) \right)^{y^3} = 0,$$

since, for sufficiently large y , $\cos(\frac{1}{y}) > 0$,

$$\left(\sin\left(\frac{1}{2y}\right) \right) / (1/2y) > \frac{1}{2} \quad \text{and} \quad \left(1 - 2 \sin^2\left(\frac{1}{2y}\right) \right)^{-\frac{1}{2 \sin^2(\frac{1}{2y})}} > 2$$

so that

$$0 < y^3 \left(\cos\left(\frac{1}{y}\right) \right)^{y^3} = y^3 \left(\left(1 - 2 \sin^2\left(\frac{1}{2y}\right) \right)^{-\frac{1}{2 \sin^2(\frac{1}{2y})}} \right)^{-y^3 2 \sin^2(\frac{1}{2y})} < y^3 2^{-y \cdot 2 \cdot \frac{1}{4} \cdot (\frac{1}{2})^2} \rightarrow 0$$

as $y \rightarrow \infty$.

Now, from Theorem 2,

$$a_n < \sqrt{\frac{2}{(1 - \frac{1}{12n^2})}} \frac{1}{\sqrt{2n+1}} \frac{\sqrt{\pi}}{2} + c_n \frac{1}{2n+1},$$

$$\text{from which } b_n < \sqrt{\frac{\pi}{2}} + c_n \frac{1}{\sqrt{2n+1}}. \quad \blacksquare$$

Note that the upper bounds given in Theorem 2 or in the proof of Proposition 4 may not improve the approximation of a_n , relative to that of Theorem 1, for small n .

Theorem 3 *Let $M > 1$ and $0 < \mu < \frac{1}{M} \sqrt{(2n+1) \left\{ \frac{1}{2} + \frac{1}{M^2} (\ln 2 - \frac{1}{2}) \right\}}$. Then, for all M, μ and $n \in \mathbb{N}$,*

$$\frac{1}{\sqrt{2n+1} \sqrt{\frac{1}{2} + \frac{1}{M^2} (\ln 2 - \frac{1}{2})}} \int_0^\mu e^{-v^2} dv + \frac{1}{2n+2} \left[\left(\frac{\pi}{2} - z \right) \left(1 - \frac{1}{2} z^2 \right)^{2n+1} \right] < a_n,$$

where $z = \mu / \left(\frac{1}{M} \sqrt{(2n+1) \left\{ \frac{1}{2} + \frac{1}{M^2} (\ln 2 - \frac{1}{2}) \right\}} \right)$.

Proof: Observe that for all x , $0 < x < \frac{1}{M}$,

$$\begin{aligned} \cos^{2n+1} x &> e^{-(2n+1) \sum_{i=1}^{\infty} \frac{(\frac{1}{2} x^2)^i}{i}} > e^{-(2n+1) \left\{ \frac{1}{2} x^2 + \frac{1}{M^2} \sum_{i=2}^{\infty} \frac{(\frac{1}{2})^i}{i} x^{2i-2} \right\}} \\ &> e^{-(2n+1) \left\{ \frac{1}{2} + \frac{1}{M^2} \left(\sum_{i=2}^{\infty} \frac{(1/2)^i}{i} \right) \right\} x^2} = e^{-(2n+1) \left\{ \frac{1}{2} + \frac{1}{M^2} (\ln 2 - \frac{1}{2}) \right\} x^2}. \end{aligned}$$

Noting that $0 < z < 1$, follow the steps at the end of theorem's 1 proof. $\blacksquare$

Proposition 5 $\liminf b_n \geq \sqrt{\frac{\pi}{2}}.$

Proof: Let $M = \sqrt[4]{2n+1}$ and $\mu = \frac{1}{M} \sqrt{(2n+1) \left\{ \frac{1}{2} + \frac{1}{M^2} (\ln 2 - \frac{1}{2}) \right\}} - \frac{1}{n}$.

Then $\mu \sim \sqrt{\frac{1}{2}} \sqrt[4]{2n+1} \rightarrow \infty$ and $y \rightarrow 0$ as $n \rightarrow \infty$.

Now, from Theorem 3,

$$\begin{aligned} \liminf b_n &\geq \lim_{n \rightarrow \infty} \sqrt{2n+1} \left(\frac{1}{\sqrt{2n+1} \sqrt{\left\{ \frac{1}{2} + \frac{1}{M^2} (\ln 2 - \frac{1}{2}) \right\}}} \int_0^\mu e^{-v^2} dv \right. \\ &\quad \left. + \frac{1}{2n+2} \left[\left(\frac{\pi}{2} - z \right) \left(1 - \frac{1}{2} z^2 \right)^{2n+1} \right] \right) \\ &= \sqrt{2} \int_0^\infty e^{-v^2} dv = \sqrt{\frac{\pi}{2}}. \end{aligned} \quad \blacksquare$$

Propositions 4 and 5 tell us that there exists the limit $\lim_{n \rightarrow \infty} b_n$ and that it is equal to $\sqrt{\pi/2}$.

Theorem 4 (*asymptotic behaviour*)

$$\sum_{i=1}^n \binom{n}{i} \frac{(-1)^i}{2i+1} = \int_0^{\pi/2} \cos^{2n+1} x \, dx \sim \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{2n+1}} \quad \text{as } n \rightarrow \infty.$$

Proof: Propositions 4 and 5. $\blacksquare$

3 Comments

We observe that the bounds can be improved by dividing the interval $(0, \pi/2)$ in more than two pieces as we have done here and the most direct approach seems to be piecewise linearly approximating the cosine function both from above and below except at a neighbourhood of the origin where it is approximated by Taylor's formula. Although this procedure will improve the bounds for small n , of course it will not change the asymptotic result.

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References

- [1] Nicholas, C.P. Yates, R.C. The Probability Integral, Amer. Math. Monthly, Vol 57, No 6, 412-413, 1950.

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