



On the Schrödinger–Born–Infeld System

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Abstract

In this paper we study a system which we propose as a model to describe the interaction between matter and electromagnetic field from a dualistic point of view. This system arises from a suitable coupling of the Schrödinger and the Born–Infeld agrangians, this latter replacing the role that, classically, is played by the Maxwell Lagrangian. We use a variational approach to find an electrostatic radial ground state solution by means of suitable estimates on the functional of the action.

Keywords Schrödinger–Born–Infeld equation · Nonlinear electromagnetic theory

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1 Introduction

In the recent years, several models have been proposed to provide a mathematical description of the interaction between a charged particle and the electromagnetic field

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generated by itself. According to two different philosophical concepts, the way to perform a mathematical formulation can follow two different and, in some way, antithetical approaches.

The theory developed by Born and Infeld (see 1933, 1934) introduced the idea that both the matter and the electromagnetic field were expression of a unique physical entity. According to this unitarian point of view, the system giving a complete description of the dynamics arose variationally starting from a nonlinear version of the Maxwell Lagrangian. This unitarian approach was also taken up by Benci and Fortunato in Benci and Fortunato (2004) [see also Azzollini et al. (2006) and D'Aprile and Siciliano (2011)].

On the other hand there is the dualistic point of view, based on the idea that the dynamics can be described coupling equations related with particles and equations related with the electromagnetic field through a suitable combination of the lagrangians. Starting from the results obtained by Benci and Fortunato (1998), the literature is rich of papers studying models based on this latter point of view.

In the past, the duality matter-electromagnetic field was usually carried out by means of either Schrödinger or Klein–Gordon Lagrangian to provide the mathematical description of the particle, and of the Maxwell Lagrangian, or higher order approximations (in the sense of Taylor series) of the Born–Infeld Lagrangian [see for example d'Avenia and Pisani (2002) and Benmilh and Kavian (2008)] to represent the electromagnetic field.

Recently, Yu has proposed in (2010) a dualistic model obtained coupling Klein–Gordon and Born–Infeld lagrangians and has studied the electrostatic case expressed by the following system

$$\begin{cases} -\Delta u + (m^2 - (\omega + \phi)^2)u = |u|^{p-1}u & \text{in } \mathbb{R}^3, \\ \operatorname{div} \left(\frac{\nabla \phi}{\sqrt{1 - |\nabla \phi|^2}} \right) = u^2(\omega + \phi) & \text{in } \mathbb{R}^3, \\ u(x) \rightarrow 0, \phi(x) \rightarrow 0, & \text{as } x \rightarrow \infty. \end{cases} \quad (KGBT)$$

As a consequence of the form of the differential operator in the second equation, a variational approach to the problem can not be performed in the usual functional spaces. In particular, the quantity $1/\sqrt{1 - |\nabla \phi(x)|^2}$ makes sense when $x \in \mathbb{R}^3$ is such that $|\nabla \phi(x)| < 1$, being this inequality a necessary constraint to be considered in the functional setting.

Inspired by Yu (2010), our aim is to propose and study a new model which represents a variant of the well-known Schrödinger–Maxwell system as it was introduced in D'Aprile and Mugnai (2004a). Indeed we replace the usual Maxwell Lagrangian with the Born–Infeld one and we look for the electrostatic solutions. The system in this case becomes

$$\begin{cases} -\Delta u + u + \phi u = |u|^{p-1}u & \text{in } \mathbb{R}^3, \\ -\operatorname{div} \left(\frac{\nabla \phi}{\sqrt{1 - |\nabla \phi|^2}} \right) = u^2 & \text{in } \mathbb{R}^3, \\ u(x) \rightarrow 0, \phi(x) \rightarrow 0, & \text{as } x \rightarrow \infty, \end{cases} \quad (SBT)$$

and we will refer to it as Schrödinger–Born–Infeld system.

At least formally, the system (SBI) comes variationally from the action functional F defined by

$$F(u, \phi) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) + \frac{1}{2} \int_{\mathbb{R}^3} \phi u^2 - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} - \frac{1}{2} \int_{\mathbb{R}^3} (1 - \sqrt{1 - |\nabla \phi|^2}).$$

Dealing with this functional presents evident difficulties for several reasons, starting with the definition of the functional setting. Indeed we observe that, being on the one hand natural to consider $u \in H^1(\mathbb{R}^3)$, on the other the presence of the term $\int_{\mathbb{R}^3} (1 - \sqrt{1 - |\nabla \phi|^2})$ forces us to restrict the setting of admissible functions ϕ .

We define

$$\mathcal{X} := \mathcal{D}^{1,2}(\mathbb{R}^3) \cap \{\phi \in C^{0,1}(\mathbb{R}^3) : \|\nabla \phi\|_{\infty} \leq 1\} \quad (1)$$

where $\mathcal{D}^{1,2}(\mathbb{R}^3)$ is the completion of $C_c^\infty(\mathbb{R}^3)$ with respect to the norm $\|\nabla \cdot\|_2$. Hereafter we denote by $\|\cdot\|_q$ the norm in $L^q(\mathbb{R}^3)$, for $q \in [1, +\infty]$.

We are looking for *weak solutions* in the following sense.

Definition 1.1 A *weak solution* of (SBI) is a couple $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{X}$ such that for all $(v, \psi) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3)$, we have

$$\begin{cases} \int_{\mathbb{R}^3} \nabla u \cdot \nabla v + uv + \phi uv = \int_{\mathbb{R}^3} |u|^{p-1} uv \\ \int_{\mathbb{R}^3} \frac{\nabla \phi \cdot \nabla \psi}{\sqrt{1 - |\nabla \phi|^2}} = \int_{\mathbb{R}^3} u^2 \psi. \end{cases}$$

Observe that the boundary condition at infinity is encoded in the functional space.

Of course, the fact that the setting $H^1(\mathbb{R}^3) \times \mathcal{X}$ is not a vector space is a nontrivial obstacle to our variational approach. In particular, to compute variations with respect to ϕ along the direction established by a generic smooth and compactly supported function, we need to require in advance that $\|\nabla \phi\|_{\infty} < 1$. This fact brings with it a concrete complication, for example in dealing with the reduction method which is a standard tool used in this kind of problems [see, for example, Benci and Fortunato (1998, 2002); Yu (2010)]. Indeed, the strongly indefinite nature of the functional can be classically removed showing that, for any radial $u \in H^1(\mathbb{R}^3)$ fixed, there exists a unique $\phi_u \in \mathcal{X}$ solution of the second equation of system (SBI) and reducing the problem to that of finding critical points of the (no more strongly indefinite) one-variable functional $I(u) = F(u, \phi_u)$, defined on $H^1(\mathbb{R}^3)$ (see Sect. 2 for more details).

As a consequence, we are led to consider a preliminary minimizing problem on the set \mathcal{X} and then, because of the bad properties of \mathcal{X} itself, we have to study the relation between solutions of this minimizing problem and solutions of the second equation (with respect to ϕ , being u fixed). This second step is one of the questions left open for $(KGBI)$ in Yu (2010), which has been recently solved in Bonheure et al. (2016) in a radial setting. For this reason, and also in order to overcome difficulties related

with compactness, we will restrict our study to radial solutions. So, let us introduce our functional framework: we set

$$H_r^1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) \mid u \text{ is radially symmetric}\}$$

and

$$\mathcal{X}_r = \{\phi \in \mathcal{X} \mid \phi \text{ is radially symmetric}\}.$$

Our main results are the following

Theorem 1.2 *For any $p \in (5/2, 5)$, the problem (SBI) possesses a radial ground state solution, namely a solution $(u, \phi) \in H_r^1(\mathbb{R}^3) \times \mathcal{X}_r$ minimizing the functional F among all the nontrivial radial solutions. Moreover both u and ϕ are of class $C^2(\mathbb{R}^3)$.*

What immediately stands out is the unusual range where p varies. It follows from the fact that, in view of the application of the Mountain Pass Theorem, we need to find a point with a sufficiently large norm where the functional is negative. In order to do this, usually one computes the reduced one-variable functional I on curves of the type

$$t \in (0, +\infty) \mapsto u_t := t^\alpha u(t^\beta \cdot) \in H_r^1(\mathbb{R}^3),$$

and look for suitable values of α and β for which $I(u_t) < 0$ for large values of t . However, in our case, because of the lack of homogeneity and since a precise expression of ϕ_{u_t} is not available, we need to proceed by means of estimates of ϕ_u which lead, as a consequence, to lose something in terms of powers p .

Summing up, our aim in this paper is to propose the new model problem (SBI) and give a positive answer concerning the existence of solutions, at least for $p \in (5/2, 5)$. We leave as an open problem the case of smaller p and the existence of non-radial solutions.

The paper is organized as follows: in Sect. 2 we introduce the functional setting and present some preliminary results, while in Sect. 3 we prove Theorem 1.2.

We finish this section with some notations. In the following we denote by $\|\cdot\|$ the norm in $H^1(\mathbb{R}^3)$ and by c, c_i, C, C_i arbitrary fixed positive constants which can vary from line to line.

2 Functional Setting and Preliminary Results

We start recalling some properties of the ambient space \mathcal{X} defined in (1).

Lemma 2.1 (Lemma 2.1 of Bonheure et al. (2016)) *The following assertions hold:*

- (i) \mathcal{X} is continuously embedded in $W^{1,p}(\mathbb{R}^3)$, for all $p \in [6, +\infty)$;
- (ii) \mathcal{X} is continuously embedded in $L^\infty(\mathbb{R}^3)$;
- (iii) if $\phi \in \mathcal{X}$, then $\lim_{|x| \rightarrow \infty} \phi(x) = 0$;

- (iv) \mathcal{X} is weakly closed;
- (v) if $(\phi_n)_n \subset \mathcal{X}$ is bounded, there exists $\bar{\phi} \in \mathcal{X}$ such that, up to a subsequence, $\phi_n \rightharpoonup \bar{\phi}$ weakly in \mathcal{X} and uniformly on compact sets.

As already observed in the Introduction, the functional F is strongly indefinite on $H^1(\mathbb{R}^3) \times \mathcal{X}$ from above and from below, and so we will consider a reduced one-variable functional, solving the second equation of (SBI), for any fixed $u \in H_r^1(\mathbb{R}^3)$. Let us start considering the functional $E : H^1(\mathbb{R}^3) \times \mathcal{X} \rightarrow \mathbb{R}$ defined as

$$E(u, \phi) = \int_{\mathbb{R}^3} (1 - \sqrt{1 - |\nabla \phi|^2}) - \int_{\mathbb{R}^3} \phi u^2.$$

The following lemma holds.

Lemma 2.2 *For any $u \in H^1(\mathbb{R}^3)$ fixed, there exists a unique $\phi_u \in \mathcal{X}$ such that the following properties hold:*

- (i) ϕ_u is the unique minimizer of the functional $E(u, \cdot) : \mathcal{X} \rightarrow \mathbb{R}$ and $E(u, \phi_u) \leq 0$, namely
- $$\int_{\mathbb{R}^3} \phi_u u^2 \geq \int_{\mathbb{R}^3} (1 - \sqrt{1 - |\nabla \phi_u|^2}); \quad (2)$$
- (ii) $\phi_u \geq 0$ and $\phi_u = 0$ if and only if $u = 0$;
 - (iii) if ϕ is a weak solution of the second equation of system (SBI), then $\phi = \phi_u$ and it satisfies the following equality

$$\int_{\mathbb{R}^3} \frac{|\nabla \phi_u|^2}{\sqrt{1 - |\nabla \phi_u|^2}} = \int_{\mathbb{R}^3} \phi_u u^2. \quad (3)$$

Moreover, if $u \in H_r^1(\mathbb{R}^3)$, then $\phi_u \in \mathcal{X}_r$ is the unique weak solution of the second equation of system (SBI).

Proof Points (i), (ii) and (iii) are an immediate consequence of Theorems 1.3 and Lemma 2.12 of Bonheure et al. (2016). For the second part of the statement we refer to [Bonheure et al. (2016), Theorem 1.4]. \square

Remark 2.3 We point out that, as stated in [Bonheure et al. (2016), Remark 5.5], if $w_n \rightarrow w$ in $L^p(\mathbb{R}^3)$, with $p \in [1, +\infty)$ then $\phi_{w_n} \rightarrow \phi_w$ in $L^\infty(\mathbb{R}^3)$.

By Lemma 2.2, we can deal with the following one-variable functional defined on $H^1(\mathbb{R}^3)$ as

$$\begin{aligned} I(u) &= F(u, \phi_u) \\ &= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) + \frac{1}{2} \int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} - \frac{1}{2} \int_{\mathbb{R}^3} (1 - \sqrt{1 - |\nabla \phi_u|^2}) \\ &= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} - \frac{1}{2} E(u, \phi_u). \end{aligned}$$

Proposition 2.4 *The functional I is of class C^1 and for every $u, v \in H^1(\mathbb{R}^3)$,*

$$I'(u)[v] = \int_{\mathbb{R}^3} \nabla u \cdot \nabla v + \int_{\mathbb{R}^3} uv + \int_{\mathbb{R}^3} \phi_u uv - \int_{\mathbb{R}^3} |u|^{p-1} uv.$$

Proof Arguing as in Yu (2010), let us show that

$$I(u+v) - I(u) - DI(u)[v] = o(v), \quad \text{as } v \rightarrow 0,$$

where

$$DI(u)[v] := \int_{\mathbb{R}^3} \nabla u \cdot \nabla v + \int_{\mathbb{R}^3} uv + \int_{\mathbb{R}^3} \phi_u uv - \int_{\mathbb{R}^3} |u|^{p-1} uv$$

which is trivially linear and continuous in v .

We set

$$I(u+v) - I(u) - DI(u)[v] = A_1 + A_2 + A_3$$

where

$$\begin{aligned} A_1 &:= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^3} v^2 \\ A_2 &:= -\frac{1}{p+1} \int_{\mathbb{R}^3} |u+v|^{p+1} + \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} + \int_{\mathbb{R}^3} |u|^{p-1} uv, \\ A_3 &:= -\frac{1}{2} E(u+v, \phi_{u+v}) + \frac{1}{2} E(u, \phi_u) - \int_{\mathbb{R}^3} \phi_u uv. \end{aligned}$$

Clearly

$$A_1 = o(v), \quad A_2 = o(v).$$

Now observe that by point (i) of Lemma 2.2 we have $E(u+v, \phi_u) \geq E(u+v, \phi_{u+v})$, so that an explicit computation gives

$$A_3 \geq -\frac{1}{2} E(u+v, \phi_u) + \frac{1}{2} E(u, \phi_u) - \int_{\mathbb{R}^3} \phi_u uv = \frac{1}{2} \int_{\mathbb{R}^3} \phi_u v^2 = o(v)$$

being $\left| \int_{\mathbb{R}^3} v^2 \phi_u \right| \leq C \|v\|^2 \|\phi_u\|_\infty$. Analogously, once again by point (i) of Lemma 2.2, being $E(u, \phi_u) \leq E(u, \phi_{u+v})$, we get

$$\begin{aligned} A_3 &= \int_{\mathbb{R}^3} \phi_{u+v} uv + \frac{1}{2} \int_{\mathbb{R}^3} \phi_{u+v} v^2 - \int_{\mathbb{R}^3} \phi_u uv \\ &= \frac{1}{2} \int_{\mathbb{R}^3} \phi_{u+v} v^2 + \int_{\mathbb{R}^3} (\phi_{u+v} - \phi_u) uv \\ &\leq C \|v\|^2 \|\phi_{u+v}\|_\infty + C \|u\| \|v\| \|\phi_{u+v} - \phi_u\|_\infty = o(v), \end{aligned}$$

in view of Remark 2.3. Hence $A_3 = o(v)$ and the differentiability of I is proved.

Finally, let us prove the continuity of the map

$$u \in H^1(\mathbb{R}^3) \mapsto \phi_u u \in \mathcal{L}(H^1(\mathbb{R}^3); \mathbb{R}),$$

from which we easily deduce the continuity of $DI : H^1(\mathbb{R}^3) \rightarrow \mathcal{L}(H^1(\mathbb{R}^3); \mathbb{R})$.

Let $u_n \rightarrow u$ in $H^1(\mathbb{R}^3)$. Observe that uniformly in $v \in H^1(\mathbb{R}^3)$, with $\|v\| \leq 1$,

$$\int_{\mathbb{R}^3} |\phi_{u_n} u_n - \phi_u u| |v| \leq \int_{\mathbb{R}^3} |\phi_{u_n}| |u_n - u| |v| + \int_{\mathbb{R}^3} |\phi_{u_n} - \phi_u| |u| |v| = o_n(1),$$

again by Remark 2.3. The conclusion follows. \square

Proposition 2.5 *If $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{X}$ is a weak nontrivial solution of (SBI) , then $\phi = \phi_u$ and u is a critical point of I . On the other hand, if $u \in H_r^1(\mathbb{R}^3) \setminus \{0\}$ is a critical point of I , then (u, ϕ_u) is a weak nontrivial solution of (SBI) .*

Proof The first part of the statement is a consequence of [Bonheure et al. (2016), Proposition 2.6] and Proposition 2.4, while the second part follows by Lemma 2.2 and Proposition 2.4. \square

In the next proposition we are going to prove that $H_r^1(\mathbb{R}^3)$ is a natural constraint for the functional I .

Proposition 2.6 *If $u \in H_r^1(\mathbb{R}^3)$ is a critical point of $I|_{H_r^1(\mathbb{R}^3)}$, then u is a critical point of I .*

Proof Denote by $O(3)$ the group of rotations in \mathbb{R}^3 and for any $g \in O(3)$ consider the action induced on $H^1(\mathbb{R}^3)$, that is

$$T_g : u \in H^1(\mathbb{R}^3) \mapsto u \circ g \in H^1(\mathbb{R}^3).$$

Clearly $H_r^1(\mathbb{R}^3)$ is the set of the fixed points for the group $T = \{T_g\}_{g \in O(3)}$ namely

$$H_r^1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) \mid T_g u = u \text{ for all } g \in O(3)\}.$$

Then the conclusion can be achieved by the Palais' Principle of Symmetric Criticality, if we show that I is invariant under the action of T , that is

$$I(T_g u) = I(u), \quad \text{for all } g \in O(3), u \in H^1(\mathbb{R}^3).$$

Actually it is sufficient to show that $\phi_{T_g u} = T_g \phi_u$ for any $u \in H^1(\mathbb{R}^3)$ and for all $g \in O(3)$. To this aim, by Lemma 2.2, we have

$$E(u, T_{g^{-1}} \phi_{T_g u}) = E(T_g u, \phi_{T_g u}) \leq E(T_g u, T_g \phi_u) = E(u, \phi_u)$$

and so, by the uniqueness of the minimizer of $E(u, \cdot)$, we conclude that $\phi_u = T_{g^{-1}} \phi_{T_g u}$ as desired. \square

The following technical lemma will be useful to study the geometry of the functional I .

Lemma 2.7 *Let q be in $[2, 3)$. Then there exist positive constants C and C' such that, for any $u \in H^1(\mathbb{R}^3)$, we have*

$$\|\nabla\phi_u\|_2^{\frac{q-1}{q}} \leq C\|u\|_{2(q^*)'} \leq C'\|u\|,$$

where q^* is the critical Sobolev exponent related to q and $(q^*)'$ is its conjugate exponent, namely

$$q^* = \frac{3q}{3-q} \quad \text{and} \quad (q^*)' = \frac{3q}{4q-3}.$$

Proof Since $\|\nabla\phi_u\|_\infty \leq 1$ and $q \in [2, 3)$

$$\|\phi_u\|_{q^*} \leq C\|\nabla\phi_u\|_q = C\left(\int_{\mathbb{R}^3} |\nabla\phi_u|^2 |\nabla\phi_u|^{q-2}\right)^{\frac{1}{q}} \leq C\|\nabla\phi_u\|_2^{\frac{2}{q}},$$

so, by (2) and being $2(q^*)' \in [2, 6]$, we have

$$\begin{aligned} \|\nabla\phi_u\|_2^2 &\leq C \int_{\mathbb{R}^3} (1 - \sqrt{1 - |\nabla\phi_u|^2}) \leq C \int_{\mathbb{R}^3} \phi_u u^2 \\ &\leq C\|\phi_u\|_{q^*}\|u\|_{2(q^*)'}^2 \leq C\|\nabla\phi_u\|_2^{\frac{2}{q}}\|u\|_{2(q^*)'}^2 \end{aligned}$$

and we get the conclusion. \square

We conclude this section showing that the radial weak solutions of (SBI) are actually classical and satisfy a Pohozaev type identity.

Proposition 2.8 *If $(u, \phi) \in H_r^1(\mathbb{R}^3) \times \mathcal{X}_r$ is a weak solution of (SBI) , then both u and ϕ are of class $C^2(\mathbb{R}^3)$.*

Proof Since $u \in H_r^1(\mathbb{R}^3)$, by [Bonheure et al. (2016), Theorem 3.2] we deduce that $\phi \in C^1(\mathbb{R}^3)$. Looking at the first equation in the system and by using a bootstrap argument, we conclude that $u \in C^2(\mathbb{R}^3)$. We define $\varphi : [0, +\infty[\rightarrow \mathbb{R}$ such that for any $r \geq 0$: $\varphi(r) = \phi(|x|)$ where $x \in \mathbb{R}^3$ is arbitrarily chosen in such a way that $|x| = r$.

From now on, we proceed as in [Berestycki and Lions (1983), Lemma 1, page 329]. Since ϕ is radial and satisfies the second equation in a weak sense, we deduce that

$$D\left(\frac{\varphi' r^2}{\sqrt{1 - |\varphi'|^2}}\right) = -u^2 r^2, \quad \text{in } (0, +\infty)$$

where the symbol D denotes the derivative in the sense of distributions.

Since on the right hand side we have a continuous function, the derivative actually has to be meant in the classical sense. So, integrating in $(0, r)$ and since $\varphi'(0) = 0$,

$$\frac{\varphi'(r)}{\sqrt{1 - |\varphi'(r)|^2}} = -\frac{1}{r^2} \int_0^r u^2(s) s^2 ds =: f(r) \in C^1((0, +\infty)). \quad (4)$$

On the one hand, by (4), we deduce that, for $r > 0$, we have

$$f'(r) = \frac{2}{r^3} \int_0^r u^2(s) s^2 ds - u^2(r)$$

and then $\lim_{r \rightarrow 0} f'(r) = -\frac{1}{3}u^2(0)$.

On the other hand, again by (4),

$$\lim_{r \rightarrow 0} \frac{f(r)}{r} = \lim_{r \rightarrow 0} -\frac{1}{r^3} \int_0^r u^2(s) s^2 ds = -\frac{1}{3}u^2(0).$$

We conclude that there exists $f'(0)$ and $\lim_{r \rightarrow 0} f'(r) = f'(0)$. Then $f \in C^1([0, +\infty))$. By some computations and by (4), we have

$$\varphi'(r) = \frac{f(r)}{\sqrt{1 + f^2(r)}} \in C^1([0, +\infty))$$

and we are done. \square

Proposition 2.9 *If (u, ϕ) is a solution of (SBI) of class $C^2(\mathbb{R}^3)$, then the following Pohozaev type identity is satisfied:*

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{3}{2} \int_{\mathbb{R}^3} u^2 + 2 \int_{\mathbb{R}^3} \frac{|\nabla \phi|^2}{\sqrt{1 - |\nabla \phi|^2}} \\ - \frac{3}{2} \int_{\mathbb{R}^3} \left(1 - \sqrt{1 - |\nabla \phi|^2}\right) = \frac{3}{p+1} \int_{\mathbb{R}^3} |u|^{p+1}. \end{aligned} \quad (5)$$

Proof Arguing as in D'Aprile and Mugnai (2004b), for every $R > 0$, we have

$$\int_{B_R} -\Delta u (x \cdot \nabla u) = -\frac{1}{2} \int_{B_R} |\nabla u|^2 - \frac{1}{R} \int_{\partial B_R} |x \cdot \nabla u|^2 + \frac{R}{2} \int_{\partial B_R} |\nabla u|^2, \quad (6)$$

$$\int_{B_R} u (x \cdot \nabla u) = -\frac{3}{2} \int_{B_R} u^2 + \frac{R}{2} \int_{\partial B_R} u^2, \quad (7)$$

$$\int_{B_R} \phi u (x \cdot \nabla u) = -\frac{1}{2} \int_{B_R} u^2 (x \cdot \nabla \phi) - \frac{3}{2} \int_{B_R} \phi u^2 + \frac{R}{2} \int_{\partial B_R} \phi u^2, \quad (8)$$

$$\int_{B_R} |u|^{p-1} u (x \cdot \nabla u) = -\frac{3}{p+1} \int_{B_R} |u|^{p+1} + \frac{R}{p+1} \int_{\partial B_R} |u|^{p+1}, \quad (9)$$

where B_R is the ball of \mathbb{R}^3 centered in the origin and with radius R .

Moreover, denoting by δ_{ij} the Kronecker symbols, since for any $i, j = 1, 2, 3$,

$$\begin{aligned} \int_{B_R} \partial_i \left(\frac{\partial_i \phi}{\sqrt{1 - |\nabla \phi|^2}} \right) x_j \partial_j \phi &= - \int_{B_R} \frac{\partial_i \phi \partial_j \phi}{\sqrt{1 - |\nabla \phi|^2}} \delta_{ij} - \int_{B_R} \frac{\partial_i \phi \partial_{i,j}^2 \phi}{\sqrt{1 - |\nabla \phi|^2}} x_j \\ &\quad + \int_{\partial B_R} \frac{\partial_i \phi \partial_j \phi}{\sqrt{1 - |\nabla \phi|^2}} \frac{x_i x_j}{|x|}, \end{aligned}$$

we have

$$\begin{aligned} \int_{B_R} -\operatorname{div} \left(\frac{\nabla \phi}{\sqrt{1 - |\nabla \phi|^2}} \right) (x \cdot \nabla \phi) &= - \sum_{i,j=1}^3 \int_{B_R} \partial_i \left(\frac{\partial_i \phi}{\sqrt{1 - |\nabla \phi|^2}} \right) x_j \partial_j \phi \\ &= \int_{B_R} \frac{|\nabla \phi|^2}{\sqrt{1 - |\nabla \phi|^2}} + \sum_{j=1}^3 \int_{B_R} \partial_j (1 - \sqrt{1 - |\nabla \phi|^2}) x_j \\ &\quad - \sum_{i,j=1}^3 \int_{\partial B_R} \frac{\partial_i \phi \partial_j \phi}{\sqrt{1 - |\nabla \phi|^2}} \frac{x_i x_j}{|x|} \\ &= \int_{B_R} \frac{|\nabla \phi|^2}{\sqrt{1 - |\nabla \phi|^2}} - 3 \int_{B_R} (1 - \sqrt{1 - |\nabla \phi|^2}) \\ &\quad + R \int_{\partial B_R} (1 - \sqrt{1 - |\nabla \phi|^2}) - \sum_{i,j=1}^3 \int_{\partial B_R} \frac{\partial_i \phi \partial_j \phi}{\sqrt{1 - |\nabla \phi|^2}} \frac{x_i x_j}{|x|}. \end{aligned} \quad (10)$$

Multiplying the first equation of (SBI) by $x \cdot \nabla u$ and the second equation by $x \cdot \nabla \phi$ and integrating on B_R , by (6), (7), (8), (9) and (10) we get, respectively,

$$\begin{aligned} &-\frac{1}{2} \int_{B_R} |\nabla u|^2 - \frac{1}{R} \int_{\partial B_R} |x \cdot \nabla u|^2 + \frac{R}{2} \int_{\partial B_R} |\nabla u|^2 - \frac{3}{2} \int_{B_R} u^2 + \frac{R}{2} \int_{\partial B_R} u^2 \\ &-\frac{1}{2} \int_{B_R} u^2 (x \cdot \nabla \phi) - \frac{3}{2} \int_{B_R} \phi u^2 + \frac{R}{2} \int_{\partial B_R} \phi u^2 \\ &= -\frac{3}{p+1} \int_{B_R} |u|^{p+1} + \frac{R}{p+1} \int_{\partial B_R} |u|^{p+1}, \end{aligned} \quad (11)$$

and

$$\begin{aligned} \int_{B_R} u^2 (x \cdot \nabla \phi) &= \int_{B_R} \frac{|\nabla \phi|^2}{\sqrt{1 - |\nabla \phi|^2}} - 3 \int_{B_R} (1 - \sqrt{1 - |\nabla \phi|^2}) \\ &\quad + R \int_{\partial B_R} (1 - \sqrt{1 - |\nabla \phi|^2}) - \sum_{i,j=1}^3 \int_{\partial B_R} \frac{\partial_i \phi \partial_j \phi}{\sqrt{1 - |\nabla \phi|^2}} \frac{x_i x_j}{|x|}. \end{aligned} \quad (12)$$

Substituting (12) into (11), since all the boundary integrals go to zero as $R \rightarrow +\infty$ (we can repeat the arguments of Berestycki and Lions (1983)), by (3) we get the conclusion. \square

3 Proofs of the Main Results

Using an idea from Jeanjean (1999); Struwe (1985), we look for bounded Palais–Smale sequences of the following perturbed functionals

$$I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) + \frac{1}{2} \int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{2} \int_{\mathbb{R}^3} (1 - \sqrt{1 - |\nabla \phi_u|^2}) - \frac{\lambda}{p+1} \int_{\mathbb{R}^3} |u|^{p+1},$$

for almost all λ near 1. Then we will deduce the existence of a non-trivial critical point v_λ of the functional I_λ at the Mountain Pass level. Afterward, we study the convergence of the sequence $(v_\lambda)_\lambda$, as λ goes to 1 (observe that $I_1 = I$).

We begin applying a slightly modified version of the monotonicity trick due to Jeanjean (1999); Struwe (1985).

Proposition 3.1 *Let $(X, \|\cdot\|)$ be a Banach space and $J \subset \mathbb{R}^+$ an interval. Consider a family of C^1 functionals I_λ on X defined by*

$$I_\lambda(u) = A(u) - \lambda B(u), \quad \text{for } \lambda \in J,$$

with B non-negative and either $A(u) \rightarrow +\infty$ or $B(u) \rightarrow +\infty$ as $\|u\| \rightarrow +\infty$ and such that $I_\lambda(0) = 0$. For any $\lambda \in J$, we set

$$\Gamma_\lambda := \{\gamma \in C([0, 1], X) \mid \gamma(0) = 0, I_\lambda(\gamma(1)) < 0\}.$$

Assume that for every $\lambda \in J$, the set Γ_λ is non-empty and

$$c_\lambda := \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0, 1]} I_\lambda(\gamma(t)) > 0.$$

Then for almost every $\lambda \in J$, there is a sequence $(v_n)_n \subset X$ such that

- (i) $(v_n)_n$ is bounded in X ;
- (ii) $I_\lambda(v_n) \rightarrow c_\lambda$, as $n \rightarrow +\infty$;
- (iii) $I'_\lambda(v_n) \rightarrow 0$ in the dual space X^{-1} of X , as $n \rightarrow +\infty$.

In our case $X = H_r^1(\mathbb{R}^3)$

$$A(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) + \frac{1}{2} \int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{2} \int_{\mathbb{R}^3} (1 - \sqrt{1 - |\nabla \phi_u|^2}),$$

$$B(u) = \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1}.$$

Observe that, by (2), $A(u) \rightarrow +\infty$ as $\|u\| \rightarrow +\infty$.

Proposition 3.2 For all $\lambda \in [1/2, 1]$, the set Γ_λ is not empty.

Proof Fix $\lambda \in [1/2, 1]$ and $u \in H_r^1(\mathbb{R}^3) \setminus \{0\}$, then, by Lemma 2.7 and for $q \in [2, 3)$, we have

$$\begin{aligned} I_\lambda(u) &\leq \frac{1}{2}\|u\|^2 + \frac{1}{2} \int_{\mathbb{R}^3} \phi_u u^2 - \frac{\lambda}{p+1} \|u\|_{p+1}^{p+1} \\ &\leq \frac{1}{2}\|u\|^2 + c\|\phi_u\|_6 \|u\|_{12}^2 - \frac{\lambda}{p+1} \|u\|_{p+1}^{p+1} \\ &\leq \frac{1}{2}\|u\|^2 + c\|\nabla \phi_u\|_2 \|u\|^2 - \frac{\lambda}{p+1} \|u\|_{p+1}^{p+1} \\ &\leq \frac{1}{2}\|u\|^2 + c\|u\|^{\frac{3q-2}{q-1}} - \frac{\lambda}{p+1} \|u\|_{p+1}^{p+1}. \end{aligned}$$

Therefore, if $\lambda \in [1/2, 1]$ and $u \in H_r^1(\mathbb{R}^3) \setminus \{0\}$ and $t > 0$, we infer that

$$I_\lambda(tu) \leq c_1 t^2 + c_2 t^{\frac{3q-2}{q-1}} - c_3 \lambda t^{p+1}.$$

Since $p \in (5/2, 5)$, we can find $q \in [2, 3)$ such that $I_\lambda(tu) < 0$, for t sufficiently large. \square

Proposition 3.3 For any $\lambda \in [1/2, 1]$, there exist $\alpha > 0$ and $\rho > 0$, sufficiently small, such that $I_\lambda(u) \geq \alpha$, for all $u \in H^1(\mathbb{R}^3)$, with $\|u\| = \rho$. As a consequence $c_\lambda \geq \alpha$.

Proof The conclusion follows easily by Lemma 2.2. \square

Proposition 3.4 For almost every $\lambda \in J$, there exists $u_\lambda \in H_r^1(\mathbb{R}^3)$, $u_\lambda \neq 0$, such that $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) = c_\lambda$.

Proof By Propositions 3.2 and 3.3 we can apply the monotonicity trick (Proposition 3.1) and we argue that, for almost every $\lambda \in J$ there exists a bounded Palais-Smale sequence $(u_n)_n \subset H_r^1(\mathbb{R}^3)$ for the functional I_λ at level c_λ , namely as $n \rightarrow +\infty$,

$$I_\lambda(u_n) \rightarrow c_\lambda, \quad I'_\lambda(u_n) \rightarrow 0.$$

Fix such a $\lambda \in J$. Exploiting compactness results holding for $H_r^1(\mathbb{R}^3)$, we have that there exists $u_\lambda \in H_r^1(\mathbb{R}^3)$ such that, up to subsequences,

$$\begin{aligned} u_n &\rightharpoonup u_\lambda \quad \text{weakly in } H_r^1(\mathbb{R}^3), \\ u_n &\rightarrow u_\lambda \quad \text{in } L^s(\mathbb{R}^3), \quad 2 < s < 6, \\ u_n &\rightarrow u_\lambda \quad \text{a.e. in } \mathbb{R}^3. \end{aligned} \tag{13}$$

By [Bonheure et al. (2016), Remark 5.5], we infer that $\phi_n := \phi_{u_n} \rightarrow \phi_{u_\lambda} =: \phi_\lambda$, weakly in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ (and uniformly in \mathbb{R}^3) so we conclude that, for every $v \in H_r^1(\mathbb{R}^3)$,

$$\lim_n I'_\lambda(u_n)[v] = I'_\lambda(u_\lambda)[v] = 0$$

that is u_λ is a critical point of I_λ .

Moreover, since the following convergence holds

$$\left| \int_{\mathbb{R}^3} \phi_n u_n^2 - \int_{\mathbb{R}^3} \phi_\lambda u_\lambda^2 \right| \leq \left| \int_{\mathbb{R}^3} \phi_n (u_n^2 - u_\lambda^2) \right| + \left| \int_{\mathbb{R}^3} (\phi_n - \phi_\lambda) u_\lambda^2 \right| \xrightarrow{n \rightarrow +\infty} 0, \quad (14)$$

taking into account that $I'_\lambda(u_n)[u_n] = o_n(1)$ and $I'_\lambda(u_\lambda)[u_\lambda] = 0$, by Proposition 2.4 it follows

$$\lim_n \|u_n\|^2 = \lim_n \left(\int_{\mathbb{R}^3} |u_n|^{p+1} - \int_{\mathbb{R}^3} \phi_n u_n^2 \right) = \int_{\mathbb{R}^3} |u_\lambda|^{p+1} - \int_{\mathbb{R}^3} \phi_\lambda u_\lambda^2 = \|u_\lambda\|^2.$$

By this and (13) we deduce that $u_n \rightarrow u_\lambda$ in $H_r^1(\mathbb{R}^3)$ and then, by (14),

$$0 < c_\lambda = \lim_n I_\lambda(u_n) = I_\lambda(u_\lambda)$$

which concludes the proof. \square

Now we are ready to prove our main result.

Proof of Theorem 1.2 By Proposition 3.4, there exists a sequence $(\lambda_n)_n \subset J$ such that $\lambda_n \nearrow 1$ and, for all $n \in \mathbb{N}$, there exists $u_n \in H_r^1(\mathbb{R}^3) \setminus \{0\}$ such that

$$\begin{aligned} I_{\lambda_n}(u_n) &= c_{\lambda_n}, \\ I'_{\lambda_n}(u_n) &= 0 \quad \text{in } (H^1(\mathbb{R}^3))'. \end{aligned} \quad (15)$$

For the sake of brevity, we will denote $\phi_n := \phi_{u_n}$. By (3), (5) and since $I'_{\lambda_n}(u_n)[u_n] = 0$, we have

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{3}{2} \int_{\mathbb{R}^3} u_n^2 + 2 \int_{\mathbb{R}^3} \frac{|\nabla \phi_n|^2}{\sqrt{1 - |\nabla \phi_n|^2}} - \frac{3}{2} \int_{\mathbb{R}^3} (1 - \sqrt{1 - |\nabla \phi_n|^2}) \\ &= \frac{3\lambda_n}{p+1} \int_{\mathbb{R}^3} |u_n|^{p+1} \int_{\mathbb{R}^3} |\nabla u_n|^2 + \int_{\mathbb{R}^3} u_n^2 + \int_{\mathbb{R}^3} \frac{|\nabla \phi_n|^2}{\sqrt{1 - |\nabla \phi_n|^2}} = \lambda_n \int_{\mathbb{R}^3} |u_n|^{p+1}. \end{aligned}$$

Multiplying the first equation by $\alpha/3$ and the second one by $\beta/(p+1)$ and summing, we have

$$\begin{aligned} & \frac{(\alpha + \beta)\lambda_n}{p+1} \int_{\mathbb{R}^3} |u_n|^{p+1} = \left(\frac{\alpha}{6} + \frac{\beta}{p+1} \right) \int_{\mathbb{R}^3} |\nabla u_n|^2 + \left(\frac{\alpha}{2} + \frac{\beta}{p+1} \right) \int_{\mathbb{R}^3} u_n^2 \\ & + \left(\frac{2\alpha}{3} + \frac{\beta}{p+1} \right) \int_{\mathbb{R}^3} \frac{|\nabla \phi_n|^2}{\sqrt{1 - |\nabla \phi_n|^2}} - \frac{\alpha}{2} \int_{\mathbb{R}^3} (1 - \sqrt{1 - |\nabla \phi_n|^2}). \end{aligned}$$

Assuming, in particular, $\alpha = 1 - \beta$, we get

$$\begin{aligned} & \frac{\lambda_n}{p+1} \int_{\mathbb{R}^3} |u_n|^{p+1} = \left(\frac{1}{6} + \frac{\beta(5-p)}{6(p+1)} \right) \int_{\mathbb{R}^3} |\nabla u_n|^2 + \left(\frac{1}{2} + \frac{\beta(1-p)}{2(p+1)} \right) \int_{\mathbb{R}^3} u_n^2 \\ & + \left(\frac{2}{3} + \frac{\beta(1-2p)}{3(p+1)} \right) \int_{\mathbb{R}^3} \frac{|\nabla \phi_n|^2}{\sqrt{1 - |\nabla \phi_n|^2}} - \left(\frac{1}{2} - \frac{\beta}{2} \right) \int_{\mathbb{R}^3} (1 - \sqrt{1 - |\nabla \phi_n|^2}). \end{aligned} \quad (16)$$

Therefore, since for all $t \in [0, 1[$

$$1 - \sqrt{1-t} \leq \frac{1}{2} \frac{t}{\sqrt{1-t}},$$

substituting (16) into (15), we have

$$\begin{aligned} c_{\text{In}} = I_{\text{In}}(u_n) &= \left(\frac{1}{3} - \frac{\beta(5-p)}{6(p+1)} \right) \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{\beta(p-1)}{2(p+1)} \int_{\mathbb{R}^3} u_n^2 \\ &\quad + \left(\frac{\beta(2p-1)}{3(p+1)} - \frac{1}{6} \right) \int_{\mathbb{R}^3} \frac{|\nabla \phi_n|^2}{\sqrt{1-|\nabla \phi_n|^2}} - \frac{\beta}{2} \int_{\mathbb{R}^3} (1 - \sqrt{1-|\nabla \phi_n|^2}) \\ &\geq \left(\frac{1}{3} - \frac{\beta(5-p)}{6(p+1)} \right) \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{\beta(p-1)}{2(p+1)} \int_{\mathbb{R}^3} u_n^2 \\ &\quad + \left(\frac{\beta(2p-1)}{3(p+1)} - \frac{1}{6} - \frac{\beta}{4} \right) \int_{\mathbb{R}^3} \frac{|\nabla \phi_n|^2}{\sqrt{1-|\nabla \phi_n|^2}}. \end{aligned}$$

Since $p > 2$, there exists a constant β such that all the coefficients in the previous inequality are positive and so, by the boundedness of $(c_{\text{In}})_n$ (indeed the map $\lambda \mapsto c_\lambda$ is non-increasing), we infer the boundedness of the sequence $(u_n)_n$ in $H_r^1(\mathbb{R}^3)$, too.

Now, arguing similarly as in the proof of Proposition 3.4, we can easily prove the existence of a nontrivial critical point u of I . Hence we have

$$\mathcal{S}_r := \{u \in H_r^1(\mathbb{R}^3) \setminus \{0\} \mid I'(u) = 0\} \neq \emptyset.$$

Moreover, any $u \in \mathcal{S}_r$ satisfies

$$\|u\|^2 \leq \int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} u^2 + \int_{\mathbb{R}^3} \frac{|\nabla \phi_u|^2}{\sqrt{1-|\nabla \phi_u|^2}} = \int_{\mathbb{R}^3} |u|^{p+1} \leq C \|u\|^{p+1},$$

and therefore

$$\inf_{u \in \mathcal{S}_r} \|u\| > 0.$$

Since we have that $I(u) \geq c \|u\|^2$ for all $u \in \mathcal{S}_r$, we conclude that

$$\sigma_r := \inf_{u \in \mathcal{S}_r} I(u) > 0.$$

Let $(u_n)_n \subset \mathcal{S}_r$ such that $I(u_n) \rightarrow \sigma_r$. Arguing as before we have that the sequence is bounded. Finally, as in the proof of Proposition 3.4, there exists $u \in H_r^1(\mathbb{R}^3)$ critical point of I such that, up to subsequences, $u_n \rightarrow u$ in $H_r^1(\mathbb{R}^3)$. Then (u, ϕ_u) is a radial ground state solution by Proposition 2.5.

Finally, by Proposition 2.8 we conclude that u and ϕ_u are of class $C^2(\mathbb{R}^3)$. \square

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