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A NEW SUFFICIENT ALGEBRAIC CONDITION FOR THE CONTROLLABILITY AND OBSERVABILITY OF LINEAR TIME-VARYING SYSTEMS

HUGO LEIVA

Nº 63

NOTAS

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Resumo

Neste artigo damos uma condição algébrica suficiente para a controlabilidade e observabilidade do seguinte sistema

$$x'(t) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t)$$

onde $x \in I\!R^n$, $u \in I\!R^\ell$, e as matrizes $A(t) - n \times n$, $B(t) - n \times \ell$ e $C(t) - R \times n$ são $(n-2)$, $(n-1)$ e $(n-1)$ vezes continuamente diferenciáveis respectivamente. Todas as condições apresentadas aqui são em termos de quantidades conhecidas e portanto, facilmente verificadas. Muitas restrições complicadas, tal como a matriz $A(t)$ ser expandida na álgebra de Lie gerada, com espeito a uma base, suavidade dos coeficientes de $A(t)$ dadas em [4] são completamente relaxadas. Este trabalho é motivado por um resultado profundo de Silverman and Meadows [3]

A New Sufficient Algebraic Condition for the Controllability and Observability of Linear Time-Varying Systems

HUGO LEIVA

Department of Mathematics

Universidad de Los Andes

Merida 5101-Venezuela

Instituto de Ciências Matemáticas de São Carlos

Universidade de São Paulo

São Carlos, Brasil

Abstract

In this paper we give a sufficient algebraic conditions for controllability and observability of the following system

$$x'(t) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t)$$

where $x \in \mathbb{R}^n$. $u \in \mathbb{R}^\ell$, and the matrices $A(t) - n \times n$, $B(t) - n \times \ell$ and $C(t) - R \times n$ are $(n-2)$, $(n-1)$ and $(n-1)$ times continuously differentiable respectively. All conditions presented here are in terms of known quantities and therefore, easily verified. Many complicated restrictions, such as the matrix $A(t)$ is expanded in the generated Lie algebra, with respect to a basis, smoothness of the coefficients of $A(t)$ or differential algebraic conditions for the coefficients of $A(t)$ given in [4] are completely relaxed. This work is motivated by the deep result of Silverman and Meadows [3].

Key words. time-varying systems, controllability, observability, algebraic rank condition.

AMS(MOS) subject classifications. primary: 93B05; secondary: 93C25.

1 Introduction

In this paper we present a sufficient algebraic condition for the controllability and observability associated with the time-varying control system

$$x'(t) = A(t)\,x(t) + B(t)\,u(t), \quad y(t) = C(t)\,x(t), \tag{1.1}$$

where $x \in I\!R^n$, $A(t) \in I\!R^{n\times n}$, $B(t) \in I\!R^{n\times \ell}$, $C(t) \in I\!R^{R\times n}$ and the control $u \in I\!R^\ell$. The main Hypothesis in this Note is that:

H) The matrices $A(t)$, $B(t)$ and $C(t)$ are $(n-2), (n-1)$ and $(n-1)$ times continuously differentiable respectively.

Assuming the control $u(\cdot)$ is a locally integrable function on $I\!R$, the unique solution of the differential equation (1.1) with the initial condition x_0 at $t = 0$, is given by

$$x(t) = \Phi(t)\,x_0 + \int_0^t \Phi(t)\,\Phi^{-1}(s)\,B(s)\,u(s)\,ds\ ,\ t \in I\!R\ , \tag{1.2}$$

where $\Phi(t)$ is the fundamental matrix of the linear system $x' = A(t)\,x$. i.e.,

$$\Phi'(t) = A(t)\,\Phi(t)\ \ ,\ \Phi(0) = I.$$

Definition 1.1 *The system (1.1) is:*

i) *Completely controllable on* $(0,T)$ *if for any initial state* x_0 *at* $t = 0$, *and any desired final state* x_1 *at* $t = T$, *there exists a control* $u(\cdot)$ *defined on* $[0.T]$ *such that the corresponding solution* $x(\cdot)$ *of (1.1) satisfies* $x(T) = x_1$.

ii) *The system (1.1) is totally controllable on* $(0,T)$ *if it is completely controllable on every subinterval of* $(0,T)$.

Definition 1.2 *The system (1.1) is:*

i) *Completely observable on* $(0, T)$ *if any initial state* x_0 *at* $t = 0$ *can be determined from knowledge of the system out put* $y(t)$ *and the control* $u(\cdot)$ *over* $[0.T]$.

ii) *The system (1.1) is totally observable on* $(0, T)$ *if it is completely observable on every subinterval of* $(0, T)$.

It is well known that controllability and observability properties of the system (1.1) in $[0, T]$, $T > 0$, can be fully determined by analyzing the following Gramian matrices

$$\begin{aligned} W(T) &= \int_0^T \Phi^{-1}(t) B(t) B^*(t) \Phi^{-1*}(t)\, dt \\ Z(T) &= \int_0^T \Phi^*(t) C^*(t) C(t) \Phi(t)\, dt \end{aligned}$$

respectively (see [5,9]).

However, the fundamental matrix $\Phi(t)$ is rarely known in closed form and computing $W(t)$, $Z(t)$ is not a happy prospect. Therefore, while necessary and sufficient controllability and observability conditions for (1.1) are easily established, often times these conditions are of limited practicality.

The breakthrough work of Silverman and Meadows[3] shows under the Hypothesis **H)** that the controllability and observability property of (1.1) can be characterized in terms of $A(t)$, $B(t)$, $C(t)$ and their appropriate derivatives. Since the matrices $A(t)$, $B(t)$ and $C(t)$ are a priori known, these results provide a means of verifying controllability and observability test that are easily attainable.

For the sake of convenience, we formulate here the Silverman-Meadows results. To this end, we define the controllability matrix of (1.1):

$$Q_c(t) = \left[P_0(t) \vdots P_1(t) \vdots \cdots \vdots P_{n-1}(t)\right] \tag{1.3}$$

where

$$P_{k+1}(t) = -A(t) P_k(t) + P_k'(t),\ P_0(t) = B(t)\ . \tag{1.4}$$

The observability matrix is defined as follows

$$Q_0(t) = \left[S_0(t) \vdots S_1(t) \vdots \cdots \vdots S_{n-1}(t)\right] \tag{1.5}$$

where

$$S_{k+1}(t) = -A^*(t) S_k(t) + S_k'(t),\ S_0(t) = C^*(t)\ . \tag{1.6}$$

Theorem 1.1 a) *System (1.1) is completely controllable on* $(0,T)$ *if* $Q_c(t)$ *has rank* n *for some* $t \in (0,T)$.

b) *System (1.1) is totally controllable on* $(0,T)$ *if and only if* $Q_c(t)$ *does not have rank less than* n *on any subinterval of* $(0,T)$.

By duality, the following criteria for observability of (1.1) holds.

Theorem 1.2 a) *System (1.1) is completely observable on the interval* $(0,T)$ *if* $Q_0(t)$ *has rank* n *for some* $t \in (0,T)$.

b) *System (1.1) is totally observable on the interval* $(0,T)$ *if and only if* $Q_0(t)$ *does not have rank less than* n *on any subinterval of* $(0,T)$.

Although the work of Silverman and Meadows[3] was performance more than thirty years ago, there has been (to the author knowledge) very limited improvement on their results. One might even ask, how it could be possible to improve on results which are so easily to verify?

In fact, the Silverman and Meadows results have became so highly regarded that they can be found in virtually every graduate level Linear System textbook.

In Leiva and Lehman[1,2] they tray to improve Silverman and Meadows results in the following particular case:

Suppose the matrix $A(t)$ is only locally integrable on $\mathbb{R}$ and matrices B and C are constants. Every matrix $A(t)$ can be written as follow

$$A(t) = \sum_{i=1}^{m} a_i(t) A_i \tag{1.7}$$

where $a_i(t)$, s are scalar functions only integrable on compact set of $\mathbb{R}$.

It was proved by Wei and Norman in [7] that: If the Lie algebra generated by the set $\{A_1, \ldots, A_m\}$ under the commuter product $[A_i, A_j] = A_iA_j - A_jA_i$ has dimension equal m. then the fundamental matrix $\Phi(t)$ can be written as the product of exponential matrices

$$\Phi(t) = e^{g_1(t)A_1} \cdot e^{g_2(t)A_2} \cdot \ldots \cdot e^{g_m(t)A_m}, \quad t \in [0, T] \tag{1.8}$$

for some $T > 0$, where the functions $g_i : [0, T] \to \mathbb{R}$, $i = 1, 2, \ldots, m$ are the solutions of a very complicated nonlinear system of differential equations depending on a_i, $i = 1, 2, \ldots, m$:

$$\frac{dg}{dt} = f(a, g), \ g(0) = 0, \ a = (a_1, \ldots, a_m), \ g = (g_1, \ldots, g_m). \tag{1.9}$$

Even when $A(t)$ is a 2×2 matrix, it is almost impossible knowing the function g_i, s. It should be pointed out that the representation (1.8) is valid for all $t \in \mathbb{R}$ in the 2×2 case.

Using the representation (1.8) of $\Phi(t)$ in [1] they give the following necessary algebraic conditions for the controllability and observability of the system (1.1).

Theorem 1.3 a) *If system (1.1) is completely controllable on* $(0,T)$, *then the following algebraic rank condition holds*

$$rank\left[A_m^{k_m}A_{m-1}^{k_{m-1}}\dots A_1^{k_1}B : k_i=0,\dots,n-1; i=1,\dots,m\right]=n \qquad (1.10)$$

b) *If system (1.1) is completely observable on* $(0,T)$, *then the following algebraic rank condition holds*

$$rank\left[A_m^{*k_m}A_{m-1}^{*k_{m-1}}\dots A_1^{*k_1}C^* : k_i=0,\dots,n-1; i=1,\dots,m\right]=n \qquad (1.11)$$

It seem to be that we have to add the terms

$$A_1^{k_1}A_2^{k_2}\dots A_m^{k_m}B,\ A_1^{*k_1}A_2^{*k_2}\dots A_m^{*k_m}C^*$$

to the algebraic conditions (1.10) and (1.11) respectively. If we do so, then (1.10) and (1.11) can be written respectively as follow

$$rank\left[A_{i_k}A_{i_{k-1}}\dots A_{i_1}B : 0\le k\le n-1;\ \imath_{r\neq 0}=1,2,\dots,m\right]=n \qquad (1.12)$$

$$rank\left[A_{i_k}^*\dots A_{i_1}^*C^* : 0\le k\le n-1;\ \imath_{r\neq 0}=1,2,\dots,m\right]=n \qquad (1.13)$$

where $i_0=0$ *and* $A_0=I$.

One of the goal of this note, is to use Theorems 1.1 and 1.2 of Silverman-Meadows to find a checkable additional conditions in terms of the coefficients for the matrices $A(t)$, $B(t)$ and $C(t)$ to make the algebraic rank test conditions (1.12), (1.13) sufficient for the controllability and observability of the system (1.1) respectively.

2 Main Theorems

In this section we shall present and prove the main theorems of this paper. To this end, we shall write the matrices $A(t)$, $B(t)$ and $C(t)$ as follows

$$A(t) = \sum_{i=1}^{m} a_i(t) A_i, \quad B(t) = \sum_{j=1}^{J} b_j(t) B_j, \quad C(t) = \sum_{\ell=1}^{L} c_\ell(t) C_\ell \tag{2.1}$$

where $a_i(\cdot)$, $b_j(\cdot)$ and $c_\ell(\cdot)$ are $(n-2)$, $(n-1)$ and $(n-1)$ times continuously differentiable scalar functions in $(0,T)$, and A_i, B_j, C_ℓ are constants matrices. Also, we shall define $i_0 = 0$ and $A_0 = I$.

Theorem 2.1 a) *System (1.1) is completely controllable on* $(0,T)$ *if for some* $t \in (0,T)$

$$a_i(t) \neq 0 \text{ and } b_j(t) \neq 0; i = 1, 2, \ldots, m-1; j = 1, 2, \ldots, J \tag{2.2}$$

and

$$\begin{aligned} &rank\left[A_{i_k} A_{i_{k-1}} \ldots A_{i_1} B_j : k = 0, 1, \ldots, n-1; j = 1, \ldots, J;\right. \\ &\left. i_{r \neq 0} = 1, \ldots, m\right] = n. \end{aligned} \tag{2.3}$$

b) *System (1.1) is completely observable on* $(0,T)$ *if for some* $t \in (0,T)$

$$a_i(t) \neq 0 \text{ and } c_\ell(t) \neq 0; i = 1, 2, \ldots, m; \ell = 1, 2, \ldots, L \tag{2.4}$$

and

$$\begin{aligned} &rank\left[A^*_{i_k} A^*_{i_{k-1}} \ldots A^*_{i_1} C^*_\ell : k = 0, 1, \ldots, n-1; \ell = 1, \ldots, L;\right. \\ &\left. i_{r \neq 0} = 1, 2, \ldots, m\right] = n. \end{aligned} \tag{2.5}$$

Theorem 2.2 a) *System (1.1) is totally controllable on* $(0,T)$ *if condition (2.2) holds for all* $t \in (0,T)$ *and the algebraic rank condition (2.3) is satisfied.*

b) *System (1.1) is totally observable on* $(0,T)$ *if condition (2.4) holds for all* $t \in (0,T)$ *and the algebraic rank condition (2.5) is satisfied.*

Before proving Theorems 2.1 and 2.2 we shall consider a Proposition and a Lemma.

Proposition 2.1 *The algebraic rank conditions (2.3) and (2.5) are equivalent respectively to the following conditions*

$$Sp\Big\{A_{i_k}A_{i_{k-1}}\dots A_{i_1}B_j\mathbb{R}^{\ell} : k = 0,1,\dots,n-1; j = 1,2,\dots,J; \\ i_{r\neq 0} = 1,2,\dots,m \ \Big\} = \mathbb{R}^n \tag{2.6}$$

and

$$Sp\Big\{A^*_{i_k}A^*_{i_{k-1}}\dots A^*_{i_1}C_\ell\mathbb{R}^{R} : k = 0,1,2,\dots,n-1; \ell = 1,2,\dots,L; \\ i_{r\neq 0} = 1,2,\dots,m \ \Big\} = \mathbb{R}^n \tag{2.7}$$

where $Sp\{W\}$ *denotes the linear space generated by the set* W.

Lemma 2.1 *There exist two families of scalar polynomial depending on* $a = (a_1,\dots,a_m)^T$, $b = (b_1,\dots,b_J)^T$ *and* $c = (c_1,\dots,c_L)^T$ *given by*

$$\Big\{q^k(a;b;i_r,\dots,i_1;j) : k = 0,1,2,\dots n-1; 0 \le s < k; \\ i_{r\neq 0} = 1,2,\dots,m; 1 \le r \le k-s; j = 1,2,\dots,J\Big\} \tag{2.8}$$

$$\Big\{h^k(a;c;i_r,\dots,i_1;\ell) : k = 0,1,2,\dots n-1; 0 \le s < k; \\ i_{r\neq 0} = 1,2,\dots,m; 1 \le r \le k-s; \ell = 1,2,\dots,L\Big\} \tag{2.9}$$

Such that the polynomial $P_k(t)$ and $S_k(t)$ given by (1.4) and (1.6) respectively can be written as follows

$$P_k(t) = \sum_{j=1}^{J} \left\{ \sum_{i_k, i_{k-1}, \ldots, i_1 = 1}^{m} q^k(a(t); b(t); i_k, \ldots i_1; j) A_{i_k} \ldots A_{i_1} B_j \right.$$

$$+ \sum_{i_{k-1}, \ldots, i_1 = 1}^{m} q^k(a(t); b(t); i_{k-1}, \ldots i_1; j) A_{i_{k-1}} \ldots A_{i_1} B_j + \ldots$$

$$\left. + \sum_{i_1=1}^{m} q^k(a(t); b(t); i_1; j) A_{i_1} B_j + q^k(a(t); b(t); j) B_j \right\}$$

and

$$S_k(t) = \sum_{\ell=1}^{L} \left\{ \sum_{i_k, \ldots, i_1 = 1}^{m} h^k(a(t); c(t); i_k, \ldots i_j; \ell) A^*_{i_k} \ldots A^*_1 C^*_\ell \right.$$

$$+ \sum_{i_{k-1}, \ldots, i_1 = 1}^{m} h^k(a(t); c(t); i_{k-1}, \ldots i_1; \ell) A^*_{i_{k-1}} \ldots A^*_1 C^*_\ell + \ldots$$

$$\left. + \sum_{i_1=1}^{m} h^k(a(t); c(t); i_1; \ell) A^*_{i_1} C^*_\ell + h^k(a(t); c(t); \ell) C^*_\ell \right\}.$$

Proof. By simplicity, we shall suppose that $B(t) = B$ and $C(t) = C$ are constants. Therefore, from (1.4) we have that $P_0(t) = B$. Then $q^0(a, 0) = 1$ and

$$P_1(t) = -A(t) P_0(t) + P_0'(t) = -\sum_{i_1=1}^{m} a_{i_1}(t) A_{i_1} B.$$

So $q^1(a, i_1) = -a_{i_1}(t)$.

$$\begin{aligned} P_2(t) &= -A(t) P_1(t) + P_1'(t) \\ &= + \sum_{i_2=1}^{m} a_{i_2}(t) A_{i_2} \sum_{i_1=1}^{m} a_{i_1}(t) A_{i_1} B - \sum_{i_1=1}^{m} a'_{i_1}(t) A_{i_1} B \\ &= \sum_{i_2, i_1 = 1}^{m} a_{i_2}(t) a_{i_1}(t) A_{i_2} A_{i_1} B - \sum_{i_1=1}^{m} a'_{i_1}(t) A_{i_1} B. \end{aligned}$$

Hence,

$$q^2(a, i_2, i_1) = a_{i_2} a_{i_1}, \quad q^2(a, i_1) = -a'_{i_1}.$$

For P_3 we have that

$$q^3(a,i_3,i_2,i_1) = -a_{i_3}a_{i_2}a_{i_1},\ q^3(a,i_2,i_1) = 2a_{i_2}a'_{i_1} + a'_{i_2}a_{i_1},\ q^3(a,i_1) = -a_i^2 .$$

In this way, the polynomial $q^k(a,i_r,\ldots,i_1)$ are determined by recurrence. Similarly we can compute the polynomial $h^k(a,i_r,\ldots,i_1)$ ■

Proof of Theorem 2.1. Since part (b) is similar to part (a), we only will prove part (a). In fact, from Theorem 1.1 we only have to prove that *rank* of $Q_c(t)$ is equal to n. For the purpose of contradiction, we will suppose that $rank Q_c(t) < n$. As in Proposition 2.1, this is equivalent to

$$Sp\left\{P_k(t)\,\mathbb{R}^\ell : k = 0,1,\ldots,n-1\right\} =$$

$$Sp\left\{q^k(a;b;i_r,\ldots,i_1;j)\,A_{i_r}\ldots A_{i_1}B_j\mathbb{R}^\ell : k = 0,\ldots n-1;\ 0 \le s < k;\right.$$

$$\left. i_{r\neq 0} = 1,\ldots,m;\ 1 \le r \le k-s; j = 1,\ldots,J\right\} \subsetneq \mathbb{R}^n$$

From the construction of $q^k(a;b;i_r,\ldots,i_1;j)$ we can see that

$$q^k(a;b;i_k,i_{k-1},\ldots,i_1;j) = a_{i_k}a_{i_{k-1}}\ldots a_{i_1}b_j. \tag{2.10}$$

Hence, from condition (2.2) we get that

$$q^k(a(t);b(t);i_k,\ldots,i_1;j) = a_{i_k}(t)\,a_{i_{k-1}}(t)\ldots a_{i_1}(t)\,b_j(t) \neq 0.$$

Therefore, for $k = 0,1,2,\ldots,n-1$ and $j = 1,2,\ldots,J$ we have

$$q^k(a(t);b(t);i_k,i_{k-1},\ldots,i_1;j)\,A_{i_k}A_{i_{k-1}}\ldots A_{i_1}B_j\ \mathbb{R}^\ell = A_{i_k}A_{i_{k-1}}\ldots A_{i_1}B_j\ \mathbb{R}^\ell.$$

Hence,

$$Sp\left\{A_{i_k}\ldots A_{i_1}B_j\mathbb{R}^\ell : k = 0,1,\ldots,n-1;\ j = 1,2,\ldots,J;\right.$$

$$\left. i_{r\neq 0} = 1,2,\ldots,m\right\} \subsetneq \mathbb{R}^n$$

From Proposition 2.1, we get a contradiction with condition (2.3). ∎

Proof of Theorem 2.2. The Proof follows in the same way as Theorem 2.1, using Proposition 2.1, Lemma 2.1 and Theorems 1.1 and 1.2.

Theorem 2.3 a) *If system (1.1) is totally controllable on* $(0, T)$*, then the following algebraic rank condition holds*

$$rank\left[A_{i_r} \dots A_{i_1} B_j I\!R^\ell : k = 0, 1, \dots, n-1; j = 1, 2, \dots, J; \right. \\ \left. i_{r \neq 0} = 1, 2, \dots, m;\ 1 \leq r \leq k-s;\ 0 \leq s < k\right] = n. \tag{2.11}$$

b) *If system (1.1) is totally observable on* $(0, T)$*, then the following algebraic rank condition holds*

$$rank\left[A^*_{i_r} \dots A^*_{i_1} C^*_\ell I\!R^R : k = 0, 1, 2, \dots, n-1; j = 1, 2, \dots, J; \right. \\ \left. i_{r \neq 0} = 1, 2, \dots, m;\ 1 \leq r \leq k-s;\ 0 \leq s < k\right] = n \tag{2.12}$$

Proof. Since part (b) is similar to part (a), we only will prove part (a). From Theorem 1.1 part (b), for all subinterval of $(0, T)$ we have

$$rankQ_c(t) = rank\left[P_0(t) \vdots P_1(t) \vdots \cdots \vdots P_{n-1}(t)\right] =$$

$$= rank\left[q^k(a; b; i_r, \dots, i_1; j) A_{i_r} \dots A_{i_1} B_j :\ k = 0, 1, 2, \dots n-1; \right.$$

$$\left. j = 1, 2, \dots, J; i_{r \neq 0} = 1, 2, \dots, m; 1 \leq r \leq k-s; 0 \leq s < k\right] = n$$

But as in Proposition 2.1 this is equivalent to

$$I\!R^n = Sp\left\{q^k(a; b; i_r, \dots, i_1; j) A_{i_r} \dots A_{i_1} B_j I\!R^\ell : k = 0, 1, 2, \dots n-1; \right.$$

$$\left. j = 1, 2, \dots, J; i_{r \neq 0} = 1, 2, \dots, m; 1 \leq r \leq k-s; 0 \leq s < k\right\}$$

$$\subseteq Sp\left\{A_{i_r} \dots A_{i_1} B_j I\!R^\ell : k = 0, 1, \dots, n-1; j = 1, 2, \dots, J; i_{r \neq 0} = 1, 2, \dots, m; \right.$$

$$\left. 1 \leq r \leq k-s\ ;\ 0 \leq s < k\right\} = I\!R^n\ ,$$

which is equivalent to condition (2.11). ∎

3 Application

In this section we shall consider some example as an application of our results.

Example 3.1 *Consider the nth-order controlled linear differential equation*

$$y^{(n)}(t) + b_{n-1}(t)\, y^{(n-1)}(t) + \ldots + b_0(t)\, y = u(t), \tag{3.1}$$

where $b_i(t)$, *s are* $(n-2)$ *times continuously differentiable scalar functions in* $(0,\infty)$ *with* $b_i(t) \neq 0$. *Then the equivalent system* $x'(t) = A(t)\,x(t) + B(t)\,u$, *where*

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \vdots & \vdots & \ldots & 1 \\ -b_0 & -b_1 & \vdots & \vdots & \ldots & -b_{n-1} \end{pmatrix}, \; B(t) = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

is totally controllable on $(0,\infty)$. *For simplicity, we shall consider the case* $n = 3$.

$$A(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -b_o & -b_1 & -b_2 \end{pmatrix}, \; B(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Hence,

$$A(t) = a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4 + a_5A_5$$

where

$$a_1 = -b_0,\; a_2 = -b_1,\; a_3 = -b_2,\; a_4 = a_5 = 1$$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; A_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

In this case, the rank condition (2.3) can be written as follow

$$rank\left[B\vdots A_{i_1}B\vdots A_{i_2}A_{i_1}B : i_1, i_2 = 1, 2, \ldots, 5\right] = 3,$$

which is trivial to verify. Therefore, this system is totally controllable on $(0, \infty)$.

Example 3.2 *Consider the following Mechanical System*

$$\begin{cases} m_1\,\ddot{y}_1 = -k_1(t)\,y_1 + c(\dot{y}_2 - \dot{y}_1) = +u_1(t) \\ m_2\,\ddot{y}_2 = -k_2(t)\,y_2 + c(\dot{y}_2 - \dot{y}_1) + u_2(t) \end{cases} \tag{3.2}$$

where m_1, m_2 *are masses of the two bodies;* $k_1(t)$, $k_2(t)$ *are the forces due to the stretching springs;* $c > 0$ *corresponds to the damping forces and* $u_1(t)$, $u_2(t)$ *are external forces view as the control.*

For simplicity we shall suppose that $m_1 = m_2 = 1$. *Of course,* k_1 , k_2 *and* u_1 , u_2 *are smooth enough to apply Theorems 2.1, 2.2 and 2.3. The system (3.1) can be written as a first order system as follows*

$$\begin{cases} x_1' = x_2 \\ x_2' = -k_1(t)\,x_1 + cx_4 - cx_2 + u_1(t) \\ x_3' = x_4 \\ x_4' = -k_2(t)\,x_3 + cx_4 - cx_2 + u_2(t) \end{cases} \tag{3.3}$$

In this case we have:

$$x' = A(t)\,x + Bu,\ x \in I\!R^4, \tag{3.4}$$

where

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -k_1(t) & -c & 0 & c \\ 0 & 0 & 0 & 1 \\ 0 & c & -k_2(t) & -c \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and $u = (u_1, u_2)^T$.

$$A(t) = a_1(t)\,A_1 + a_2(t)\,A_2 + a_3(t)\,A_3$$

with $a_1(t) = -k_1(t)$, $a_2(t) = -k_2(t)$, $a_3(t) = 1$ *and*

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \; A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \; A_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -c & 0 & c \\ 0 & 0 & 0 & 1 \\ 0 & c & 0 & -c \end{pmatrix};$$

We shall suppose that $k_i(t) > 0$, $t \in [0, \infty)$, $i = 1, 2$. *Then,*

$$a_i(t) \neq 0, \; t \in [0, \infty), \; i = 1, 2, 3.$$

Therefore, the condition (2.3) of Theorem 2.1 can be written as

$$rank\,[A_{i_k} \dots A_{i_1} B : k = 0, 1, 2, 3; i_{r \neq 0} = 1, 2, 3] = 4,$$

i.e.

$$rank\left[B \vdots A_1 B \vdots A_2 B \vdots A_3 B \vdots A_1^2 B \vdots \dots A_3^2 B | A_1 A_2 B \dots A_3^3 B\right] = 4.$$

This is trivial to verify. Hence, system (3.1) is controllable over any $(0, T)$, $T > 0$.

Example 3.3 *Determine the controllability of the following systems*

a) $x' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ -e^{2t} \end{pmatrix} u$

b) $x' = \begin{pmatrix} -1 & e^{2t} \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$

c) $x' = \begin{pmatrix} -1 & e^{2t} \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$

Solution *a) and b) are totally controllable. c) is not controllable.*

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