

CRITICAL COVERING MAPS WITHOUT ABSOLUTELY CONTINUOUS INVARIANT PROBABILITY MEASURE

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ABSTRACT. We consider the dynamics of smooth covering maps of the circle with a single critical point of order greater than 1. By directly specifying the combinatorics of the critical orbit, we show that for an uncountable number of combinatorial equivalence classes of such maps, there is no periodic attractor nor an ergodic absolutely continuous invariant probability measure.

1. Introduction. In the study of one-dimensional dynamical systems, the question of the existence of an absolutely continuous invariant probability (acip) plays a central role. The importance stems from the fact that an acip provides a complete description of the asymptotic distribution of typical orbits of the system.

The existence of an acip has been much studied for unimodal maps of the interval. Jakobson [8] showed that maps in the quadratic family have an acip for a set of parameters of positive measure. For example, parameters for which the critical point is strictly preperiodic, as studied by Misiurewicz [14]. Keller [10] proved that the existence of an acip is equivalent to almost everywhere positivity of the Lyapunov exponent. Collet and Eckmann [5] showed that exponential growth of the size of the derivative along the critical orbit is a sufficient condition for the existence of an acip. This growth condition was weakened to a summability condition by Nowicki and van Strien [15], and weakened still further to a lower bound condition by Bruin, Shen and van Strien [4]. This result was later extended to multimodal maps [3].

There are also results on non-existence. Arnold [1] showed that for a dense set of irrational rotation numbers, the conjugacy from an analytic diffeomorphism of the circle to a rotation with the same rotation number is not absolutely continuous. In particular, such a diffeomorphism has no acip. Johnson [9] was the first to construct quadratic interval maps without an acip and topologically conjugate to a tent map.

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This was further developed by Hofbauer and Keller [7] using the tools of kneading theory.

Nevertheless, existence of an acip has been shown to be the more prevalent situation for certain families of maps. Herman [6] proved that for almost every irrational rotation number, the conjugacy to a rotation is smooth, and so an acip exists. Lyubich [11] proved that for almost every map in the quadratic family, there is either an acip or an attracting periodic cycle. This result was extended to real analytic quadratic unimodal maps by Avila, Lyubich and de Melo [2].

In this article, we consider smooth critical covering maps of the circle without an acip. As in the non-existence results for other classes of maps, we specify the dynamics at the topological level. For clarity, we specify suitable combinatorics for the critical orbit directly. The fact that the critical point of a covering map is of inflection type rather than a turning point means that the techniques used are significantly different to the unimodal case because of the lack of dynamical symmetry. In the next section we state the main results after introducing the required notation.

2. Notation and results.

2.1. Covering maps. Consider the circle S^1 to be defined as the set of complex numbers with modulus one equipped with the topology, orientation and differentiable structure of the real numbers induced by the exponential map $\tau : \mathbb{R} \rightarrow S^1$ given by $\tau(t) = e^{2\pi it}$. A distance on S^1 can be defined in the following way: given $x = \tau(t_1)$ and $y = \tau(t_2)$, with $t_1, t_2 \in [0, 1]$, the distance between x and y , denoted by $|x - y|$ is the minimum of $|t_1 - t_2|$ and $1 - |t_1 - t_2|$. Many times in this article, we choose a convenient value of t and represent the circle by the interval $[t, t + 1]$, where we use the identification $t \sim t + 1$.

A surjective locally homeomorphic map $f : S^1 \rightarrow S^1$ is said to be a *covering map of the circle of (topological) degree d* , $|d| \geq 1$, if the pre-image of each point consists of exactly $|d|$ points and f is order-preserving or order-reversing for $d > 0$ or $d < 0$ respectively.

Given a covering map $f : S^1 \rightarrow S^1$ of degree d , we can find a *lift*: a map $F : \mathbb{R} \rightarrow \mathbb{R}$ for which $\tau \circ F = f \circ \tau$. A lift F has the property that $F(t+1) = F(t) + d$ for all $t \in \mathbb{R}$. Conversely, any homeomorphism of the real line with this property is a lift of some covering map of the circle. For example, the linear function $L_d : \mathbb{R} \rightarrow \mathbb{R}$ given by $L_d(t) = dt$ is a lift of the *uniform covering map of degree d* which we denote by $l_d : S^1 \rightarrow S^1$.

We introduce an equivalence relation for covering maps. Given two covering maps f and g of the circle and two marked points $a, b \in S^1$, we say that the pairs (f, a) and (g, b) are *combinatorially equivalent* if there exists a homeomorphism $h : S^1 \rightarrow S^1$ such that $h(f^n(a)) = g^n(b)$ for all $n \geq 0$.

2.2. Topological dynamics. We begin our study by recalling some of the basic topological dynamics of covering maps. If $d \in \{-1, 1\}$, then f is a homeomorphism. For $d = -1$, f is an order-reversing homeomorphism, which necessarily has a fixed point and so the dynamics are well understood. For $d = 1$, f may have no periodic points, and the dynamical behaviour is an area of current interest.

For $d \notin \{-1, 1\}$, covering maps are not injective and possess periodic points of all periods and also many compact invariant sets: that is, compact sets $\Lambda \subset S^1$ such that $f(\Lambda) \subseteq \Lambda$. Given $x \in S^1$, the set of all accumulation points of the forward

orbit $(f^n(x))_{n \geq 0}$ is called the ω -limit set of x and is denoted by $\omega(x)$. The ω -limit set of a point is a compact invariant set that is important for the analysis of the dynamics. If x is periodic, then $\omega(x)$ is a finite set, but there are other possibilities for ω -limit sets, such as Cantor sets or the whole circle S^1 .

For $|d| \geq 2$, the uniform covering map l_d is expanding and points in a residual set have orbits that are dense on the whole circle S^1 . Covering maps that are not expanding may have periodic attractors. Also there may exist intervals $I \subset S^1$ whose orbit consists of pairwise disjoint intervals: that is, $f^n(I) \cap f^m(I) = \emptyset$ for $0 \leq n < m$. Such an interval could be contained in the basin of a periodic attractor. If this is not the case, then it is called a *wandering interval*. The existence of wandering intervals is a factor that complicates the topological understanding of the dynamics.

There exists a semi-conjugacy between a covering map f of degree d and the uniform covering map l_d : that is, a monotone surjective map $h : S^1 \rightarrow S^1$ such that $h \circ f = l_d \circ h$. The map h is locally constant on wandering intervals and on connected components of the basin of a periodic attractor, if either of these exist. In the absence of wandering intervals, periodic attractors, and intervals consisting entirely of periodic points, the semi-conjugacy is, in fact, a conjugacy: h is a homeomorphism. In which case, we can conclude that for f , orbits of points in a residual set are dense on the whole circle S^1 . Also, we can conclude that if $\text{Per}(f)$ is the set of all periodic points, then the closure $\overline{\text{Per}(f)}$ is the whole circle.

2.3. Measure-theoretic properties. The second step in understanding the dynamics of critical covering maps depends on the ergodic aspects. Let us assume from now on that there are no wandering intervals, periodic attractors nor intervals consisting entirely of periodic points.

The Lebesgue measure on the circle, which we denote by λ , is invariant by the uniform covering map l_d : that is, for any Borel set $B \subset S^1$, $\lambda(l_d^{-1}(B)) = \lambda(B)$. Using the homeomorphism $h : S^1 \rightarrow S^1$ conjugating a critical covering map f to l , we can define an invariant probability measure μ for f : for every Borel set $B \subset S^1$, $\mu(B) = \lambda(h(B))$. This can also be formulated in terms of the regularity of the conjugating homeomorphism.

Let ν be a Borel probability measure on the circle that is invariant for f . Assume that ν is *ergodic* with respect to f : that is, for any Borel set $B \subset S^1$, the ν -measure of the symmetric difference $f^{-1}(B) \Delta B$ is either 0 or 1. The *basin* of ν is the set

$$\mathcal{B}(\nu) = \left\{ x \in S^1 : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \int_{S^1} \varphi \, d\nu \quad \text{for all } \varphi \in C^0(S^1, \mathbb{R}) \right\},$$

where $C^0(S^1, \mathbb{R})$ denotes the class of continuous functions from S^1 to \mathbb{R} . The *support* of ν is a compact invariant set for f . The basin of ν is a *totally invariant set*: that is, $f(\mathcal{B}(\nu)) \subseteq \mathcal{B}(\nu)$ and $f^{-1}(\mathcal{B}(\nu)) \subseteq \mathcal{B}(\nu)$. By the Ergodic Theorem of Birkhoff, the ν -measure of $\mathcal{B}(\nu)$ is 1. However, the Lebesgue measure of $\mathcal{B}(\nu)$ could be smaller, even zero. If the invariant probability measure ν is absolutely continuous with respect to Lebesgue measure λ (or an acip measure, for short), then the support of ν has positive Lebesgue measure. Moreover, under our assumptions the map f is ergodic with respect to λ (see [16]) and this implies that the basin of ν has Lebesgue measure 1. This guarantees that the acip measures are of great relevance for the description of the dynamics of f .

2.4. Critical covering maps. A *critical covering map* of the circle is a covering map $f : S^1 \rightarrow S^1$ of class C^r , $r \geq 1$, with a unique critical point of inflection type, which we denote by c_f or simply by c if no confusion will occur. The local inverse of a critical covering map is also C^r , except at the critical value $f(c)$. If f and g are critical covering maps for which (f, c_f) is combinatorially equivalent to (g, c_g) , where c_f and c_g are the critical points of f and g respectively, then we shall say that f is combinatorially equivalent to g .

As a source of examples of critical covering maps, we consider the *Arnold family* of maps $g_\alpha : S^1 \rightarrow S^1$, $\alpha \in [0, 1]$, with lift $G_\alpha : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$G_\alpha(t) = \alpha + 2 \left(t + \frac{1}{2\pi} \sin(2\pi t) \right).$$

As we show below the Arnold family is an example of a full family of critical covering maps. Consider a family $\{g_\alpha\}_{\alpha \in \Delta}$ of critical covering maps, where Δ is an interval and $\alpha \mapsto g_\alpha(x)$ is continuous for each $x \in S^1$. Such a family is said to be a *full family* if, any critical covering map f is combinatorially equivalent to g_α , for some $\alpha \in \Delta$.

Let us show some basic properties of each map g_α in the Arnold family. A straightforward computation shows that G_α has negative Schwarzian derivative, which implies that g_α has at most one periodic attractor, see [13]. The point $c = 1/2$ and its translations by integers are the critical points of G_α and all of them project to the unique critical point of g_α , which is a fixed point for $g_{-1/2}$. For $\beta = \left(-\frac{1}{6} + \frac{\sqrt{3}}{2\pi}\right)$ it is easy to see that $g_{-1/2-\beta}$ and $g_{-1/2+\beta}$ have an indifferent fixed point which is attracting from one side and repelling from the other. For $\alpha \in (-1/2-\beta, -1/2+\beta)$ the map g_α has an attracting hyperbolic fixed point whose immediate basin of attraction, bounded by two repelling hyperbolic fixed points, contains the critical point. For $\alpha \in (-1/2+\beta, 1/2-\beta)$, the map g_α has only one fixed point which is a hyperbolic repeller, say p_α . This fixed point and its pre-image define a partition of the circle with two arcs, each of which are mapped onto the circle by g_α . So we can conclude that g_α is topologically conjugate to the shift on $\{0, 1\}^{\mathbb{N}}$. Now we observe that the derivative with respect to α is 1 for the critical value $G_\alpha(1/2)$ and is negative for the repelling fixed point p_α . Then we can conclude that for any sequence in $\{0, 1\}^{\mathbb{N}}$, there is some value of $\alpha \in [-1/2+\beta, 1/2-\beta]$ such that the itinerary of the critical value of g_α realises this sequence. Thus the Arnold family $\{g_\alpha\}_{\alpha \in \Delta}$, with $\Delta = [-1/2-\beta, 1/2-\beta]$, is an example of a full family of critical covering maps of the circle.

In order to ensure that a critical covering map has no wandering intervals, see [13] and also [16], it is enough to assume that it is of class C^2 (outside the critical point where it is C^1) and the critical point has finite order: that is, there exist a C^1 map ψ defined near c satisfying $\lim_{x \rightarrow c} \psi(x) = 0$ and real constants $\vartheta > 0$ and $\beta > 1$ such that

$$f(x) = f(c) + \vartheta \operatorname{sgn}(x - c) |x - c|^\beta (1 + \psi(x)), \quad (1)$$

for every x in a neighbourhood of c . The constant β , when it exists for some ψ as above, is called the *order* of the critical point.

From now on, we shall concentrate on full families of C^1 critical covering maps $\{g_\alpha\}_{\alpha \in \Delta}$ with the following properties for all $\alpha \in \Delta$:

- (H1) the critical point c has order $\beta > 1$;
- (H2) g_α has topological degree 2;

(H3) Restricted to $S^1 \setminus \{c\}$, the map g_α is C^3 and has negative Schwarzian derivative.

Remark 1. Assumptions (H2) and (H3) are to simplify the exposition: the same methods can be used for any degree $d \geq 2$, and the arguments could be adapted so as not to use the negative Schwarzian assumption, but this would increase the technicality of the proofs.

2.5. Statement of results. Our main result, Theorem 2.1 below, concerns the absence of acip measures, which we will prove in the Section 5. According to this theorem, the measure μ defined above may not describe the statistical behaviour of a significant set of orbits. This property depends on a strong recurrence of the critical point and to understand it, we will describe in Section 3 an uncountable set of combinatorics of the critical orbit associated to it.

Theorem 2.1. *Within any full family of critical covering maps satisfying the above hypotheses (H1)–(H3), there are uncountably many combinatorially non-equivalent maps with no absolutely continuous invariant measure and no periodic attractor.*

In the case of any full family, we have the following corollary.

Corollary 1. *Within any full family of critical covering maps $\{g_\alpha\}_{\alpha \in \Delta}$ satisfying the above hypotheses (H1)–(H3), there are uncountably many values of $\alpha \in \Delta$ for which g_α has no absolutely continuous invariant measure and no periodic attractor.*

We specify the combinatorics used in Theorem 2.1 above in the following section.

3. Combinatorics.

3.1. Kneading sequences. For our results we need to find critical covering maps with strongly recurrent critical points. Then, when iterating the map, the strong contraction near to the critical point can overcome the expansion that occurs on the rest of the circle. We specify the strongly recurrent behaviour at the combinatorial level.

There are different choices of partition that can be made for defining the kneading sequence of a critical covering map. For example, the partition defined by the unique fixed point p and its pre-image $q \neq p$ is a Markov partition with two intervals. However, the itineraries given by this partition are linked with the recurrence of the critical point in a complicated way.

Given a critical covering map f consider instead the circle represented by the interval $[c, c + 1]$, where c and $c + 1$ are identified. Then choose the partition $c < z_1^+ < z_1^- + 1 < c + 1$, where $z_1^- < c < z_1^+$ are the two pre-images of c . This partition has three intervals $I_0 = [c, z_1^+]$, $I_1 = (z_1^+, z_1^- + 1)$ and $I_2 = [z_1^- + 1, c + 1)$. We shall use this partition to define a coding of the orbits of f . Let A denote the alphabet consisting of the three symbols 0, 1 and 2.

We say that $x \in S^1$ is coded by the symbol $i \in A$ if $x \in I_i$. Similarly, given $x \in S^1$ and $n \geq 1$, the piece of orbit $(x, f(x), \dots, f^{n-1}(x))$ is coded by the finite word $i_1 \dots i_n \in A^n$ if $f^{j-1}(x) \in I_{i_j}$ for all $1 \leq j \leq n$.

Extending to forward orbits, we have the coding map $\kappa : S^1 \rightarrow A^\mathbb{N}$, where $\kappa(x)$ is the infinite word $i_1 i_2 \dots$ with j th term i_j , where $f^{j-1}(x) \in I_{i_j}$ for $j \in \mathbb{N}$. If we denote by $\mathcal{S} : A^\mathbb{N} \rightarrow A^\mathbb{N}$ the left shift, then we have $\mathcal{S} \circ \kappa = \kappa \circ f$.

Of special importance is the coding $\kappa(f(c))$ of the forward orbit of the critical value $f(c)$, which is called the *kneading sequence* of f and denoted by κ_f . Clearly, combinatorially equivalent critical covering maps have the same kneading sequence

and vice-versa. However, as the partition I_0, I_1, I_2 is not Markov, not every element of $A^{\mathbb{N}}$ is the kneading sequence of some critical covering map. We say a sequence $\omega \in A^{\mathbb{N}}$ is *admissible* if there exists some critical covering map f with $\kappa_f = \omega$.

We now define the kneading sequences required for Theorem 2.1. Given any two sequences $(\ell_n)_{n=1}^{\infty}$ and $(m_n)_{n=1}^{\infty}$ of natural numbers with $(\ell_n)_{n=1}^{\infty}$ strictly increasing and $m_n \geq 2$, we inductively define a sequence $\omega \in A^{\mathbb{N}}$. We first define the words $\omega_1 = 0$ and $\nu_1 = 1^{\ell_1}0$. Then, assuming that the words ω_n and ν_n are already defined, we define the words ω_{n+1} and ν_{n+1} by setting

$$\omega_{n+1} = \omega_n^{m_n} \nu_n \quad \text{and} \quad \nu_{n+1} = \omega_n^{m_n} \cdots \omega_1^{m_1} 1^{\ell_{n+1}} 0, \quad (2)$$

where, for a finite word u and integer $k \geq 1$, u^k denotes the word obtained by concatenating k copies of the word u . Similarly, we will write u^{∞} for the infinite word obtained by concatenating countably many copies of a finite word u .

Notice that the symbol 2 is forbidden from occurring in the kneading sequences. As a result, the critical orbit avoids the interval I_2 and so the ω -limit set $\omega(c)$ is a Cantor set (see Theorem 4.1).

Denoting the number of symbols in a word by $|\cdot|$, we have that

$$|\omega_{n+1}| = m_n |\omega_n| + |\nu_n| \quad \text{and} \quad |\nu_{n+1}| = m_n |\omega_n| + \cdots + m_1 |\omega_1| + \ell_{n+1} + 1.$$

We also define $r_n = |\omega_n|$. We denote by $[\omega_n] \subset A^{\mathbb{N}}$ the *cylinder* of all infinite words beginning with the word ω_n . By construction, for each n the word ω_{n+1} begins with the word ω_n , and so we have a nested sequence $[\omega_1] \supset [\omega_2] \supset \cdots$. As $|\omega_n| \rightarrow \infty$, the intersection $\bigcap_{n=1}^{\infty} [\omega_n]$ consists of a unique infinite word, which we denote by $\omega \in A^{\mathbb{N}}$.

As there are uncountably many distinct choices for the sequences $(\ell_n)_{n=1}^{\infty}$ and $(m_n)_{n=1}^{\infty}$, and each pair of sequences gives rise to a different sequence in $A^{\mathbb{N}}$, we obtain an uncountable collection of infinite words ω that we denote by \mathcal{K} .

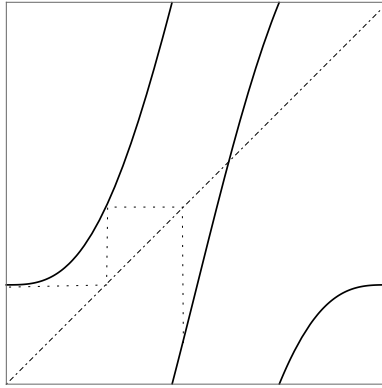
We denote by $\mathcal{H}(\omega_n) = \mathcal{H}_{\beta, \vartheta, \psi}(\omega_n)$ the set of all critical covering maps which satisfy hypotheses (H1)–(H3) with $\beta > 1$, $\vartheta > 0$, ψ fixed (see Equation 1) and having kneading sequence in the cylinder $[\omega_n]$.

Remark 2. We will show below that the sets $\mathcal{H}(\omega_n)$ are non-empty and that they nest down to the non-empty set $\mathcal{H}(\omega) = \bigcap_{n=1}^{\infty} \mathcal{H}(\omega_n)$. Every critical covering map in this set has the same kneading sequence ω in \mathcal{K} . We also can see that $\mathcal{H}(\omega_n)$ contains $\mathcal{H}(\omega_n^{\infty})$ and, as a consequence, any critical covering map in $\mathcal{H}(\omega)$ can be approached by critical covering maps in $\mathcal{H}(\omega_n^{\infty})$. The critical point of each map in $\mathcal{H}(\omega_n^{\infty})$ is asymptotic to a periodic attractor of period $r_n = |\omega_n|$.

3.2. Admissibility of combinatorics. We will now show that $\mathcal{H}(\omega)$ is non-empty. We start by showing that $\mathcal{H}(\omega_n)$ is non-empty: indeed, we will construct by induction, a sequence of critical covering maps $f_n \in \mathcal{H}(\omega_n)$, each with critical point $c = 0$. The way we construct f_n , the sequence will converge to a map in $\mathcal{H}(\omega)$, showing that this set is also non-empty. Alternatively, each f_n is topologically conjugate to a critical covering map g_{α_n} in a given full family $\{g_{\alpha}\}_{\alpha \in \Delta}$, then g_{α_n} converges to a map in $\mathcal{H}(\omega)$.

Before this, note that, for a critical covering map f , the open interval $I_1 = (z_1^+, z_1^- + 1)$ contains a unique fixed point p of f and

$$[z_1^+, p) = \bigcup_{n=1}^{\infty} \overline{L_n} \quad \text{and} \quad (p, z_1^- + 1] = \bigcup_{n=1}^{\infty} \overline{R_n},$$

FIGURE 1. The graph of the critical covering map f_1 .

where $L_n, R_n, n \geq 1$, are the *left and right fundamental domains* of p : that is, they are the maximal open intervals satisfying

$$f(L_{n+1}) = L_n, \quad f(R_{n+1}) = R_n, \quad f(L_1) = (c, z_1^+) \quad \text{and} \quad f(R_1) = (z_1^- + 1, c + 1).$$

Remember that we identify the intervals $(z_1^- + 1, c + 1)$ and (z_1^-, c) .

Let $V_1 = [z_1^-, z_1^+]$ denote the closed neighbourhood of c with endpoints z_1^- and z_1^+ . For any small neighbourhood U of c , we shall denote the *left side* $U \cap [z_1^-, c]$ by U^- and the *right side* $U \cap [c, z_1^+]$ by U^+ . So U^- and U^+ are intervals such that $U = U^- \cup U^+$ and $U^- \cap U^+ = \{c\}$.

Now we start the construction by induction of the sequence of critical covering maps $(f_n)_{n=1}^\infty$ in $\mathcal{H}(\omega_n)$. The easy first step of induction below is just the construction by hand shown in Figure 3.2.

3.3. Induction argument. We use an induction argument to show that all sequences in \mathcal{K} are admissible.

First step of induction. There exists a critical covering map f_1 for which the following properties hold:

- (i) There exists a closed neighbourhood V_2 of c , $V_2 \subset V_1 = [z_1^+, z_1^-]$ such that, for $r_1 = |\omega_1| = 1$, the iterate $f_1^{r_1}$ maps V_2 homeomorphically onto V_1^+ . The restriction of $f_1^{r_1}$ to V_2 is a branch of the first entry map to V_1^+ which we denote by ϕ_1 ;
- (ii) The branch ϕ_1 maps V_2 onto V_1^+ and, for $x \in V_2$, the piece of orbit $(f_1(x), \dots, f_1^{r_1}(x))$ is coded by the word ω_1 ;
- (iii) For $k > \ell_0 := 0$, there exists a branch $\sigma_{1,k} : E_1^k \rightarrow V_1^+$ of first entry map that maps $E_1^k \subset V_1^+ \setminus V_2^+$ diffeomorphically onto V_1^+ . For $y \in E_1^k$, the piece of orbit $(f_1(y), \dots, \sigma_{1,k}(y))$ is coded by the word $1^k 0$. In particular the entry time of $f_1(y)$ to V_1^+ is $t_{1,k} = k$;
- (iv) For $1 \leq j < m_1$, $\phi_1^j(c) \in V_2^+$, $\phi_1^{m_1}(c) \in E_1^{\ell_1}$ and there is a closed neighbourhood $D_1 \subset V_2$ such that $\phi_1^{m_1}(D_1) = V_1^+ \setminus V_2^+$ (that is, D_1 is a *fundamental domain* of ϕ_1) and $E_1^k \subset \phi_1^{m_1}(D_1^+)$ for all $k > \ell_1$.

Induction hypothesis. For $n \geq 1$, there exist critical covering maps f_1, \dots, f_n such that, for $1 \leq i \leq n$, the following properties hold:

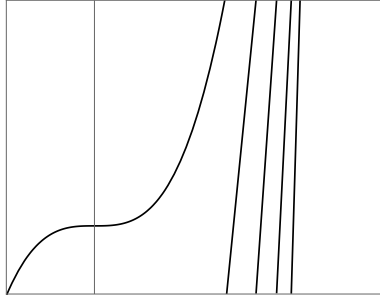


FIGURE 2. The first entry map to the interval V_{2n-1}^+ , showing the critical branch ϕ_n and some of the branches σ_{k,ℓ_k} , with $k \geq n$.

- (i) There exist nested closed neighbourhoods $V_{2n} \subset V_{2n-1} \subset \cdots \subset V_2 \subset V_1$ of c such that, for $r_i = |\omega_i|$, the iterate $f_i^{r_i}$ maps V_{2i} homeomorphically onto V_{2i-1}^+ . The restriction of $f_i^{r_i}$ to V_{2i} is a branch of the first entry map to V_{2i-1}^+ which we denote by ϕ_i ;
- (ii) The branch ϕ_i maps V_{2i} onto V_{2i-1}^+ and, for $x \in V_{2i}$, the piece of orbit $(f_i(x), \dots, f_i^{r_i}(x))$ is coded by the word ω_i ;
- (iii) For each $k > \ell_{i-1}$, there exists a branch $\sigma_{i,k} : E_i^k \rightarrow V_{2i-1}^+$ of first entry map that maps $E_i^k \subset V_{2i-1}^+ \setminus V_{2i}^+$ diffeomorphically onto V_{2i-1}^+ . For $y \in E_i^k$, the piece of orbit $(f_i(y), \dots, \sigma_{i,k}(y))$ is coded by the word $\omega_{i-1}^{m_{i-1}} \cdots \omega_1^{m_1} 1^k 0$. In particular the entry time of y to V_{2i-1}^+ is $t_{i,k} := m_{i-1}|\omega_{i-1}| + \cdots + m_1|\omega_1| + k$;
- (iv) For $1 \leq j < m_i$, $\phi_i^j(c) \in V_{2i}^+$, $\phi_i^{m_i}(c) \in E_i^{\ell_i}$ and there is a closed neighbourhood interval $D_i \subset V_{2i}$ such that $\phi_i^{m_i}(D_i) = V_{2i-1}^+ \setminus V_{2i}^+$ (that is, D_i is a *fundamental domain* of ϕ_i) and $E_i^k \subset \phi_i^{m_i}(D_i^+)$ for all $k > \ell_i$;
- (v) For $1 \leq i < n$, $f_{i+1} = f_i$ on $S^1 \setminus D_i$.

In Figure 3.3 we show the intervals $E_n^{\ell_n}$, $E_n^{\ell_{n+1}}$, V_{2n}^+ inside V_{2n-1}^+ and the branches σ_{n,ℓ_n} , $\sigma_{n,\ell_{n+1}}$ and ϕ_n of first entry map to V_{2n-1}^+ defined on them.

Induction step. We assume that the induction hypothesis holds true for $n \geq 1$ and prove it for $n+1$. Since $c \in D_n \subset V_{2n}$ and $\phi_n^{m_n}(D_n) = V_{2n-1}^+ \setminus V_{2n}^+$, we define $V_{2n+1} \subset V_{2n}$ to be the closed neighbourhood D_n . For $k > \ell_{n-1}$, there is a closed interval $I_n^k \subset E_n^k$ which is mapped diffeomorphically onto V_{2n+1}^+ by $\sigma_{n,k}$. If necessary, we modify f_n on the interior of $D_n = V_{2n+1}$ to obtain a critical covering map f_{n+1} for which $\phi_n^{m_n}(c) \in I_n^{\ell_n}$. Then we define the interval $V_{2n+2} \subset V_{2n+1}$ around the critical point such that $\phi_n^{m_n}(V_{2n+2}) = I_n^{\ell_n}$. The first entry map ϕ_{n+1} from V_{2n+2} onto V_{2n+1}^+ is defined by $\phi_{n+1} = \sigma_{n,\ell_n} \circ \phi_n^{m_n}$ and properties (i)–(ii) are satisfied for $1 \leq i \leq n+1$. For $k > \ell_n$, $I_n^k \subset E_n^k \subset \phi_n^{m_n}(V_{2n+1}^+)$ and we can define the interval $E_{n+1}^k \subset V_{2n+1}^+ \setminus V_{2n+2}^+$ such that $\phi_n^{m_n}(E_{n+1}^k) = I_n^k$. The first entry map $\sigma_{n+1,k}$ from E_{n+1}^k onto V_{2n+1}^+ is defined by $\sigma_{n+1,k} = \sigma_{n,k} \circ \phi_n^{m_n}$. In this way we have property (iii) satisfied for $1 \leq i \leq n+1$. To ensure that property (iv) is satisfied, we can adjust f_{n+1} further on the interior of V_{2n+1} a little more. Since $V_{2n+1} = D_n$, the map f_{n+1} we have constructed also satisfies property (v).

The induction above is the main step in proving Proposition 1 that guarantees that $\mathcal{H}(\omega)$ is non-empty and all sequences in \mathcal{K} are admissible.

Proposition 1. *Given a sequence $\omega \in \mathcal{K}$, in any full family of critical covering maps there exists a map f with kneading sequence ω .*

Proof. Each critical covering map f_n in the induction argument above has kneading sequence κ_{f_n} contained in the cylinder $[\omega_n]$ of all words starting with the word ω_n , $n \geq 1$. We have that $f_{n+1} = f_n$, on $S^1 \setminus D_n$. Since we have the freedom to choose the sizes of the closed neighbourhoods D_n of c so that $\bigcap_{n=1}^{\infty} D_n = \{c\}$, this guarantees that the sequence $(f_n)_{n=1}^{\infty}$ converges to a continuous covering \tilde{f} such that the kneading sequence of the marked point c is ω . In any full family there is a critical covering f with a marked critical point which is combinatorially equivalent to \tilde{f} with the point c marked. This implies that the kneading sequence of f is ω and we can conclude that ω is admissible.

Alternatively, each f_n is topologically conjugate to a critical covering map g_{α_n} in a full family $\{g_{\alpha}\}_{\alpha \in \Delta}$. Then g_{α_n} converges to a map in $\mathcal{H}(\omega)$, also showing that ω is admissible. \square

3.4. Combinatorics in full families. The sets $\mathcal{H}(\omega_n)$ and $\mathcal{H}(\omega) = \bigcap_{n=1}^{\infty} \mathcal{H}(\omega_n)$ are non-empty. In any full family $\{g_{\alpha}\}_{\alpha \in \Delta}$ of critical covering maps with g_{α} satisfying hypotheses (H1)–(H3), there is a non-degenerate interval of parameters α such that $g_{\alpha} \in \mathcal{H}(\omega_n)$. These intervals accumulate on a parameter α_{ω} such that $g_{\alpha_{\omega}} \in \mathcal{H}(\omega)$. One way to reach this interval of parameters is starting with maps g_{α_n} in $\mathcal{H}(\omega_n^{\infty})$, that is, parameters such that the critical point is asymptotic to a periodic attractor associated to the word ω_i .

The lemma below guarantees that $\mathcal{H}(\omega_{n+1}^{\infty})$ and $\mathcal{H}(\omega_n^{m_n} \nu_n^{\infty})$ are also non-empty. Therefore, the infinite words ω_{n+1}^{∞} and $\omega_n^{m_n} \nu_n^{\infty}$ are admissible.

Below, for a given $f_n \in \mathcal{H}(\omega_n)$, as in the induction above, let us use the notation E_n instead of E_{n,ℓ_n} and σ_n instead of σ_{n,ℓ_n} .

Lemma 3.1. *Given $f_n \in \mathcal{H}(\omega_n)$, the following properties hold true:*

1. *There are $x_n, y_n \in E_n$ such that $\sigma_n(x_n) = c$ and $\sigma_n(y_n) = y_n$. For $y = x_n$ or $y = y_n$, the piece of orbit $(f_n(y), \dots, \sigma_n(y))$ is coded by the word ν_n ;*
2. *f_n can be modified on D_n to obtain either a critical covering map G_n^- or G_n^+ , both in $\mathcal{H}(\omega_n)$, for which $\phi_n^{m_n}(c)$ is either x_n or y_n , respectively. The corresponding kneading sequences of G_n^- and G_n^+ are ω_n^{∞} and $\omega_{n-1}^{m_{n-1}} \nu_{n-1}^{\infty}$, respectively.*

Proof. As $E_n \subset V_{2n-1}^+$ and the map $\sigma_n : E_n \rightarrow V_{2n-1}^+$ is a surjective diffeomorphism, there are $x_n, y_n \in E_n$ such that $\sigma_n(x_n) = c$ and $\sigma_n(y_n) = y_n$. By part (iii) of the induction hypothesis, for $y \in E_n$, the piece of orbit $(f_n(y), \dots, \sigma_n(y))$ is coded by the word $\omega_{n-1}^{m_{n-1}} \dots \omega_1^{m_1} 1^{\ell_n} 0 = \nu_n$. This finishes the proof of the first part of the lemma. To prove the second part of the lemma we observe that changes of f_n in D_n do not affect the branches $\sigma_i : E_i \rightarrow V_{2i-1}^+$ and the points $x_i, y_i \in E_i$, for $1 \leq i \leq n$. There exists a closed neighbourhood H_n of c that is properly contained in D_n with the property that $\phi_n^{m_n}$ maps H_n diffeomorphically onto $[x_n, y_n]$. Moreover, the closed intervals $H_n, \phi_n(H_n), \dots, \phi_n^{m_n-1}(H_n)$ are pairwise disjoint and all are properly contained in V_{2n} . So by subtracting from f_n a smooth bump function supported on D_n we can reduce each of the points $\phi_n(c), \phi_n^2(c), \dots, \phi_n^{m_n}(c)$ so as to get a map G_n^- for which $\phi_n^{m_n}(c) = x_n$. We then have that c is a periodic point since $\phi_{n+1}(c) = \sigma_n \circ \phi_n^{m_n}(c) = c$, and so the kneading sequence for G_n^- is $(\omega_n^{m_n} \nu_n)^{\infty} = \omega_{n+1}^{\infty}$. Alternatively, by adding to f_n a smooth bump function supported on D_n we can increase each of the points $\phi_n(c), \phi_n^2(c), \dots, \phi_n^{m_n}(c)$ in

order to get a map G_n^+ for which $\phi_n^{m_n}(c) = y_n$. We then have that $\phi_n^{m_n}(c)$ is the periodic point y_n and so the kneading sequence of G_n^+ in this case is $\omega_n^{m_n} \nu_n^\infty$. \square

We have the following immediate corollary.

Corollary 2. *Given $f_n \in \mathcal{H}(\omega_n)$, there exists $G_n \in \mathcal{H}(\omega_n^\infty)$ with the property that $G_n = f_n$ on $S^1 \setminus D_n$ with critical point asymptotic to a parabolic periodic attractor.*

Proof. This is an immediate consequence of Lemma 3.1. Indeed, any continuous one parameter family in $\mathcal{H}(\omega_n)$, starting in G_n^- and ending in G_n^+ , has a critical covering map as stated. \square

Now, looking in a full family, we have the following lemma.

Lemma 3.2. *Let $\{g_\alpha\}_{\alpha \in \Delta}$ be a full family of critical covering maps that satisfies hypotheses (H1)–(H3). Then, for each $n \geq 1$, there are parameters $a(n)$ and $b(n)$ which satisfy the following:*

1. *The critical point is a periodic superattractor for $g_{a(n)}$, with period $r_n = |\omega_n|$;*
2. *The critical point is asymptotic to the orbit of a parabolic periodic attractor for $g_{b(n)}$, with period r_n ;*
3. *For α in the interval defined by $a(n)$ and $b(n)$, the map g_α has a periodic attractor of period r_n and its kneading sequence is ω_n^∞ .*

Proof. According to Lemma 3.1 there is a critical covering G_{n-1}^- whose kneading sequence is ω_n^∞ . This can be realized in any full family and then we get the parameter $a(n)$ of the statement. Instead of this, we can get a parameter $\tilde{b}(n)$ corresponding to the kneading sequence $\omega_n^{m_n} \nu_n^\infty$. Then, by an intermediate value argument, there is a parameter $b(n)$ in between a_n and \tilde{b}_n as stated. \square

3.5. Extension of first entry branches. The lemma below assures that the branches of first entry maps σ_n and ϕ_n associated to a critical covering map f_n in $\mathcal{H}(\omega_n)$ have big diffeomorphic extensions. Their ranges cover at least the interval $(p_n - 1, f_n(c) + 1)$, where p_n is the unique fixed point of f_n in $(c, c + 1)$. These extensions depend only on the combinatorial properties given by the word ω_n . In fact they depend only on the form of the word and in particular they do not depend on the sequences $(\ell_n)_{n=1}^\infty$ and $(m_n)_{n=1}^\infty$. As we will see in Lemma 5.1, as a consequence of Koebe Principles, these extensions imply a uniform control of distortion for these branches.

Lemma 3.3. *Given $f_n \in \mathcal{H}(\omega_n)$, the following properties hold true:*

1. *There exist intervals $V_{2n} \subseteq \hat{V}_n \subset \mathcal{V}_n$ which are mapped homeomorphically onto $V_{2n-1}^+ \subseteq V_1^+ \subset (p_n - 1, f_n(c) + 1)$ by $f_n^{r_n}$. Moreover, the interval \mathcal{V}_n contains only one critical point of $f_n^{r_n}$, the point c ;*
2. *If $k > \ell_{n-1}$, there exist intervals $E_n^k \subseteq \hat{E}_n^k \subset \mathcal{E}_n^k$ which are mapped diffeomorphically onto $V_{2n-1}^+ \subseteq V_1^+ \subset (p_n - 1, f_n(c) + 1)$ by $f_n^{t_{n,k}}$;*
3. *if $k > \ell_{n-1}$, then the intervals $\hat{E}_n^k, \hat{E}_n^{k+1}, \hat{E}_n^{k+2}, \dots$ are pairwise disjoint, $\hat{E}_n^k \subset \phi_n^{m_n}(V_{2n+1})$ and \mathcal{E}_n^k is contained in the convex hull of $\phi_n^{m_n-1}(c)$ and $V_{2n-1}^+ \setminus V_{2n}^+$;*
4. *If $k > \ell_n$, then $\mathcal{E}_n^k \subset \phi_n^{m_n}(V_{2n+1}^+)$;*
5. *The collections of intervals $\{\hat{E}_n^k, \dots, \hat{f}_n^{t_{n,k}}(\hat{E}_n^k)\}$ and $\{\hat{V}_n, \dots, f_n^{r_n}(\hat{V}_n)\}$ have multiplicity of intersection at most two.*

Proof. We proceed by induction on n and to start let us assume that $n = 1$. For Part 1 of the lemma we choose $\hat{V}_1 = V_2$ and to define \mathcal{V}_1 we remember that $r_1 = 1$,

f has only one critical point in $(c-1, c+1)$ and the image of this interval by f_n contains $(p_n-1, f_n(c)+1)$. Then we can choose \mathcal{V}_1 as wanted.

For the proof of Part 2 of the lemma we define $\hat{E}_1^k = E_1^k$ and to define \mathcal{E}_1^k we remember that $E_1^k \subset V_1^+ \setminus V_2^+$ and $f_n(E_1^k) \subset \mathcal{F}_k \subset f_n(V_1^+ \setminus V_2^+) = (z_1^+, c+1)$. We consider three cases:

- (i) if $k = 1$ we define \mathcal{E}_1^1 to be the convex hull of $\phi_1^{m_1-1}(c)$ and $V_1^+ \setminus V_2^+$ and get that $f_n(\mathcal{E}_1^1) = (f_n(\phi_1^{m_1-1}(c)), c+1) = (\phi_1^{m_1}(c), c+1)$. In this case $t_{1,1} = 2$ and the diffeomorphic image of \mathcal{E}_1^1 by f_n^2 is $(f_n(\phi_1^{m_1}(c)) - 1, f_n(c)+1)$;
- (ii) if $k = \ell_1 + 1$ we define $\mathcal{E}_1^{\ell_1+1} \subset V_1^+ \setminus V_2^+$ such that $f_n(\mathcal{E}_1^{\ell_1+1})$ is the minimal interval which contains $f_n(\phi_1^{m_1}(c))$ and R_{ℓ_1} . In this case $t_{1,k} = \ell_1 + 2$ and the diffeomorphic image of $\mathcal{E}_1^{\ell_1+1}$ by $f_n^{\ell_1+2}$ is $(f_n^{\ell_1+2}(\phi_1^{m_1}(c)) - 1, f_n(c)+1)$;
- (iii) if $k \geq 2$ and $k \neq \ell_1 + 1$ we define $\mathcal{E}_1^k \subset V_1^+ \setminus V_2^+$ such that $f_n(\mathcal{E}_1^k) = \mathcal{F}_{k-1}$. In this case $t_{1,k} = k+1$ and the diffeomorphic image of \mathcal{E}_1^k by f_n^{k+1} is $(f_n(c) - 1, f_n(c)+1)$.

The interval (p_n-1, c) has no points from the forward orbit of the critical point and, for $k > \ell_0$, $t_{1,k} = k+1$. In any one of the three cases above, we can shrink \mathcal{E}_1^k so that $f_n^{k+1}(\mathcal{E}_1^k) = (p_n-1, f_n(c)+1)$ as required.

To prove Part 3 of the lemma we remember that the interval \hat{E}_1^k is a connected component of the first entry map to V_1^+ and, for $y \in \hat{E}_1^k$, the piece of orbit $\{f_n(y), \dots, \sigma_{1,k}(\hat{E}_1^k)\}$ corresponds to the word $1^k 0$. This implies that the intervals $\hat{E}_1^k, \hat{E}_1^{k+1}, \hat{E}_1^{k+2}, \dots$ are pairwise disjoint. By definition, $\hat{E}_1^k = E_1^k$ and $\phi_1^{m_1}(V_3) = V_1^+ \setminus V_2^+$, it follows that, for $k > \ell_0 = 0$, $\hat{E}_1^k \subset \phi_1^{m_1}(V_3)$. The fact that \mathcal{E}_1^k is contained in the convex hull of $\phi_1^{m_1-1}(c)$ and $V_1^+ \setminus V_2^+$ also follows from the definition in one of the cases (i)–(iii) above.

To prove Part 4 of the lemma we observe that, for $k > \ell_1$, \mathcal{E}_1^k is defined by the case (ii) or case (iii) above and we get that $\mathcal{E}_1^k \subset \phi_1^{m_1}(V_3^+)$ as required.

To prove Part 5 of the lemma we remember that $\hat{E}_1^k = E_1^k$ is the domain of a branch of the first entry map to V_1^+ . Then the collection of intervals $\{\hat{E}_1^k, \dots, f_n^{\ell_1,k}(\hat{E}_1^k)\}$ has multiplicity of intersection equal to two. The collection of intervals $\{\hat{V}_1, \dots, f_n^{t_{r_1}}(\hat{V}_1)\}$ also has multiplicity of intersection at most two because $r_1 = 1$. This finishes the first step of induction.

Now we assume that the lemma is true for $n \geq 1$ and prove it for $n+1$. To define the pair $\hat{V}_{n+1} \subset \mathcal{V}_{n+1}$ required to prove Part 1 we remember that $\phi_n^{m_n}(c) = f_n^{m_n r_n}(c) \in E_n^{\ell_n}$, $\phi_n^{m_n-1}(c) \in V_{2n}^+$, $\hat{E}_n^{\ell_n} \subset \phi_n^{m_n}(V_{2n+1}) = V_{2n-1}^+ \setminus V_{2n}^+$ and $\mathcal{E}_n^{\ell_n}$ is contained in the convex hull of $\phi_n^{m_n-1}(c)$ and $V_{2n-1}^+ \setminus V_{2n}^+$. All these properties imply that we can take the pair $\hat{V}_{n+1} \subset \mathcal{V}_{n+1}$ as the intervals around the critical point such that $f_n^{m_n r_n}(\hat{V}_{n+1}) = \hat{E}_n^{\ell_n}$ and $f_n^{m_n r_n}(\mathcal{V}_{n+1}) = \mathcal{E}_n^{\ell_n}$. With this choice we have that $f_n^{r_{n+1}} = f_n^{t_{n,\ell_n}} \circ f_n^{m_n r_n}$ maps the triple $V_{2n+2} \subset \hat{V}_{n+1} \subset \mathcal{V}_{n+1}$ homeomorphically onto the triple $V_{2n+1}^+ \subset V_1^+ \subset (p_n-1, f_n(c)+1)$. We also have that $f_n^{j r_n}(\mathcal{V}_{n+1}) \subseteq V_{2n-1}^+$, for $1 \leq j \leq m_n$, therefore $f_n^{m_n r_n}$ has only one critical point in \mathcal{V}_{n+1} , the point c . Since $f_n^{t_{n,\ell_n}}$ is a diffeomorphism from $\mathcal{E}_n^{\ell_n} = f_n^{m_n r_n}(\mathcal{V}_{n+1})$ onto $(p_n-1, f_n(c)+1)$, we are done.

For Part 2 of the lemma we assume that $k > \ell_n$, in which case $E_n^k \subseteq \hat{E}_n^k \subset \mathcal{E}_n^k$ are all contained in $\phi_n^{m_n}(V_{2n+1}^+)$ and there is a closed interval $I_n^k \subset E_n^k$ that is mapped diffeomorphically onto V_{2n+1}^+ by $\sigma_{n,k}$. Then we can take $E_{n+1}^k \subset \hat{E}_{n+1}^k \subset \mathcal{E}_{n+1}^k$

to be the intervals mapped diffeomorphically onto $I_n^k \subset \hat{E}_n^k \subset \mathcal{E}_n^k$ by $\phi_n^{m_n}$. Since $f_n^{t_{n+1,k}}(y) = \sigma_{n,k} \circ \phi_n^{m_n}(y)$, for $y \in E_{n+1}^k$, Part 2 of the lemma follows.

For Part 3 we remember that, for $k > \ell_n$, we can conclude that the intervals $\hat{E}_{n+1}^k, \hat{E}_{n+1}^{k+1}, \hat{E}_{n+1}^{k+2}, \dots$ are pairwise disjoint because they are mapped by $\phi_n^{m_n}$ diffeomorphically onto the pairwise disjoint intervals $\hat{E}_n^k, \hat{E}_n^{k+1}, \hat{E}_n^{k+2}, \dots$. The interval \hat{E}_n^k is on the right of $\hat{E}_n^{\ell_n}$ and this implies that $\hat{E}_{n+1}^k \subset V_{2n+1}^+ \setminus V_{2n+2}^+ = \phi_n^{m_n}(V_{2n+3})$ as required. We still need to prove that \mathcal{E}_{n+1}^k is contained in the convex hull of $\phi_{n+1}^{m_{n+1}-1}(c)$ and $V_{2n+1}^+ \setminus V_{2n+2}^+$. By definition, $\phi_n^{m_n}(\mathcal{E}_{n+1}^k) = \mathcal{E}_n^k$ and $\mathcal{E}_{n+1}^k \subset V_{2n+1}^+$. We have that $\phi_{n+1}^{m_{n+1}-1}(c) \in V_{2n+2}^+$ and we claim that this point is not in \mathcal{E}_{n+1}^k . Indeed, the image of \mathcal{E}_{n+1}^k by $f_n^{t_{n+1,k}}$ is $(p_n - 1, f_n(c) + 1)$. Then, if the claim were not true, the interval $(p_n - 1, c)$ would contain points from the forward orbit of the critical point which is not true.

To prove Part 4 of the lemma we consider $k > \ell_{n+1}$ and by the same reasoning as above, the point $\phi_{n+1}^{m_{n+1}}(c)$ is in $E_{n+1}^{\ell_{n+1}}$, but not in \mathcal{E}_{n+1}^k . Otherwise the interval $(p_n - 1, c)$ would contain points from the forward orbit of the critical point, which is not true.

To prove Part 5 we remember that the points $x \in \hat{E}_{n+1}^k$ have entry time to V_{2n+1}^+ equal to $t_{n+1,k} = m_n r_n + t_{n,k}$ and $f_n^{t_{n+1,k}}(x) = \sigma_{n,k} \circ \phi_n^{m_n}(x)$. We also have that $\sigma_{n,k}(y) = \sigma_{1,k} \circ \phi_1^{m_1} \circ \dots \circ \phi_{n-1}^{m_{n-1}}(y)$, for all $y \in \hat{E}_n^k$. By definition, for $1 \leq i \leq n$, the interval $\hat{E}_{i+1}^k \subset V_{2i}^+$ is inside a fundamental domain D_i of $\phi_i : V_{2i} \rightarrow V_{2i-1}^+$ and $\phi_i^{m_i}(\hat{E}_{i+1}^k) = \hat{E}_i^k$. The fundamental domain D_i of ϕ_i is, by definition, the maximal interval $D_i \subset V_{2i}$ such that $D_i, \dots, \phi_i^{m_i}(D_i)$ are pairwise disjoint and $\phi_i^{m_i}(D_i) = V_{2i-1}^+ \setminus V_{2i}^+$. In fact, by the property (iv) of the induction hypotheses in the construction of f_n , we have the equality $D_i = V_{2i+1}$. Therefore, the piece of orbit from \hat{E}_{i+1}^k to $\phi_i^{m_i}(\hat{E}_{i+1}^k) = \hat{E}_i^k$ has pairwise disjoint intervals, all of them contained in $(V_{2i+1}^+ \setminus V_{2i+2}^+) \cup \dots \cup f_s^{m_i r_i}(V_{2i+1}^+ \setminus V_{2i+2}^+)$. Then, since the intervals $V_1^+ \setminus V_2, \dots, V_{2n+1}^+ \setminus V_{2n+2}$ are pairwise disjoint, if we put together all the intervals from the above pieces of orbits we get that the piece of orbit from \hat{E}_{n+1}^k to $\phi_1^{m_1} \circ \dots \circ \phi_n^{m_n}(\hat{E}_{n+1}^k) = \hat{E}_1^k$ has pairwise disjoint intervals. On the other hand the intervals of the piece of orbit from $f_n(\hat{E}_1^k)$ to $\sigma_{1,k}(\hat{E}_1^k) = V_1^+$ are pairwise disjoint. Then the collection $\{\hat{E}_{n+1}^k, \dots, f_n^{t_{n+1,k}}(\hat{E}_{n+1}^k)\}$ has multiplicity of intersection equal to two.

If we replace \hat{E}_{n+1}^k by \hat{V}_{n+1} and use the same reasoning above we conclude that the collection $\{\hat{V}_{n+1}, \dots, f_n^{r_{n+1}}(\hat{V}_{n+1})\}$ have multiplicity of intersection equal to two. Indeed, by definition, $r_{n+1} = m_n r_n + t_{n,\ell_n}$, $\hat{V}_{n+1} \subset V_{2n+1}$ and $f_n^{r_{n+1}}(x) = \sigma_{n,\ell_n} \circ \phi_n^{m_n}(x)$, for all $x \in V_{2n+1} \subset V_{2n}$. The interval V_{2n+1} is the fundamental domain of $\phi_n : V_{2n} \rightarrow V_{2n-1}^+$ such that $\phi_n^{m_n}(V_{2n+1}) = V_{2n-1}^+ \setminus V_{2n}^+$. This implies that the collection $\{\hat{V}_{n+1}, \dots, f_n^{m_n r_n}(\hat{V}_{n+1}) = \hat{E}_n^{\ell_n}\}$ has pairwise disjoint intervals, all of them contained in $V_{2n+1} \cup \dots \cup f_n^{m_n r_n}(V_{2n+1})$. The intervals of the collection $\{f_n(\hat{E}_n^{\ell_n}), \dots, \phi_1^{m_1} \circ \dots \circ \phi_{n-1}^{m_{n-1}}(\hat{E}_n^{\ell_n}) = \hat{E}_1^{\ell_n}\}$ are also pairwise disjoint and they have empty intersection with the intervals from the previous collection. If we put all these intervals together we get that the intervals of the piece of orbit from \hat{V}_{n+1} to $\phi_1^{m_1} \circ \dots \circ \phi_n^{m_n}(\hat{V}_{n+1}) = \hat{E}_1^{\ell_n}$ are pairwise disjoint. Now, adding the pairwise disjoint intervals $f_n(\hat{E}_1^{\ell_n}), \dots, f_n^{t_{1,\ell_n}}(\hat{E}_1^{\ell_n})$ we get the collection $\{\hat{V}_{n+1}, \dots, f_n^{r_{n+1}}(\hat{V}_{n+1})\}$ and conclude it has multiplicity of intersection equal to two as required. \square

4. The limit set of the critical point. In this section we study the ω -limit set $\omega(c)$ of the critical point of a critical covering map $f \in \mathcal{H}(\omega)$. The fact that the sequence $(\ell_n)_{n=1}^\infty$ is strictly increasing means that $\omega(c)$ contains the repelling fixed point p . So $\omega(c)$ is not a minimal set, since it has a proper non-trivial invariant subset $\{p\}$. We now prove that $\omega(c)$ is a Cantor set of zero Lebesgue measure.

Theorem 4.1. *If f is a critical covering map which satisfies hypotheses (H1)–(H3) and has kneading sequence in \mathcal{K} , then the ω -limit set of its critical point is a Cantor set of zero Lebesgue measure.*

For a better understanding of the set $\omega(c)$ and the proof of Theorem 4.1, observe that the symbol 2 does not occur anywhere in sequences in \mathcal{K} . This means that the positive orbit of c does not visit the arc of the circle corresponding to $[p, c+1]$. Therefore, it will follow that $\omega(c)$ is a Cantor set. A more refined study of the relative sizes of the pre-images of this arc will permit us to prove that the Lebesgue measure of this Cantor set is zero and conclude the proof. In this stage we will need to use some Koebe Principles, that may be found, for example, in [13]. They are our main tool for controlling the distortion of iterates of a function, and we state them in the following lemma.

Lemma 4.2 (Koebe Principles). *Let $J \subset T \subset \mathbb{S}^1$ be a pair of intervals such that $T \setminus J$ has two non-empty connected components L and R . If h is a C^3 map defined on T with no critical point, negative Schwarzian and $\min\{|h(L)|, |h(R)|\} \geq \alpha|h(J)|$, then the following properties hold:*

1. *for all $x, y \in J$ we have that*

$$\frac{Dh(x)}{Dh(y)} \leq \left(\frac{1+\alpha}{\alpha} \right)^2;$$

2. $\min\{|L|, |R|\} \geq \alpha|J|$.

Proof. For a proof see [13]. □

Remark 3. With the notation of Lemma 4.2 we define the *distortion of h in J* , that is:

$$\text{Dist}(h, J) = \sup \left\{ \frac{Dh(x)}{Dh(y)} : x, y \in J \right\}.$$

We remark that, if α in Lemma 4.2 grows to infinity, then the distortion of h in J decreases to 1.

4.1. Measure of the limit set of the critical point. To prove that the ω -limit set of the critical point has Lebesgue measure zero, part of Theorem 4.1, we use a well-known procedure of inducing to get rid of the critical point. We consider the function $T : \bigcup_{n=0}^\infty W_n \rightarrow W$, where $W = V_1^+ = [c, z_1^+]$ and $W_0 = V_2^+$, such that the coding of the forward orbit of $f(x)$, for $x \in W_0$, start with the symbol 0. For $n \geq 1$ and $y \in W_n \subset W$, the coding of the forward orbit of $f(y)$ begins with the word $1^n 0$. We define T at $x \in W_n$ by putting $T(x) = f^{n+1}(x)$. The restriction of T to W_n is a branch of the first entry map to W . In particular, the restriction of T to W_0 coincides with ϕ_1 , which coincides with f restricted to W_0 .

Lemma 4.3. *Consider the function $T : \bigcup_{n=0}^\infty W_n \rightarrow W$ associated to a critical covering map $f \in \mathcal{H}(\omega)$ that satisfies hypotheses (H1)–(H3) for some kneading sequence $\omega \in \mathcal{K}$. Consider also the set Ω consisting of the points $x \in W$ for which*

$T^n(x)$ is well-defined, for all $n \geq 1$. Then Ω is a Cantor set of zero Lebesgue measure.

Proof. We denote by T_n the branch of T that corresponds to its restriction to the interval W_n and we begin by listing some properties of these branches:

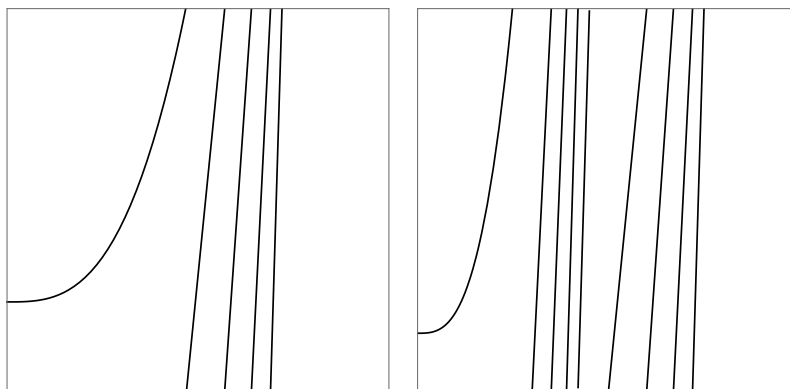
- (i) by definition, for $n \geq 0$, we have $W_{n+1} \subset [W_n, W_{n+2}] \subset W$;
- (ii) the left endpoint of W_0 is the critical point c , T_0 is the restriction to W_0 of f and coincides with the branch ϕ_1 of the first entry map from V_2^+ to $W = V_1^+$. By Lemma 3.3, there is an interval $\mathcal{W}_0 = \mathcal{V}_1$ around W_0 which is mapped f homeomorphically (with only one critical point, the point c) onto the interval $(p-1, f(c)+1)$;
- (iii) for $n \geq 1$, T_n is the restriction to W_n of f^{n+1} and coincides with the branch $\sigma_{1,n}$ of the first entry map from E_1^k to $W = V_1^+$. By Lemma 3.3, there is an interval $\mathcal{W}_n = \mathcal{E}_1^k \subset W$ which is mapped by f^{n+1} diffeomorphically onto the interval $(p-1, f(c)+1)$;
- (iv) for $z \in [p^+, z_1^+]$, the coding of the forward orbit of $f(z)$ includes the symbol 2, and so the forward orbit of the critical point doesn't visit this interval and the map T isn't defined in it.

We are going to substitute the function T , which has a critical point at c , for another function \mathcal{T} that has no critical points. This function is induced and obtained through an inductive process that generates a sequence $(\mathcal{T}_n)_{n=1}^\infty$ of functions \mathcal{T}_n that possess a critical branch (with critical point c) defined on intervals that nest down to c as $n \geq 1$ increases. The sequence $(\mathcal{T}_n)_{n=1}^\infty$ has a limit \mathcal{T} that has an infinite number of branches that map their respective domains diffeomorphically onto W . In the initial step of this induction we put $\mathcal{T}_1 = T$. This function \mathcal{T}_1 has infinitely many branches that are diffeomorphisms of their respective domains onto W and a single branch with a critical point at c and defined on $\hat{W}_1 = W_0$. Then we compose this critical branch with the diffeomorphic branches. In this way we obtain new branches that are diffeomorphisms onto W and a new critical branch that is defined on an interval $\hat{W}_2 \subset \hat{W}_1$. Continuing in this way we obtain a sequence $(\mathcal{T}_n)_{n=1}^\infty$ and a sequence of nested closed intervals $(\hat{W}_n)_{n=1}^\infty$ with the following properties:

- (i) $\bigcap_{n=1}^\infty \hat{W}_n = \{c\}$;
- (ii) \mathcal{T}_n has diffeomorphic branches mapping onto W , except for a single critical branch with critical point c and defined on the interval \hat{W}_n ;
- (iii) the branches of \mathcal{T}_n have extensions mapping diffeomorphically onto $(p-1, f(c)+1)$, except for the critical branch, which has an extension mapping homeomorphically onto $(p-1, f(c)+1)$ with critical point c ;
- (iv) outside of the interval \hat{W}_n we have that $\mathcal{T}_{n+1} = \mathcal{T}_n$;
- (v) the sequence $(\mathcal{T}_n)_{n=1}^\infty$ has a unique limit, which has no critical points and is the function \mathcal{T} described above.

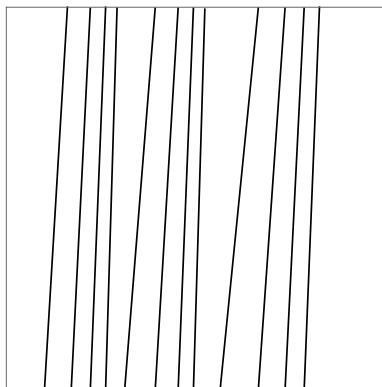
Figure 4.1 shows the graphs of \mathcal{T}_1 and \mathcal{T}_2 . Figure 4.1 shows the graph of \mathcal{T} .

Each branch of \mathcal{T} is a diffeomorphism that coincides with some iterate f^t that maps a neighbourhood of the domain of a branch of \mathcal{T} diffeomorphically onto $(p-1, f(c)+1)$. Then, by the Koebe Principle in Lemma 4.2, the branches of \mathcal{T} and their iterates are diffeomorphisms onto W and have uniformly bounded distortion. Because of this, together with the fact that the complement of the domain of \mathcal{T} in W contains the interval $[p^+, z_1^+]$, we can conclude that the set Ω of points $x \in W$ such that $\mathcal{T}^n(x)$ is defined for all $n \geq 1$, is a Cantor set and has zero Lebesgue measure and the lemma is complete. \square

FIGURE 3. Comparing graphs of the functions \mathcal{T}_1 (left) and \mathcal{T}_2 (right).

Proof of Theorem 4.1. It is clear that $\omega(c)$ is a set that is closed and totally disconnected. As c is recurrent, we also have that $\omega(c)$ is perfect. Thus $\omega(c)$ is a Cantor set. To conclude that the Lebesgue measure of $\omega(c)$ is zero, we observe that the forward orbit of the critical point has empty intersection with the interval $[p^+, z_1^+]$. Here $p^+ \in [c, z_1^+]$ is the point for which $f(p^+) = f(p) = p$. Then $f(c) \in \Omega$ and $W \cap \omega(c) \subset \Omega$ and, by Lemma 4.3, the Lebesgue measure of $W \cap \omega(c)$ is zero. So we conclude that the Lebesgue measure of $\omega(c)$ is zero, proving the theorem. \square

5. Absence of acip measures. As we have already observed, a critical covering map f has many invariant measures and this is the topic of this section. We know that under our hypotheses (H1)–(H3), with the absence of periodic attractors and non-degenerate intervals consisting of periodic points, f is topologically conjugate to the uniform covering map $l = l_2$ by a homeomorphism $h : S^1 \rightarrow S^1$: that is, $h \circ f = l \circ h$. As l preserves Lebesgue measure on the circle, denoted by λ , we can define the measure $h^*\lambda$ by setting $h^*\lambda(B) = \lambda(h(B))$ for any Borel subset $B \subset S^1$. This measure $h^*\lambda$ is always invariant for f and we could expect that it would be

FIGURE 4. The graph of \mathcal{T}

an acip measure. Depending on the speed that m_n grows with n , this might not be the case as we will show.

Let us study a little more the critical covering maps f which have kneading sequence in \mathcal{K} . Recall that on the combinatorial level $\mathcal{K} = \bigcap_{n=1}^{\infty} [\omega_n]$, where $[\omega_n] \subset A^{\mathbb{N}}$ is the cylinder of all infinite words beginning with the word ω_n , see Equation 2. The sequence of words $(\omega_n)_{n=1}^{\infty}$ is defined in terms of two sequences $(\ell_n)_{n=1}^{\infty}$ and $(m_n)_{n=1}^{\infty}$ of natural numbers, with $(\ell_n)_{n=1}^{\infty}$ strictly increasing.

On the dynamical level the critical covering map f is obtained as a limit of a sequence $(f_n)_{n=1}^{\infty}$ of critical covering maps $f_n \in \mathcal{H}(\omega_n)$, where $\mathcal{H}(\omega_n)$ is the set of all critical covering maps which satisfy hypotheses (H1)–(H3) with $\beta > 1$, $\vartheta > 0$, ψ fixed (see Equation 1) and having kneading sequence in $[\omega_n]$.

Associated to $f_n \in \mathcal{H}(\omega_n)$, there are branches of first entry maps to V_{2n-1}^+ (see the induction hypothesis in Subsection 3.2),

$$\phi_n : V_{2n} \rightarrow V_{2n-1}^+ \quad \text{and} \quad \sigma_{n,k} : E_n^k \rightarrow V_{2n-1}^+, \quad \text{for all } k > \ell_{n-1}.$$

The special branches $\phi_n : V_{2n} \rightarrow V_{2n-1}^+$ and $\sigma_{n,\ell_n} : E_n^{\ell_n} \rightarrow V_{2n-1}^+$ are associated to the words ω_n and ν_n , respectively (see Equation 2). Figure 3.3 is an illustration of the graphs of these special branches.

Using Lemmas 3.3 and 4.2, we prove that: for $\beta > 1$, $\vartheta > 0$ and ψ fixed in Equation 1, the shape of the branches ϕ_n and $\sigma_{n,k}$ are controlled by the combinatorics given by the finite word ω_n . In fact this shape is uniform in the set of all words in $[\omega_n]$ and, in particular, this shape does not depend on ℓ_i and m_i , for $i \geq n$. This is the content of the next lemma.

Lemma 5.1. *There exist $\theta_n > 0$ and $K_n < \infty$ such that, for every $n \geq 1$, every finite word ω_n as before and every $f_n \in \mathcal{H}(\omega_n)$, the following properties are satisfied:*

1. *For all $x \in V_{2n}$,*

$$K_n^{-1} \beta \theta_n |x - c|^{\beta-1} \leq |D\phi_n(x)| \leq K_n \beta \theta_n |x - c|^{\beta-1},$$

2. *The distortion of $\sigma_{n,k} : E_n^k \rightarrow V_{2n-1}^+$ is bounded by K_n ;*

3. *If $\sigma_{n,k} = f_n^{t_{n,k}}$ in E_n^k , then the distortion of $f_n^{t_{n,k}} : \hat{E}_n^k \rightarrow V_1^+$ is bounded by K_1 ;*

4. *The constants K_n and θ_n depend only on ω_n , $\beta > 1$ and $\vartheta > 0$. Moreover, $K_n \rightarrow 1$ and $\theta_n \rightarrow \infty$, as $n \rightarrow \infty$.*

Proof. If we write $\phi_n = f_n^{r_n-1} \circ f_n$, Lemma 3.3 implies that $f_n^{r_n-1}$ maps the pair $f_n(V_{2n}) \subset f_n(V_n)$ diffeomorphically onto the pair $V_{2n-1}^+ \subset (p_n - 1, f_n(c) + 1)$, where p_n is the unique fixed point of f_n . Let τ_n be the size of the smaller connected component of $(p_n - 1, f_n(c) + 1) \setminus V_{2n-1}^+$. For $f_n \in \mathcal{H}(\omega_n)$, hypotheses (H1)–(H3) are satisfied with $\beta > 1$, $\vartheta > 0$ and ψ from Equation 1 fixed. The Koebe Principles in Lemma 4.2 imply that the distortion of $f_n^{r_n-1}$ restricted to $f_n(V_{2n})$ is bounded by $K_n = (1 + \alpha_n)^2 / \alpha_n^2$, where $\alpha_n = \tau_n / |V_{2n-1}^+|$. Since τ_n is bounded away from zero, it follows that $\alpha_n \rightarrow \infty$ as $n \rightarrow \infty$. We set $\theta_n = |V_{2n-1}^+| / |f_n(V_{2n})|$, use the mean value theorem and the hypothesis that the critical point has order β to conclude the first part the lemma. For the second and third parts of the lemma we recall that $\sigma_{n,k} = f_n^{t_{n,k}}$ in E_n^k and Lemma 3.3 implies that $f_n^{t_{n,k}}$ maps the triple $E_n^k \subset \hat{E}_n^k \subset \mathcal{E}_n^k$ diffeomorphically onto the triple $V_{2n-1}^+ \subset V_1^+ \subset (p_n - 1, f_n(c) + 1)$. For the same K_n as above, the Koebe Principles in Lemma 4.2 imply that the distortion of $f_n^{t_{n,k}}$ restricted to E_n^k is bounded by K_n and restricted to \hat{E}_n^k is bounded by K_1 . This completes the proof of the lemma. \square

An important step in our proof of the non-existence of an acip measure is the following lemma which, in particular, implies that the Lebesgue measure of $\bigcup_{i=1}^{r_n} f^i(V_{2n})$ tends to zero as $n \rightarrow \infty$, where $r_n = |\omega_n|$.

Lemma 5.2. *Given $K_1 < \infty$ as in Lemma 5.1, if $n \geq 1$, ω_n is a finite word as before and $f_n \in \mathcal{H}(\omega_n)$, then*

$$\left| \bigcup_{i=0}^{r_n} f_n^i(V_{2n}) \right| \leq 2K_1 \frac{|V_{2n-1}^+|}{|V_1^+|}.$$

In particular the Lebesgue measure of $\bigcup_{i=0}^{r_n} f_n^i(V_{2n})$ tends to zero when n grows to infinity.

Proof. One reason for this lemma to be true is the fact that, for $1 \leq i \leq r_n$, we can pull back the space $V_1^+ \setminus V_{2n-1}^+$ on the side of V_{2n-1}^+ to a space on the side of $f_n^i(V_{2n})$ maintaining the proportion up to a uniform distortion constant. To be more precise observe that Lemma 3.3 implies that there are intervals $V_{2n} \subset \hat{V}_n \subset \mathcal{V}_n$ such that, for $1 \leq i \leq r_n$, the map $f_n^{r_n-i}$ is a diffeomorphism from $f_n^i(\hat{V}_{2n}) \subset f_n^i(\hat{V}_n) \subset f_n^i(\mathcal{V}_n)$ onto $V_{2n-1}^+ \subset V_1^+ \subset (p_n - 1, f_n(c) + 1)$, where p_n is unique fixed point of f_n . By Lemma 4.2 the distortion of $f_n^{r_n-i}$ on $f_n^i(\hat{V}_n)$ is uniformly bounded by K_1 and

$$|f_n^i(V_{2n})| \leq K_1 \frac{|V_{2n-1}^+|}{|V_1^+|} |f_n^i(\hat{V}_n)|.$$

Taking the sum with $1 \leq i \leq r_n$ we get that

$$\sum_{i=1}^{r_n} |f_n^i(V_{2n})| \leq K_1 \frac{|V_{2n-1}^+|}{|V_1^+|} \sum_{i=1}^{r_n} |f_n^i(\hat{V}_n)|.$$

By Lemma 3.3, the multiplicity of intersection of the collection $\{\hat{V}_n, \dots, f_n^{r_n}(\hat{V}_n)\}$ is at most two. Therefore

$$\sum_{i=1}^{r_n} |f_n^i(V_{2n})| \leq 2 K_1 \frac{|V_{2n-1}^+|}{|V_1^+|}$$

and the lemma follows. \square

To follow the same ideas as Johnson in [9], we observe that as set out in Lemma 5.2, the Lebesgue measure of the union $\bigcup_{i=1}^{r_n} f_n^i(V_{2n})$ tends to zero as n tends to infinity. On the other hand, since the interval V_{2n} contains the critical point, using the theorem of Mañé (see [12] and [13]) it is easy to conclude that, for every n fixed, the Lebesgue measure of $\bigcup_{i=0}^N f_n^{-i}(V_{2n})$ tends to 1, when N tends to infinity. Therefore an arbitrarily small set is visited by an arbitrarily large set. Moreover, we would like to have the property that the set of points of $\bigcup_{i=0}^N f_n^{-i}(V_{2n})$ that leave $\bigcup_{i=0}^{r_n} f_n^i(V_{2n})$ in fewer than N iterates could be made arbitrarily small by increasing m_n . For this to hold, we will substitute the interval V_{2n} by the interval $U_n \subset V_{2n}$. First we consider the point $\gamma_n \in V_{2n}$ such that the derivative $D\phi_n(\gamma_n)$ is equal to 1. Then we define U_n to be the connected component of $V_{2n} \setminus \{\gamma_n\}$ which contains c . For the map G_n from the Corollary 2 or for the map $g_{b(n)}$ from Lemma 3.2, there is a parabolic periodic attractor and $m_n = \infty$. In this case the point γ_n is the parabolic periodic attractor.

The next lemma guarantees that the size of the interval U_n has a positive lower bound that is uniform for every $f_n \in \mathcal{H}(\omega_n)$ with ω_n fixed. In particular, U_n does not degenerate to a point when we keep $\ell_1, \dots, \ell_{n-1}, m_1, \dots, m_{n-1}$ fixed and

increase m_n to infinity. Observe, however, that V_{2n} and U_n degenerate if one of the numbers $\ell_1, \dots, \ell_{n-1}$ tends to infinity.

Lemma 5.3. *Given a finite word ω_n as before, there exists $\zeta_n > 0$ such that, for every $f_n \in \mathcal{H}(\omega_n)$, the corresponding interval U_n defined above satisfies $|U_n| \geq \zeta_n$.*

Proof. This lemma is an immediate consequence of the first part of Lemma 5.1 applied to the branch $\phi_n : V_{2n} \rightarrow V_{2n-1}^+$. Indeed, given the word ω_n the numbers $\ell_1, \dots, \ell_{n-1}, m_1, \dots, m_{n-1}$ are fixed and if we increase $m_n \geq 1$ to infinity, then $|\gamma_n - \phi_n(\gamma_n)|$ goes down to zero. \square

Since the interval U_n contains the critical point, it absorbs almost every point. On the other hand, if we keep $\ell_1, \dots, \ell_{n-1}, m_1, \dots, m_{n-1}$ fixed and increase m_n to infinity, $|\gamma_n - \phi_n(\gamma_n)|$ goes down to zero. This implies that less and less points escape from U_n . All of this implies that forward orbits concentrate near the orbit of the critical point more and more. This is the main reason for the non-existence of an acip measure for the limit map of the sequence $\{f_n\}_{n=1}^\infty$. See the lemma below for a precise statement.

Lemma 5.4. *Given $\epsilon > 0$, there exists $n_0 = n_0(\epsilon)$ such that, for every $n \geq n_0$ and every finite word ω_n as before, there exists $N_n = N(\epsilon, n, \omega_n)$ such that, for any $f_n \in \mathcal{H}(\omega_n)$, the following properties are satisfied:*

1. $\left| \bigcup_{i=0}^{r_n} f_n^i(U_n) \right| < \epsilon;$
2. $\left| \bigcup_{i=0}^N f_n^{-i}(U_n) \right| \geq 1 - \epsilon$, for every $N \geq N_n$ and every $m_n \geq 1$;
3. For every $N \geq N_n$, there exists $M_n = M(\epsilon, n, \omega_n, N)$ such that

$$\left| \left\{ x \in \bigcup_{i=0}^N f_n^{-i}(U_n) : f_n^N(x) \notin \bigcup_{i=0}^{r_n} f_n^i(U_n) \right\} \right| < \epsilon,$$

for every $m_n \geq M_n$.

Proof. The first part of the lemma is a direct consequence of Lemma 5.2 simply because $U_n \subset V_{2n}$. Since U_n is a neighbourhood of the critical point we get the second part using the theorem of Mañé, see [12] and [13]. The uniform estimate on $m_n \geq 1$ follows from Lemmas 5.1 and 5.3.

For the third part of the lemma we assume that we are given $\epsilon > 0$, $n \geq n_0$, ω_n and $N \geq N_0$ satisfying the first and second parts of the lemma for every $m_n \geq 1$. Then we observe that, if we keep $\ell_1, \dots, \ell_{n-1}, m_1, \dots, m_{n-1}$ fixed and $m_n = \infty$, we get the map G_n from Corollary 2 and the point γ_n is a parabolic periodic attractor for G_n . In this case $\bigcup_{i=0}^{r_n} G_n^i(U_n)$ is forward invariant. This implies that, for $\ell_1, \dots, \ell_{n-1}, m_1, \dots, m_{n-1}$ fixed, if $m_n \rightarrow \infty$, then $|\gamma_n - \phi_n(\gamma_n)| \rightarrow 0$. Then we can take $M_0 = M(\epsilon, n, \omega_n, N)$ sufficiently large to satisfy the third part of the lemma. \square

5.1. Proof of Theorem 2.1. Let us suppose that the sequences $(m_n)_{n=1}^\infty$ and $(\ell_n)_{n=1}^\infty$ are given and then we take the sequence $(\omega_n)_{n=1}^\infty$ of finite words as defined by Equation 2. We also take a sequence $(f_n)_{n=1}^\infty$ such that $f_n \in \mathcal{H}(\omega_n)$ converges to $f \in \mathcal{H}(\omega)$. Then, for $\epsilon_k = 1/2^{k+2}$, we consider $n_k = n_0(\epsilon_k)$ and the sequence of

constants $(N_n)_{n=n_k}^\infty$ and $(M_n)_{n=n_k}^\infty$ given by Lemma 5.4. For simplicity of notation we set

$$\begin{aligned}\mathcal{A}_n &= \bigcup_{i=0}^{r_n} f_n^i(U_n), \\ \mathcal{Q}_n &= \left\{ x \in \bigcup_{i=0}^{N_n} f_n^{-i}(U_n) : f_n^{N_n}(x) \notin \mathcal{A}_n \right\} \\ \mathcal{P}_n &= \bigcup_{i=0}^{N_n} f_n^{-i}(U_n) \setminus \mathcal{Q}_n.\end{aligned}$$

By assumption the sequence $(\ell_n)_{n=1}^\infty$ is strictly increasing and we take $(m_n)_{n=1}^\infty$ increasing fast enough in order to have $m_n \geq M_n$. Then, for the subsequence $(n_k)_{k=1}^\infty$, we have that

$$|\mathcal{A}_{n_k}| < \frac{1}{2^{k+1}}, \quad |\mathcal{Q}_{n_k}| < \frac{1}{2^{k+1}}, \quad |\mathcal{P}_{n_k}| \geq 1 - \frac{1}{2^{k+1}} \quad \text{and} \quad |\mathcal{P}_{n_k}^c \cup \mathcal{A}_{n_k}| \leq \frac{1}{2^k}.$$

Since $f_{n_k}^{N_{n_k}}$ maps \mathcal{P}_{n_k} into \mathcal{A}_{n_k} , it follows that

$$f_{n_k}^{N_{n_k}}(\mathcal{P}_{n_k} \setminus \mathcal{A}_{n_k}) \subset \mathcal{P}_{n_k}^c \cup \mathcal{A}_{n_k}.$$

So, if μ is an invariant probability measure for f_{n_k} , then $\mu(\mathcal{P}_{n_k}^c \cup \mathcal{A}_{n_k}) \geq 1/2$.

To finish the proof of Theorem 2.1 we take $f \in \mathcal{H}(\omega)$ to be the limit map of the subsequence $(f_{n_k})_{k=1}^\infty$. Since the sequence of sets $\mathcal{H}(\omega_{n_k})$ nests down to the set $\mathcal{H}(\omega)$, we have that $f \in \mathcal{H}(\omega_{n_k})$ for every $k \geq 1$. So we may consider the constant subsequence $f_{n_k} = f$ and apply the above reasoning to conclude that $\mu(\mathcal{P}_{n_k}^c \cup \mathcal{A}_{n_k}) \geq 1/2$, for every $k \geq 1$. It follows that, for

$$\Lambda_f = \bigcap_{i=1}^\infty \bigcup_{k=i}^\infty (\mathcal{P}_{n_k}^c \cup \mathcal{A}_{n_k}),$$

$\mu(\Lambda_f) \geq 1/2$ and, by the Borel–Cantelli Lemma, the Lebesgue measure of Λ_f is zero. Therefore μ cannot be an acip measure for f . \square

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