Journal of Pure and Applied Algebra ••• (••••) ••••••

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Journal of Pure and Applied Algebra

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Gelfand-Tsetlin representations of finite W-algebras

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ARTICLE INFO

Article history:
Received 2 June 2019
Received in revised form 26 August 2019
Available online xxxx
Communicated by D. Nakano

MSC:
17B10; 17B37

ABSTRACT

We construct explicitly a large family of new simple modules for an arbitrary finite W-algebra of type A. A basis of these modules is given by the Gelfand-Tsetlin tableaux whose entries satisfy certain sets of relations. Characterization and an effective method of constructing such admissible relations are given. In particular we describe a family of simple infinite dimensional highest weight relation modules. We also prove a sufficient condition for the simplicity of tensor product of two highest weight relation modules and establish the simplicity of the tensor product any number of relation modules with generic highest weights. This extends the results of Molev to infinite dimensional highest weight modules.

1. Introduction

W-algebras were first introduced in the work of Zamolodchikov in the 80's in the study of two-dimensional conformal field theories. They play an important role in string theory, integrable systems and field theories [3]. General definition of W-algebras was given in the work of Feigin and Frenkel [6] via quantized Drinfeld-Sokolov reduction. This was later generalized by Kac, Roan and Wakimoto [16], Kac and Wakimoto [15] and De Sole and Kac [26].

W-algebras can be viewed as affinizations of *finite* W-algebras, certain finitely generated structures underlying W-algebras. Their concept goes back to the papers of Kostant [17], Lynch [19] and Premet [24]. Classical finite W-algebras are constructed as Poisson reductions of Kirillov-Poisson structures on simple Lie algebras. They have a very rich theory related to the Yangians [25], [4]. In type A, Brundan and Kleshchev [4], [5] showed that finite W-algebras are isomorphic to certain quotients of the *shifted Yangians*.

If $\pi = \pi(p_1, \dots, p_n)$ is a pyramid with $N = p_1 + \dots + p_n$ boxes distributed in n rows with p_i boxes in i-th row (counting from the bottom), then the finite W-algebra $W(\pi)$ is associated with $\mathfrak{gl}_N$ and the nilpotent

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https://doi.org/10.1016/j.jpaa.2019.106226

matrix in $\mathfrak{gl}_N$ of Jordan type $(p_1, \ldots, p_n)$. In particular, $W(\pi)$ is the universal enveloping algebra of $\mathfrak{gl}_n$ if the pyramid π has one column with n boxes.

Since finite W-algebras can be embedded into the universal enveloping algebra of some (semi)simple Lie algebra via the Miura transform, representations have many features of representations of finite dimensional Lie algebras. For example, Fock space realizations of finite W-algebras were obtained in [3]. For basic representation theory of W-algebras we refer to [1] and [2].

Theory of Gelfand-Tsetlin representations for finite W-algebras of type A was developed in [9]. In such representations the Gelfand-Tsetlin subalgebra of $W(\pi)$ has a common generalized eigenspace decomposition. For an irreducible representation this is equivalent to require the existence of a common eigenvector for the Gelfand-Tsetlin subalgebra Γ. Such an eigenvector is annihilated by some maximal ideal of Γ. The main problem is to construct explicitly (with the action of algebra generators) simple Gelfand-Tsetlin modules for $W(\pi)$ generated by a vector annihilated by a fixed maximal ideal of Γ. Recent results of [7] allow to construct a certain cyclic Gelfand-Tsetlin module for $W(\pi)$ for a fixed maximal ideal of Γ. When this module is simple (sufficient condition is given in [7]) the problem of explicit construction is solved. On the other hand, even for $\mathfrak{gl}_n$ not all simple subquotients of the universal module have a tableaux basis. Hence, the difficult problem of explicit construction of simple Gelfand-Tsetlin modules remains open.

A new technique of constructing certain simple Gelfand-Tsetlin modules was developed in [12] in the case of the universal enveloping algebra of $\mathfrak{gl}_n$ generalizing the work of Gelfand and Graev [13,14] and the work of Lemire and Patera [18]. The main objective of this paper is to adapt and apply the technique of [12] in the case of finite W-algebras of type A. We obtain:

- Effective removal of relations method (the RR-method) for constructing admissible sets of relations (Theorem 3.6);
- Characterization of admissible sets of relations (Theorem 3.16);
- Explicit construction of Gelfand-Tsetlin $W(\pi)$-modules for a given admissible set of relations (Definition 3.3).

Our main result is the following:

Theorem 1.1. *For a given admissible set of relations $\mathcal{C}$ and any tableau $[l]$ satisfying $\mathcal{C}$, the space $V_{\mathcal{C}}([l])$ (see Definition 3.2) is a Gelfand-Tsetlin $W(\pi)$-module with diagonal action of the Gelfand-Tsetlin subalgebra.*

As a consequence we construct a large new family of Gelfand-Tsetlin $W(\pi)$-modules with explicit basis and action of the generators of the algebra. If $\mathcal{C}$ is an admissible set of relations and $[l]$ is any tableau satisfying $\mathcal{C}$, then we have a Gelfand-Tsetlin $W(\pi)$-module $V_{\mathcal{C}}([l])$ which we call the *relation module* associated with $\mathcal{C}$ and $[l]$. We have the following criteria for simplicity of relation modules (Theorem 4.4):

Theorem 1.2. *The Gelfand-Tsetlin module $V_{\mathcal{C}}([l])$ is simple if and only if $\mathcal{C}$ is the maximal admissible set of relations satisfied by $[l]$.*

Next we consider highest weight relation modules (in particular, for Yangians). Proposition 4.6 provides a family of simple infinite dimensional highest weight relation modules.

Finally, we consider a tensor product of relation modules. If $V_1, \ldots, V_l$ are $\mathfrak{gl}_n$-modules then $V_1 \otimes \ldots \otimes V_l$ is a module for the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$. For finite dimensional $\mathfrak{gl}_n$-modules the criterion of simplicity of such tensor product was established in [21]. We consider tensor product of infinite dimensional highest weight relation modules for $\mathrm{Y}(\mathfrak{gl}_n)$ and establish simplicity of tensor product of any number of highest weight relation modules with *generic* highest weights (Theorem 6.2), and also give sufficient conditions for the simplicity of tensor product of two highest weight relation modules (Theorem 6.3). These results extend the results of Molev [21] and Brundan and Kleshchev [5] to infinite dimensional highest weight modules for the Yangians. We observe that we do not fully cover the above mentioned results since not all finite dimensional $\mathrm{Y}(\mathfrak{gl}_n)$-modules are relation modules.

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2. Finite *W*-algebras

The ground field will be the field of complex numbers $\mathbb{C}$.

Fix a tuple $(p_1, \dots, p_n)$ such that $1 \leqslant p_1 \leqslant \cdots \leqslant p_n$. Associate with this tuple the pyramid $\pi = \pi(p_1, \dots, p_n)$, where p_i is the number of unit squares in the ith row of the pyramid counting from the bottom. We will assume that the rows of π are left-justified. From now on we set $N := p_1 + \cdots + p_n$.

Given such pyramid π, the corresponding *shifted Yangian* $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ [4] is the associative algebra over $\mathbb{C}$ defined by generators

$$\begin{aligned} &d_i^{(r)}, \quad i = 1, \dots, n, && r \geqslant 1, \\ &f_i^{(r)}, \quad i = 1, \dots, n-1, && r \geqslant 1, \\ &e_i^{(r)}, \quad i = 1, \dots, n-1, && r \geqslant p_{i+1} - p_i + 1, \end{aligned} \tag{1}$$

subject to the following relations:

$$\begin{aligned} &[d_i^{(r)}, d_j^{(s)}] = 0, \\ &[e_i^{(r)}, f_j^{(s)}] = -\,\delta_{ij} \sum_{t=0}^{r+s-1} d_i^{\prime\,(t)}\, d_{i+1}^{(r+s-t-1)}, \\ &[d_i^{(r)}, e_j^{(s)}] = (\delta_{ij} - \delta_{i,j+1}) \sum_{t=0}^{r-1} d_i^{(t)}\, e_j^{(r+s-t-1)}, \\ &[d_i^{(r)}, f_j^{(s)}] = (\delta_{i,j+1} - \delta_{ij}) \sum_{t=0}^{r-1} f_j^{(r+s-t-1)}\, d_i^{(t)}, \\ &[e_i^{(r)}, e_i^{(s+1)}] - [e_i^{(r+1)}, e_i^{(s)}] = e_i^{(r)} e_i^{(s)} + e_i^{(s)} e_i^{(r)}, \\ &[f_i^{(r+1)}, f_i^{(s)}] - [f_i^{(r)}, f_i^{(s+1)}] = f_i^{(r)} f_i^{(s)} + f_i^{(s)} f_i^{(r)}, \\ &[e_i^{(r)}, e_{i+1}^{(s+1)}] - [e_i^{(r+1)}, e_{i+1}^{(s)}] = -e_i^{(r)} e_{i+1}^{(s)}, \\ &[f_i^{(r+1)}, f_{i+1}^{(s)}] - [f_i^{(r)}, f_{i+1}^{(s+1)}] = -f_{i+1}^{(s)} f_i^{(r)}, \\ &[e_i^{(r)}, e_j^{(s)}] = 0, && \text{if} \quad |i-j| > 1, \\ &[f_i^{(r)}, f_j^{(s)}] = 0, && \text{if} \quad |i-j| > 1, \\ &[e_i^{(r)}, [e_i^{(s)}, e_j^{(t)}]] + [e_i^{(s)}, [e_i^{(r)}, e_j^{(t)}]] = 0, && \text{if} \quad |i-j| = 1, \\ &[f_i^{(r)}, [f_i^{(s)}, f_j^{(t)}]] + [f_i^{(s)}, [f_i^{(r)}, f_j^{(t)}]] = 0, && \text{if} \quad |i-j| = 1, \end{aligned}$$

for all possible i, j, r, s, t, where $d_i^{(0)} = 1$ and the elements $d_i^{\prime\,(r)}$ are obtained from the relations

$$\sum_{t=0}^{r} d_i^{(t)}\, d_i^{\prime\,(r-t)} = \delta_{r0}, \qquad r = 0, 1, \dots.$$

Note that the algebra $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ depends only on the differences $p_{i+1} - p_i$ (see (1)), and our definition corresponds to the left-justified pyramid π, as compared to [4]. In the case of a rectangular pyramid π with $p_1 = \cdots = p_n$, the algebra $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ is isomorphic to the *Yangian* $\mathrm{Y}(\mathfrak{gl}_n)$; cf. [20]. Moreover, for an arbitrary pyramid π, the shifted Yangian $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ can be regarded as a subalgebra of $\mathrm{Y}(\mathfrak{gl}_n)$.

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Following [4], the *finite W-algebra* $W(\pi)$, associated with the pyramid π, can be defined as the quotient of $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ by the two-sided ideal generated by all elements $d_1^{(r)}$ with $r \geqslant p_1+1$. In the case of the one-column pyramids π we obtain the universal enveloping algebra of $\mathfrak{gl}_n$. We refer the reader to [4,5] for a description and the structure of the algebra $W(\pi)$, including an analog of the Poincaré–Birkhoff–Witt theorem as well as a construction of algebraically independent generators of the center of $W(\pi)$.

2.1. Gelfand-Tsetlin modules

Recall that the pyramid π has left-justified rows $(p_1,\dots,p_n)$. Denote π_k the pyramid associated with the tuple $(p_1,\dots,p_k)$, and let $W(\pi_k)$ be the corresponding finite W-algebra, $k=1,\dots,n$. Then we have the following chain of subalgebras

$$W(\pi_1) \subset W(\pi_2) \subset \cdots \subset W(\pi_n) = W(\pi). \tag{2}$$

Denote by Γ the commutative subalgebra of $W(\pi)$ generated by the centers of the subalgebras $W(\pi_k)$ for $k=1,\dots,n$, which is called the *Gelfand–Tsetlin subalgebra* of $W(\pi)$ [5].

A finitely generated module M over $W(\pi)$ is called a *Gelfand-Tsetlin module* (with respect to Γ) if

$$M = \bigoplus_{\mathbf{m}\in \operatorname{Specm}\Gamma} M(\mathbf{m})$$

as a Γ-module, where

$$M(\mathbf{m}) \;=\; \{x \in M \mid \mathbf{m}^k x = 0 \quad \text{for some} \quad k \geqslant 0\}$$

and $\operatorname{Specm}\Gamma$ denotes the set of maximal ideals of Γ.

Theory of Gelfand-Tsetlin modules for $W(\pi)$ was developed in [8], [9], [10]. In particular, it was shown

Theorem 2.1. *[[10], Theorem II] Given any $\mathbf{m} \in \operatorname{Specm}\Gamma$ the number $F(n)$ of non-isomorphic simple Gelfand-Tsetlin modules M over $W(\pi)$ with $M(\mathbf{m}) \neq 0$ is non-empty and finite.*

The proof of this result is based on the important fact that the finite W-algebra $W(\pi)$ is a Galois order [11] (or equivalently, integral Galois algebra) ([10], Theorem 3.6). Moreover, in particular cases of one-column pyramids [23] and two-row pyramids [10], the number $F(n)$ is bounded by $p_1!(p_1+p_2)!\dots(p_1+\dots+p_{n-1})!$. This remains a conjecture in general.

2.2. Finite-dimensional representations of $W(\pi)$

Set

$$f_i(u) = \sum_{r=1}^{\infty} f_i^{(r)}\, u^{-r}, \qquad e_i(u) = \sum_{r=p_{i+1}-p_i+1}^{\infty} e_i^{(r)}\, u^{-r}$$

and denote

$$A_i(u) = u^{p_1}\,(u-1)^{p_2}\,\dots(u-i+1)^{p_i}\,a_i(u)$$

for $i=1,\dots,n$ with $a_i(u) = d_1(u)\,d_2(u-1)\dots d_i(u-i+1)$, and

$$B_i(u) = u^{p_1}\,(u-1)^{p_2}\,\dots(u-i+2)^{p_{i-1}}\,(u-i+1)^{p_{i+1}}\,a_i(u)\,e_i(u-i+1),$$
$$C_i(u) = u^{p_1}\,(u-1)^{p_2}\,\dots(u-i+1)^{p_i}\,f_i(u-i+1)\,a_i(u)$$

for $i = 1,\dots,n-1$. Then $A_i(u)$, $B_i(u)$, and $C_i(u)$, $i = 1,\dots,n$ are polynomials in u, and their coefficients are generators of $W(\pi)$ [9]. Define the elements $a_r^{(k)}$ for $r = 1,\dots,n$ and $k = 1,\dots,p_1+\dots+p_r$ through the expansion

$$A_r(u) = u^{p_1+\dots+p_r} + \sum_{k=1}^{p_1+\dots+p_r} a_r^{(k)}\,u^{p_1+\dots+p_r-k}.$$

Thus, the elements $a_r^{(k)}$ generate the Gelfand–Tsetlin subalgebra Γ of $W(\pi)$.

Fix an n-tuple $\lambda(u) = \big(\lambda_1(u),\dots,\lambda_n(u)\big)$ of monic polynomials in u, where $\lambda_i(u)$ has degree p_i. Let $L(\lambda(u))$ denote the irreducible highest weight representation of $W(\pi)$ with highest weight $\lambda(u)$. Then $L(\lambda(u))$ is a Gelfand-Tsetlin module generated by a nonzero vector ξ such that

$$\begin{aligned} B_i(u)\xi &= 0 && \text{for} \quad i = 1,\dots,n-1, \text{and} \\ u^{p_i}\,d_i(u)\xi &= \lambda_i(u)\xi && \text{for} \quad i = 1,\dots,n. \end{aligned}$$

Let

$$\lambda_i(u) = (u+\lambda_i^{(1)})\,(u+\lambda_i^{(2)})\dots(u+\lambda_i^{(p_i)}), \qquad i = 1,\dots,n.$$

The explicit construction of a family of finite-dimensional irreducible representations of $W(\pi)$ was given in [9]. As it will play an important role in the arguments of this paper. We recall this construction below.

2.3. Gelfand-Tsetlin basis for finite-dimensional representations

Consider a family of finite-dimensional representations of $W(\pi)$ by imposing the condition

$$\lambda_i^{(k)} - \lambda_j^{(m)} \notin \mathbb{Z}\,, \qquad \text{for all} \;\; i,j \quad \text{and all} \;\; k \neq m$$

on a highest weight $\lambda(u)$. A *standard Gelfand–Tsetlin tableau* $\mu(u)$ associated with the highest weight $\lambda(u)$ is an array of rows $(\lambda_{r1}(u),\dots,\lambda_{rr}(u))$ of monic polynomials in u for $r = 1,\dots,n$, where

$$\lambda_{ri}(u) = (u+\lambda_{ri}^{(1)})\dots(u+\lambda_{ri}^{(p_i)}), \qquad 1 \leqslant i \leqslant r \leqslant n,$$

with $\lambda_{ni}^{(k)} = \lambda_i^{(k)}$, such that the top row coincides with $\lambda(u)$, and

$$\lambda_{r+1,i}^{(k)} - \lambda_{ri}^{(k)} \in \mathbb{Z}_{\geq 0} \qquad \text{and} \qquad \lambda_{ri}^{(k)} - \lambda_{r+1,i+1}^{(k)} \in \mathbb{Z}_{\geq 0}$$

for $k = 1,\dots,p_i$ and $1 \leqslant i \leqslant r \leqslant n-1$.

The following result was shown in [9].

Theorem 2.2. *The representation $L(\lambda(u))$ of the algebra $W(\pi)$ allows a basis $\{\xi_\mu\}$ parametrized by all standard tableaux $\mu(u)$ associated with $\lambda(u)$ such that the action of the generators is given by the formulas*

$$A_r(u)\,\xi_\mu = \lambda_{r1}(u)\dots\lambda_{rr}(u-r+1)\,\xi_\mu, \quad r = 1,\dots,n \tag{3}$$

and

$$B_r(-l_{ri}^{(k)})\,\xi_\mu = -\lambda_{r+1,1}(-l_{ri}^{(k)})\cdots\lambda_{r+1,r+1}(-l_{ri}^{(k)}-r)\,\xi_{\mu+\delta_{ri}^{(k)}},\ 1\le r\le n-1, \tag{4}$$

$$C_r(-l_{ri}^{(k)})\,\xi_\mu = \lambda_{r-1,1}(-l_{ri}^{(k)})\cdots\lambda_{r-1,r-1}(-l_{ri}^{(k)}-r+2)\,\xi_{\mu-\delta_{ri}^{(k)}},\ 1\le r\le n-1, \tag{5}$$

where $l_{ri}^{(k)} = \lambda_{ri}^{(k)} - i + 1$ *and* $\xi_{\mu\pm\delta_{ri}^{(k)}}$ *corresponds to the tableau obtained from* $\mu(u)$ *by replacing* $\lambda_{ri}^{(k)}$ *by* $\lambda_{ri}^{(k)}\pm 1$*, while the vector* ξ_μ *is set to be zero if* $\mu(u)$ *is not a standard tableau associated with* $\lambda(u)$*.*

The action of the operators $B_r(u)$ and $C_r(u)$ for an arbitrary value of u can be calculated using the Lagrange interpolation formula.

For convenience we identify ξ_μ with the tableau $[l]$ with entries $l_{ij}^{(k)}$, and $\xi_{\mu\pm\delta_{ri}^{(k)}}$ with the tableau $[l\pm\delta_{ri}^{(k)}]$, which corresponds to the tableau obtained from $[l]$ by replacing $l_{ri}^{(k)}$ by $l_{ri}^{(k)}\pm 1$. Set

$$l_{ri}(u) = (u+l_{ri}^{(1)})\dots(u+l_{ri}^{(p_i)}),\qquad 1\leqslant i\leqslant r\leqslant n,$$

with $l_{ni}^{(k)} = \lambda_i^{(k)} - i + 1$, as ξ_μ is standard, we have

$$l_{r+1,i}^{(k)} - l_{ri}^{(k)} \in \mathbb{Z}_{\geq 0} \qquad\text{and}\qquad l_{ri}^{(k)} - l_{r+1,i+1}^{(k)} \in \mathbb{Z}_{>0} \tag{6}$$

for $k = 1,\dots,p_i$ and $1\leqslant i\leqslant r\leqslant n-1$. If $\lambda_{ri}(u) = l_{ri}(u+i-1)$, the Gelfand-Tsetlin formulas can be rewritten as follows:

$$A_r(u)[l] = l_{r1}(u)\dots l_{rr}(u)\,[l], r=1,\dots,n, \tag{7}$$

and

$$B_r(-l_{ri}^{(k)})\,[l] = -l_{r+1,1}(-l_{ri}^{(k)})\dots l_{r+1,r+1}(-l_{ri}^{(k)})\,[l+\delta_{ri}^{(k)}],\ r=1,\dots,n-1 \tag{8}$$

$$C_r(-l_{ri}^{(k)})\,[l] = l_{r-1,1}(-l_{ri}^{(k)})\dots l_{r-1,r-1}(-l_{ri}^{(k)})\,[l-\delta_{ri}^{(k)}],\ r=1,\dots,n-1, \tag{9}$$

where a vector $[\tilde{l}]$ is set to be zero if it does not satisfy (6).

By the Lagrange interpolation formula we have

$$\begin{aligned} A_r(u)[l] &= l_{r1}(u)\dots l_{rr}(u)\,[l],\\ B_r(u)\,[l] &= -\sum_{i,k}\left(\frac{\prod_{j,t}(l_{r+1,j}^{(t)}-l_{r,i}^{(k)})}{\prod_{(j,t)\neq(i,k)}(l_{r,j}^{(t)}-l_{r,i}^{(k)})}\right)\prod_{(j,t)\neq(i,k)}(u+l_{r,j}^{(t)})\,[l+\delta_{ri}^{(k)}],\\ C_r(u)\,[l] &= \sum_{i,k}\left(\frac{\prod_{j,t}(l_{r-1,j}^{(t)}-l_{r,i}^{(k)})}{\prod_{(j,t)\neq(i,k)}(l_{r,j}^{(t)}-l_{r,i}^{(k)})}\right)\prod_{(j,t)\neq(i,k)}(u+l_{r,j}^{(t)})\,[l-\delta_{ri}^{(k)}]. \end{aligned} \tag{10}$$

It is easy to see that $d_r(u) = a_{r-1}^{-1}(u)a_r(u) = (u-r+1)^{-p_r}A_{r-1}^{-1}A_r$. Then the action of $d_r(u)$ is given by

$$d_r(u)[l] = \frac{l_{r1}(u)\dots l_{rr}(u)}{(u-r+1)^{p_r}l_{r-1,1}(u)\dots l_{r-1,r-1}(u)}[l].$$

Note that the polynomials $l_{r1}(u)\cdots l_{rr}(u)$ and $(u-r+1)^{p_r}l_{r-1,1}(u)\cdots l_{r-1,r-1}(u)$ have the same degree $p_1+\cdots+p_r$. Hence $\frac{l_{r1}(u)\dots l_{rr}(u)}{(u-r+1)^{p_r}l_{r-1,1}(u)\dots l_{r-1,r-1}(u)}$ can be written as the following formal series in u:

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$$1+\sum_{t=1}^{\infty} d_r^{(t)}(l)\,u^{-t},$$

where $d_r^{(t)}(l)$ is a polynomial in $l_{r,i}^{(k)}$ and $l_{r-1,j}^{(s)}$ with $1 \le i \le r$, $1 \le k \le p_i$, $1 \le j \le r-1$, $1 \le k \le p_j$. Thus $d_r(u)[l] = d_r^{(t)}(l)[l]$.

Similarly, since

$$e_r(u) = u^{p_r - p_{r+1}} A_r^{-1}(u+r-1)B_r(u+r-1),$$
$$f_r(u) = C_r(u+r-1)A_r^{-1}(u+r-1),$$

the action of e_r and f_r is given by

$$\begin{aligned} e_r(u)\,[l] &= -\sum_{i,k}\left(\frac{\prod_{j,t}(l_{r+1,j}^{(t)} - l_{r,i}^{(k)})}{\prod_{(j,t)\neq(i,k)}(l_{r,j}^{(t)} - l_{r,i}^{(k)})}\frac{\prod_{(j,t)\neq(i,k)}(u+r-1+l_{r,j}^{(t)})}{u^{p_{r+1}-p_r}\prod_{(j,t)}(u+r-1+l_{r,j}^{(t)}+\delta_{ri}^{(k)})}\right)[l+\delta_{ri}^{(k)}], \\ f_r(u)\,[l] &= \sum_{i,k}\left(\frac{\prod_{j,t}(l_{r-1,j}^{(t)} - l_{r,i}^{(k)})}{\prod_{(j,t)\neq(i,k)}(l_{r,j}^{(t)} - l_{r,i}^{(k)})}\frac{\prod_{(j,t)\neq(i,k)}(u+r-1+l_{r,j}^{(t)})}{\prod_{(j,t)}(u+r-1+l_{r,j}^{(t)})}\right)[l-\delta_{ri}^{(k)}]. \end{aligned} \tag{11}$$

Since $\prod_{(j,t)\neq(i,k)}(u+r-1+l_{r,j}^{(t)})$ is a polynomial in u of degree $p_1+\cdots+p_r-1$ while $\prod_{(j,t)}(u+r-1+l_{r,j}^{(t)}+\delta_{ri}^{(k)})$ and $\prod_{(j,t)}(u+r-1+l_{r,j}^{(t)})$ are polynomials of degree $p_1+\cdots+p_r$, we can write the two rational functions in (11) as follows:

$$\frac{\prod_{(j,t)\neq(i,k)}(u+r-1+l_{r,j}^{(t)})}{u^{p_{r+1}-p_r}\prod_{(j,t)}(u+r-1+l_{r,j}^{(t)}+\delta_{ri}^{(k)})} = \sum_{t=p_{r+1}-p_r+1}^{\infty} b_{r,k,i}^{(t)}(l)\,u^{-t},$$

$$\frac{\prod_{(j,t)\neq(i,k)}(u+r-1+l_{r,j}^{(t)})}{\prod_{(j,t)}(u+r-1+l_{r,j}^{(t)})} = \sum_{t=1}^{\infty} c_{r,k,i}^{(t)}(l)\,u^{-t},$$

where $b_{r,k,i}^{(t)}(l)$ and $c_{r,k,i}^{(t)}(l)$ are polynomials in $l_{r,i}^{(k)}$ with $1 \le i \le r$, $1 \le k \le p_i$ and $b_{r,k,i}^{(p_{r+1}-p_r+1)}(l) = c_{r,k,i}^{(1)}(l) = 1$. Therefore the action of $e_r^{(t)}$ and $f_r^{(t)}$ can be expressed as follows:

$$\begin{aligned} e_r^{(t)}\,[l] &= -\sum_{i,k}\left(\frac{\prod_{j,t}(l_{r+1,j}^{(t)} - l_{r,i}^{(k)})}{\prod_{(j,t)\neq(i,k)}(l_{r,j}^{(t)} - l_{r,i}^{(k)})} b_{r,k,i}^{(t)}(l)\right)[l+\delta_{ri}^{(k)}], \\ f_r^{(t)}\,[l] &= \sum_{i,k}\left(\frac{\prod_{j,t}(l_{r-1,j}^{(t)} - l_{r,i}^{(k)})}{\prod_{(j,t)\neq(i,k)}(l_{r,j}^{(t)} - l_{r,i}^{(k)})} c_{r,k,i}^{(t)}(l)\right)[l-\delta_{ri}^{(k)}]. \end{aligned} \tag{12}$$

3. Admissible sets of relations

In this section we discuss admissible sets of relations and obtain their characterization. Each such set defines an infinite family of Gelfand-Tsetlin modules over $W(\pi)$. Let $a, b \in \mathbb{C}$, from now on whenever we write $a \geq b$ (respectively $a > b$) we will mean $a - b \in \mathbb{Z}_{\geq 0}$ (respectively $a - b \in \mathbb{Z}_{>0}$). Set $\mathfrak{V} := \{(k, i, j) \mid 1 \leq j \leq i \leq n,\ 1 \leq k \leq p_j\}$. From now on when we write a triple (k, i, j) we assume that $1 \leq j \leq i \leq n, 1 \leq k \leq p_j$ without mentioning this restriction.

Set

$$\mathcal{R}^+ := \{((k, i, j); (k', i-1, j'))\},$$
$$\mathcal{R}^- := \{((k, i, j); (k', i+1, j'))\},$$
$$\mathcal{R}^0 := \{((k, n, j); (k', n, j')) \mid k \neq k' \text{ or } j \neq j'\}.$$

Let us consider $\mathcal{R} := \mathcal{R}^- \cup \mathcal{R}^0 \cup \mathcal{R}^+ \subset \mathfrak{V} \times \mathfrak{V}$. From now on any $\mathcal{C} \subseteq \mathcal{R}$ will be called a *set of relations*.

Associated with any $\mathcal{C} \subseteq \mathcal{R}$ we can construct a directed graph $G(\mathcal{C})$ with set of vertices $\mathfrak{V}$ and an arrow going from (k, i, j) to (r, s, t) if and only if $((k, i, j); (r, s, t)) \in \mathcal{C}$. For convenience we will picture the set of vertex as disposed in an arrangement of p_n triangular arrays with k-th array given by $\{(k, i, j) | n - r + 1 \leq j \leq i \leq n\}$, where $p_{r-1} < k \leq p_r$ for some $r \in \{1, 2, \ldots, n\}$ (taking $p_0 = 0$).

Definition 3.1. Let $\mathcal{C}$ be any set of relations.

(i) By $\mathfrak{V}(\mathcal{C})$ we will denote the subset of $\mathfrak{V}$ consisting of all (k, i, j) which are the starting or the ending vertex of an arrow in $G(\mathcal{C})$.
(ii) $\mathcal{C}$ is called indecomposable if $G(\mathcal{C})$ is a connected graph.
(iii) $\mathcal{C}$ is called a loop if $G(\mathcal{C})$ is a loop.
(iv) Given $(k, i, j),\ (r, s, t) \in \mathfrak{V}(\mathcal{C})$ we will write $(k, i, j) \succeq_{\mathcal{C}} (r, s, t)$ if there exist a path in $G(\mathcal{C})$ starting in (k, i, j) and finishing in (r, s, t).

Given $\mathcal{C} \subseteq \mathcal{R}$, we have $\mathcal{C} = \mathcal{C}^- \cup \mathcal{C}^0 \cup \mathcal{C}^+$, where $\mathcal{C}^- := \mathcal{R}^- \cap \mathcal{C}$, $\mathcal{C}^0 := \mathcal{R}^0 \cap \mathcal{C}$ and $\mathcal{C}^+ := \mathcal{R}^+ \cap \mathcal{C}$.

Definition 3.2. Let $\mathcal{C}$ be any set of relations and $[l]$ any Gelfand-Tsetlin tableau.

(i) We will say that $[l]$ satisfies $\mathcal{C}$ if:
- $l_{ij}^{(k)} - l_{st}^{(r)} \in \mathbb{Z}_{\geq 0}$ for any $((k, i, j); (r, s, t)) \in \mathcal{C}^+ \cup \mathcal{C}^0$.
- $l_{ij}^{(k)} - l_{st}^{(r)} \in \mathbb{Z}_{>0}$ for any $((k, i, j); (r, s, t)) \in \mathcal{C}^-$.

(ii) We say that $[l]$ is a $\mathcal{C}$-realization if $[l]$ satisfies $\mathcal{C}$ and for any $1 \leq i \leq n-1$ we have, $l_{ij}^{(k)} - l_{ij'}^{(k')} \in \mathbb{Z}$ if only if (k, i, j) and (k', i, j') in the same connected component of $G(\mathcal{C})$.
(iii) Let $[l]$ be a $\mathcal{C}$-realization. By $\mathcal{B}_{\mathcal{C}}([l])$ we denote the set of all tableaux of the form $[l + z]$, $z_{ij}^{(k)} \in \mathbb{Z}$, $z_{nj}^{(k)} = 0$, satisfying $\mathcal{C}$ (in particular such tableaux are $\mathcal{C}$-realizations). By $V_{\mathcal{C}}([l])$ denote the complex vector space spanned by $\mathcal{B}_{\mathcal{C}}([l])$.

Definition 3.3. Let $\mathcal{C}$ be any subset of $\mathcal{R}$. We call $\mathcal{C}$ *admissible* if for any $\mathcal{C}$-realization $[l]$, Gelfand-Tsetlin formulas (10) define on $V_{\mathcal{C}}([l])$ a structure of $W(\pi)$-module.

Example 3.4. It follows from Theorem 2.2 that if

$$\begin{aligned} \mathcal{S}^+ &= \{((k, i+1, j); (k, i, j)) \mid i \leq n-1\}, \\ \mathcal{S}^- &= \{((k, i, j); (k, i+1, j+1)) \mid i \leq n-1\}, \end{aligned} \tag{13}$$

then $\mathcal{S} := \mathcal{S}^+ \cup \mathcal{S}^-$ is admissible.

Our goal is to determine admissible sets of relations.

Description of admissible sets is a difficult problem. Nevertheless, the *relations removal method* (*RR-method* for short), developed in [12] can be applied in the case of finite W-algebras and provides an effective tool of constructing admissible subsets of $\mathcal{R}$.

Definition 3.5. Let $\mathcal{C} \subseteq \mathcal{R}$ and $A \subseteq \mathfrak{V}(\mathcal{C})$. By $\mathcal{C}_A$ we denote the set of relations obtained from $\mathcal{C}$ by removing all pairs in $\mathcal{C}$ containing elements of A. We say that $\widetilde{\mathcal{C}} \subseteq \mathcal{C}$ is obtained from $\mathcal{C}$ by the RR-method if $\widetilde{\mathcal{C}} = \mathcal{C}_{\{v_1,\dots,v_t\}}$ where v_1 is maximal or minimal in $G(\mathcal{C})$ and v_s is maximal or minimal in $G(\mathcal{C}_{\{v_1,\dots,v_{s-1}\}})$ for any $s = 2,\dots,t$.

Let Ω_n be the free abelian group generated by the Kronecker delta's $\delta_{ij}^{(k)}$, $1 \le j \le i \le n-1$, $1 \le k \le p_j$. We can identify Ω_n with the set of integral tableaux with zero top rows.

Given pyramid π, denote by F_π the corresponding free algebra on the following generators

$$\begin{aligned}
&d_i^{(r)}, \quad i = 1,\dots,n, && r \geqslant 1,\\
&f_i^{(r)}, \quad i = 1,\dots,n-1, && r \geqslant 1,\\
&e_i^{(r)}, \quad i = 1,\dots,n-1, && r \geqslant p_{i+1} - p_i + 1.
\end{aligned}$$

Then Gelfand-Tsetlin formulas can be used to define an action of F_π on $V_{\mathcal{C}}([l])$. Denote the kernel of the canonical homomorphism $F_\pi \to W(\pi)$ by $\mathfrak{k}_\pi$.

Theorem 3.6. *Let $\mathcal{C}_1$ be any admissible subset of $\mathcal{R}$ and suppose that $\mathcal{C}_2$ is obtained from $\mathcal{C}_1$ by the RR-method, then $\mathcal{C}_2$ is admissible.*

Proof. Suppose $\mathcal{C}_2$ is obtained from $\mathcal{C}_1$ by removing the relations involving (k,i,j). To show $\mathcal{C}_2$ is admissible it is sufficient to prove that for any $\mathcal{C}_2$-realization $[l]$ and any generator $g \in \mathfrak{k}_\pi$ we have $g[l+z] = 0$, where $z \in \Omega_n$ is such that $[l+z] \in \mathcal{B}_{\mathcal{C}_2}([l])$. The proof of this fact generalizes the argument of the proof of Theorem 4.9 in [12].

Assume (k,i,j) to be maximal (resp. minimal) and m some positive (resp. negative) integer with $|m| > 3$. Let $[\gamma]$ be a $\mathcal{C}_1$-realization such that $\gamma_{st}^{(r)} = (l+z)_{st}^{(r)}$ for $(s,t,r) \neq (k,i,j)$. Then $V_{\mathcal{C}_1}([\gamma])$ is a $W(\pi)$-module and $[\gamma + m\delta_{ij}^{(k)}] \in \mathcal{B}_{\mathcal{C}_1}([\gamma])$. Therefore we have

$$g([\gamma + m\delta_{ij}^{(k)}]) = \sum_{w \in A} g_w(\gamma + m\delta_{ij}^{(k)})[\gamma + m\delta_{ij}^{(k)} + w],$$

where $A \subset \Omega_n$ is such that $[\gamma + m\delta_{ij}^{(k)} + w] \in \mathcal{B}_{\mathcal{C}_1}([\gamma])$ for all $w \in A$. We have that $[l+z+w] \in \mathcal{B}_{\mathcal{C}_2}([l])$ if and only if $[\gamma + m\delta_{ij}^{(k)} + w] \in \mathcal{B}_{\mathcal{C}_1}([\gamma])$ when $|m| > 3$. Thus,

$$g[l+z] = \sum_{w \in A} g_w([l+z])([l+z+w]).$$

Since $g_w([\gamma + m\delta_{ij}^{(l)}])$ are rational functions in the variable m and they are zero for infinitely many values of m, we conclude that $g_w([l+z+w]) = 0$ for all $w \in A$ and, hence, $\mathcal{C}_2$ is admissible. $\square$

Since empty set can be obtained from $\mathcal{S}$ applying the RR-method finitely many times, Theorem 3.6 immediately implies:

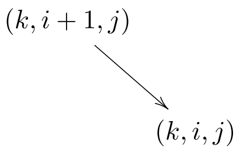
Corollary 3.7. *Empty set is admissible. In particular, if* $[l]$ *is a tableau with* $\mathbb{Z}$*-independent entries (i.e. the differences of entries on the same row are non-integers)* $l_{ij}^{(k)}$, $1 \le j \le i \le n, 1 \le k \le p_j$, $\mathcal{B}([l])$ *the set of all tableaux* $[l+z]$ *with* $z \in \Omega_n$ *and* $V([l])$ *the complex vector space spanned by* $\mathcal{B}([l])$*. Then* $V([l])$ *is a* $W(\pi)$*-module with the action of generators given by the formulas* (10).

For any fixed i, let G_i be group of permutations on the set $\{(k,i,j), 1 \le j \le i, 1 \le k \le p_j\}$. Let $G = G_1 \times G_2 \times \cdots \times G_n$. For any relation $((k,i,j);(r,s,t))$, and $\sigma \in G$, we denote $\sigma((k,i,j);(r,s,t)) = (\sigma(k,i,j);\sigma(r,s,t))$, and $\sigma\mathcal{C} = \{\sigma a \mid a \in \mathcal{C}\}$. Since the Gelfand-Tsetlin formulas (10) are G-invariant, we immediately have:

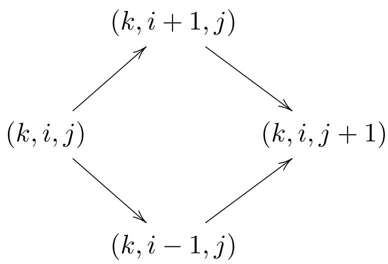
Lemma 3.8. *If* $\mathcal{C}$ *is admissible then* $\sigma\mathcal{C}$ *is admissible for any* $\sigma \in G$,

Example 3.9. The following sets are admissible by Theorem 2.2 and Theorem 3.6.

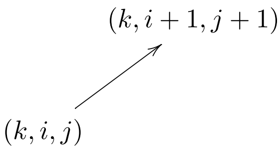
[(i)]



[(ii)]

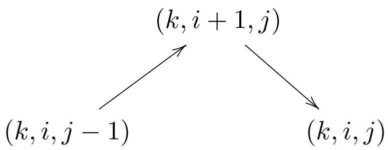


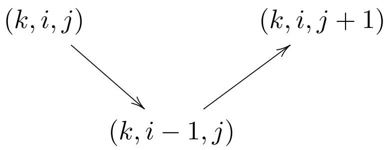
[(iii)]



It follows from Lemma 3.8 that the permutations of these sets are also admissible.

Example 3.10. The following sets are not admissible.





Hence, the permutations of these sets are not admissible either by Lemma 3.8.

Definition 3.11. $\mathcal{C} \subseteq \mathcal{R}$ is called noncritical if for any $\mathcal{C}$- realization $[l]$, and any $(k,i,j),(k,i,t) \in \mathfrak{V}(\mathcal{C})$, one has $l_{ij}^{(k)} \neq l_{it}^{(k)}$.

Definition 3.12. Let $\mathcal{C}$ be an indecomposable noncritical subset of $\mathcal{R}$. A subset of $\mathcal{C}$ of the form $\{((k_1,i,j_1);(k_4,i+1,j_4)),\ ((k_3,i+1,j_3);(k_2,i,j_2))\}$ with $k_1 < k_2$(or $k_1 = k_2, j_1 < j_2$) and $k_3 < k_4$(or $k_3 = k_4, j_3 < j_4$) will be called a *cross*.

Proposition 3.13. *Let* $\mathcal{C}$ *be an indecomposable noncritical subset of* $\mathcal{R}$*. If* $\mathcal{C}$ *contains a cross, then it is not admissible.*

Proof. Indeed, assume that $\mathcal{C}$ is admissible and contains a cross. Then, applying the RR-method to $\mathcal{C}$ we will obtain a set of relations from Example 3.10 (see details in [12]). Therefore $\mathcal{C}$ is not admissible. □

Please cite this article in press as: V. Futorny et al., Gelfand-Tsetlin representations of finite W-algebras, J. Pure Appl. Algebra (2019), https://doi.org/10.1016/j.jpaa.2019.106226

Definition 3.14. Let $\mathcal{C}$ be any noncritical set of relations. We call $\mathcal{C}$ *reduced* if for every $(k,i,j) \in \mathfrak{V}(\mathcal{C})$ the following conditions are satisfied:

(i) There exist at most one (r,t) such that $((r,i+1,t);(k,i,j)) \in \mathcal{C}$,
(ii) There exist at most one (r,t) such that $((k,i,j);(r,i+1,t)) \in \mathcal{C}$,
(iii) There exist at most one (r,t) such that $((k,i,j);(r,i-1,t)) \in \mathcal{C}$,
(iv) There exist at most one (r,t) such that $((r,i-1,t);(k,i,j)) \in \mathcal{C}$,
(v) Any relation in the top row is not implied by other relations.

The following important result follows from [12], Theorem 4.17.

Theorem 3.15. *Any noncritical set of relations is equivalent to an unique reduced set of relations.*

Let $\mathcal{C}$ be an indecomposable set and $\prec$ be the lexicographical order. We say that $\mathcal{C}$ is *pre-admissible* if it satisfies the following conditions:

(i) $\mathcal{C}$ does not contain loops.
(ii) $\mathcal{C}$ is noncritical.
(iii) For any $1 \le i \le n$, $(k,i,j) \succeq_{\mathcal{C}} (r,i,t)$ if only if $(k,i,j),(r,i,t)$ are in the same indecomposable subset of $\mathcal{C}$ and $(k,j) \prec (r,t)$.
(iv) $\mathcal{C}$ is reduced.
(v) There is not crosses in $\mathcal{C}$.

An arbitrary set $\mathcal{C}$ is pre-admissible if every indecomposable subset of $\mathcal{C}$ is pre-admissible.

The results in [12] show that in order to construct $\mathfrak{gl}_n$-modules using sets of relations it is enough to consider pre-admissible sets of relations.

Denote by $\mathfrak{F}$ the set of all indecomposable sets $\mathcal{C}$ which satisfy the following condition:

Condition 1. *for every adjoining triples* (k,i,j) *and* (r,i,s), $1 \le i \le n-1$, *one of the following is a subset of* $\mathcal{C}$

$$\begin{aligned} &\{((k,i,j);(k_1,i+1,t_1)),((k_1,i+1,t_1);(r,i,s)),\ ((k,i,j);(k_2,i,t_2)),((k_2,i,t_2),(r,i,s))\} \\ &\{((k,i,j);(k_1,i+1,j_1)),((k_2,i+1,j_2);(r,i,s))\},(k_1,j_1) \prec (k_2,j_2). \end{aligned} \tag{14}$$

It is easy to verify that if $\mathcal{C}$ is reduced and satisfies Condition 1, then $\mathcal{C}$ satisfies all other conditions in the definition of pre-admissible. Thus one has that $\mathcal{C} \in \mathfrak{F}$ if and only if $\mathcal{C}$ is reduced and it satisfies Condition 1.

The main result of this section is the following theorem which gives a characterization of admissible sets of relations. A detailed proof will be given in Section 6. For the universal enveloping algebra of $\mathfrak{gl}_n$ this result was established in [12], Theorem 4.27.

Theorem 3.16. *A pre-admissible set of relations $\mathcal{C}$ is admissible if and only if $\mathcal{C}$ is a union of indecomposable sets from $\mathfrak{F}$.*

For an admissible set of relations $\mathcal{C}$ and any $[l]$ which satisfies $\mathcal{C}$, the $W(\pi)$-module $V_{\mathcal{C}}([l])$ is a Gelfand-Tsetlin module. We will call it a *relation module*.

4. Simplicity of relation modules

In this section we establish the criterion of simplicity of the module $V_{\mathcal{C}}([l])$.

Lemma 4.1. *Let $\sum_\mu c_\mu[l_\mu]$ be a vector in $V_{\mathcal{C}}([l])$ with nonzero c_μ. Then $[l_\mu] \in V_{\mathcal{C}}([l])$ for each μ.*

Proof. Suppose $c_\mu[l_\mu] + c_\nu[l_\nu] \in V_{\mathcal{C}}([l])$ and $[l_\mu]$, $[l_\nu]$ have different entries in r-th row. By Theorem 2.2 we have $A_r(u)\,[l_\mu] = a_\mu[l_\mu]$, $A_r(u)\,[l_\mu] = a_\nu[l_\mu]$. Moreover, $a_\mu = a_\nu$ if and only if the r-th row of $[l_\nu]$ is a permutation of the corresponding row of $[l_\mu]$, which is a contradiction with the non criticality of $\mathcal{C}$. So $a_\mu \neq a_\nu$ and both $[l_\mu]$ and $[l_\nu]$ are in $V_{\mathcal{C}}([l])$. The general case follows by induction on the number of terms in the linear combination. □

Lemma 4.2. *Let $\mathcal{C}$ be an admissible set of relations, $[l]$ and $[\gamma]$ be tableaux satisfying $\mathcal{C}$ and $l_{n,i}^{(k)} = \gamma_{n,i}^{(k)}$, $l_{r,i}^{(k)} - \gamma_{r,i}^{(k)} \in \mathbb{Z}, 1 \leq r \leq n-1$. Then there exist $\{(k_t, i_t, j_t)\}_{t=1,\ldots,s} \subseteq \mathfrak{V}(\mathcal{C})$ such that for any $r \leq s$, $[l + \sum_{t=1}^{r} \epsilon_t \delta_{i_t,j_t}^{(k_t)}]$ satisfies $\mathcal{C}$ and $[l + \sum_{t=1}^{s} \epsilon_t \delta_{i_t,j_t}^{(k_t)}] = [\gamma]$, where $\epsilon_t = 1$ if $\gamma_{i_t,j_t}^{(k_t)} - l_{i_t,j_t}^{(k_t)} \geq 0$ and $\epsilon_t = -1$ if $\gamma_{i_t,j_t}^{(k_t)} - l_{i_t,j_t}^{(k_t)} < 0$.*

Proof. We prove the statement by induction on $\#\mathfrak{V}(\mathcal{C})$. It is obvious if $\#\mathfrak{V}(\mathcal{C}) = 2$. Assume $\#\mathfrak{V}(\mathcal{C}) = n$. Let (k, i, j) be maximal and $\mathcal{C}'$ be the set obtained from $\mathcal{C}$ by RR-method i.e. removing all relations that involve (k, i, j). By induction, there exist sequences (k'_t, i'_t, j'_t) $1 \leq t \leq s$ such that for any $r \leq s$, $[l + \sum_{t=1}^{r} \epsilon_t \delta_{i_t,j_t}^{(k_t)}]$ satisfies $\mathcal{C}'$ and $[l + \sum_{t=1}^{s} \epsilon_t \delta_{i_t,j_t}^{(k_t)}] = [\gamma + l_{ij}^{(k)} - \gamma_{ij}^{(k)}]$.

If $l_{ij}^{(k)} - \gamma_{ij}^{(k)} = m \geq 0$, set $(k_t, i_t, j_t) = (k'_t, i'_t, j'_t)$ for $1 \leq t \leq s$, and $(k_t, i_t, j_t) = (k, i, j)$ for $s+1 \leq t \leq t+m$.

If $l_{ij}^{(k)} - \gamma_{ij}^{(k)} = m < 0$, set $(k_t, i_t, j_t) = (k, i, j)$ for $1 \leq t \leq m$ and $(k_{m+t}, i_{m+t}, j_{m+t}) = (k'_t, i'_t, j'_t)$ for $1 \leq t \leq s$. □

Definition 4.3. We say that a set $\mathcal{C}$ is the *maximal* set of relations for $[l]$ if $[l]$ satisfies $\mathcal{C}$ and for any other set of relations $\mathcal{C}'$ satisfied by $[l]$, we have $(k, i, j) \succeq_{\mathcal{C}'} (r, i, t)$ implies $(k, i, j) \succeq_{\mathcal{C}} (r, i, t)$.

Now we can prove Theorem 1.2.

Theorem 4.4. *Let $\mathcal{C}$ be an admissible set of relations. The module $V_{\mathcal{C}}([l])$ is simple if and only if $\mathcal{C}$ is the maximal set of relations satisfied by $[l]$.*

Proof. Suppose $\mathcal{C}$ is not the maximal set of relations satisfied by $[l]$. Then there exists $l_{r+1,i}^{(s)} - l_{r,j}^{(t)} \in \mathbb{Z}$ and there is not relation between $(s, r+1, i)$ and (t, r, j). So there exists tableau $[\gamma] \in W(\pi)[l]$ such that $\gamma_{r+1,i}^{(s)} - \gamma_{r,j}^{(t)} \in \mathbb{Z}_{\geq 0}$ and $\xi \in W(\pi)[l]$ such that $\xi_{r,j}^{(t)} - \xi_{r+1,i}^{(s)} \in \mathbb{Z}_{>0}$. By Equation (11) one has that ξ is not in the submodule $W(\pi)[\gamma]$ of $V_{\mathcal{C}}([l])$ generated by $[\gamma]$, thus $V_{\mathcal{C}}([l])$ is not simple.

Conversely, let $\mathcal{C}$ be the maximal set of relations satisfied by $[l]$. By Lemma 4.2, for any tableaux $[l]$ and $[\gamma]$, there exist $\{(k_t, i_t, j_t)\}_{1\leq t\leq s}$ such that for any $r \leq s$, $[l+\sum_{t=1}^{r} \epsilon_t \delta_{i_t,j_t}^{(k_t)}]$ satisfies $\mathcal{C}$ and $[l+\sum_{t=1}^{s} \epsilon_t \delta_{i_t,j_t}^{(k_t)}] = [\gamma]$. If $[l]$ and $[l + \delta_{ij}^{(k)}]$ satisfy $\mathcal{C}$, then $l_{i,j}^{(k)} \neq \delta_{i+1,j'}^{(t)}$ for any t, j'. Similarly if $[l]$ and $[l - \delta_{ij}^{(k)}]$ satisfy $\mathcal{C}$, then $l_{i,j}^{(k)} \neq \delta_{i-1,j'}^{(t)}$ for any t, j'. Thus the coefficient of $[l + \delta_{i,j}^{(k)}]$ in $e_i^{(p_{i+1}-p_i+1)}[l]$ (resp. $[l - \delta_{i,j}^{(k)}]$ in $f_i^{(1)}[l]$ is nonzero. By Lemma 4.1, $[l \pm \delta_{i_1,j_1}^{(k_1)}] \in V_{\mathcal{C}}([l])$. By induction on s, we conclude that $[\gamma] \in V_{\mathcal{C}}([l])$. □

4.1. Highest weight relation modules

Denote by q_k the number of bricks in the column k of the pyramid π, $k = 1, \ldots, p_n$, where $l := p_n$ is the number of the columns in π. We have $N = q_1 + q_2 + \cdots + q_l = p_1 + \cdots + p_n$, with $q_1 \geq \cdots \geq q_l > 0$, moreover, if $p_{i-1} < k \leq p_i$ for some $i \in \{1, \ldots, n\}$ (taking $p_0 = 0$), then $q_k = n - i + 1$. Let $\mathfrak{g} = \mathfrak{gl}_N$, $\mathfrak{p}$ be the standard parabolic subalgebra of $\mathfrak{g}$ with the Levi factor $\mathfrak{a} = \mathfrak{gl}_{q_1} \oplus \cdots \oplus \mathfrak{gl}_{q_l}$. Then $W(\pi)$ is a subalgebra of $U(\mathfrak{p})$. We will identify $U(\mathfrak{a})$ with $U(\mathfrak{gl}_{q_1}) \otimes \cdots \otimes U(\mathfrak{gl}_{q_l})$. Let $\xi : U(\mathfrak{p}) \to U(\mathfrak{a})$ be the algebra homomorphism induced by the natural projection $\mathfrak{p} \to \mathfrak{a}$. The restriction

$$\bar{\xi} : W(\pi) \to U(\mathfrak{a})$$

of ξ to $W(\pi)$ is called the Miura transform. By [4], Theorem 11.4, $\bar{\xi}$ is an injective algebra homomorphism, allowing us to view $W(\pi)$ as a subalgebra of $U(\mathfrak{a})$.

Let M_k be a module for the Lie algebra $\mathfrak{gl}_{q_k}$, $k = 1, \dots, l$. Then using the Miura transform $\bar{\xi}$ the vector space

$$M_1 \otimes \ldots \otimes M_l$$

can be equipped with a module structure over the algebra $W(\pi)$.

For each $i = 1, \dots, n$, let $\mathcal{C}_i$ be an admissible set of relations for $\mathfrak{gl}_{q_i}$ and $[L^{(i)}]$ be a tableau such that $\mathcal{C}_i$ is the maximal set of relations satisfied by $[L^{(i)}]$. Then $V_{\mathcal{C}_1}([L^{(1)}]) \otimes \ldots \otimes V_{\mathcal{C}_l}([L^{(l)}])$ is a $\mathfrak{gl}_{q_1} \oplus \ldots \oplus \mathfrak{gl}_{q_l}$-module and thus a $W(\pi)$-module.

In the following we describe a family of highest weight modules which can be realized as relation modules $V_{\mathcal{C}}([l])$ for some admissible sets of relations $\mathcal{C}$.

Let $\lambda(u) = (\lambda_1(u), \dots, \lambda_n(u))$, where $\lambda_i(u) = \prod_{s=1}^{l} (u + \lambda_s^{(i)})$, $i = 1, \dots, n$. We identify $\lambda^{(i)}$ with the tuple $(\lambda_1^{(i)}, \dots, \lambda_l^{(i)})$.

Denote $[L]_\lambda = ([l^{(1)}], \dots, [l^{(l)}])$, where each $[l^{(k)}]$ is the tableau such that $l_{ij}^{(k)} = l_{nj}^{(k)} = \lambda_j^{(k)} - j + 1$, $i = 1, \dots, n$, $j = 1, \dots, i$, $k = 1, \dots, l$.

Definition 4.5. We will say that $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ is *good* if it satisfies the conditions: $\mu_i - \mu_j \notin \mathbb{Z}$ or $\mu_i - \mu_j > i - j$ for any $1 \leq i < j \leq n$. We say that $\lambda(u)$ is good if $\lambda^{(k)}$ is good for all $k = 1, \dots, l$. In this case $[L]_\lambda$ is also called good.

Assume $\lambda(u)$ is good. For each $k = 1, \dots, l$ let $\mathcal{C}_k$ be the maximal set of relations satisfied by $[l^{(k)}]$. Then $V_{\mathcal{C}_k}([l^{(k)}])$ is simple highest weight $\mathfrak{gl}_{q_k}$-module with highest weight $\lambda^{(k)} = (\lambda_{n-q_k+1}^{(k)}, \cdots, \lambda_n^{(k)})$ ([12] Proposition 5.7).

The following proposition follows from Theorem 3.16 and Theorem 4.4.

Proposition 4.6. *Let $\lambda(u)$ be good, $[L]_\lambda = ([l^{(1)}], \dots, [l^{(l)}])$, and $\mathcal{C}$ be the maximal set of relations satisfied by $[L]_\lambda$. If for any $[T] \in \mathcal{B}_{\mathcal{C}}([l])$ and all i, j, $r \neq s$, we have $T_{ki}^{(r)} \neq T_{kj}^{(s)}$ for $k = 1, \dots, n-1$, then $\mathcal{C}$ is admissible and $L(\lambda(u)) \simeq V_{\mathcal{C}}([l])$. Moreover, the explicit basis of $L(\lambda(u))$ is $\mathcal{B}_{\mathcal{C}}([l])$.*

In particular, if $\lambda(u)$ is a good dominant integral highest weight, then $L(\lambda(u))$ is a finite dimensional relation module. We note that not every finite dimensional $W(\pi)$-module is a relation module. For instance, if $t = 2$, $\lambda^{(1)} = \lambda^{(2)} = (5, 1)$, then we have some equal entries in the first row. Hence, the corresponding finite dimensional module is not a relation module.

5. Tensor product of highest weight relation modules

If the pyramid π has parameters $p_1 = \ldots = p_n = p$ then $W(\pi)$ is a finitely generated Yangian of level p. In this section we consider highest weight relation modules for the Yangians. The *Yangian* $\mathrm{Y}(n) := \mathrm{Y}(\mathfrak{gl}_n)$, is the complex associative algebra with the generators $t_{ij}^{(1)}, t_{ij}^{(2)}, \dots$ where $1 \leq i, j \leq n$, and the defining relations

$$[t_{ij}(u),\ t_{kl}(v)] = \frac{1}{u - v}(t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u)) \tag{15}$$

where

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)}u^{-1} + t_{ij}^{(2)}u^{-2} + \cdots \in \mathrm{Y}(n)[[u^{-1}]]$$

and u is a formal variable.

$\mathrm{Y}(n)$ is a Hopf algebra with the coproduct $\Delta : \mathrm{Y}(n) \to \mathrm{Y}(n) \otimes \mathrm{Y}(n)$ defined by

$$\Delta(t_{ij}(u)) = \sum_{a=1}^{n} t_{ia}(u) \otimes t_{aj}(u) \tag{16}$$

Given sequences $a_1, \ldots, a_r$ and $b_1, \ldots, b_r$ of elements of $\{1, \ldots, n\}$ the corresponding *quantum minor* of the matrix $[t_{ij}(u)]$ is defined by the following equivalent formulas:

$$\begin{aligned} t^{a_1\cdots a_r}_{b_1\cdots b_r}(u) &= \sum_{\sigma\in\mathfrak{S}_r} \operatorname{sgn}\sigma \cdot t_{a_{\sigma(1)}b_1}(u)\cdots t_{a_{\sigma(r)}b_r}(u-r+1) \\ &= \sum_{\sigma\in\mathfrak{S}_r} \operatorname{sgn}\sigma \cdot t_{a_1b_{\sigma(1)}}(u-r+1)\cdots t_{a_rb_{\sigma(r)}}(u). \end{aligned}$$

The series $t^{a\cdots a_r}_{b_1\cdots b_r}(u)$ is skew symmetric under permutations of the indices a_i, or b_i.

Proposition 5.1 *([22] Proposition 1.11). The images of the quantum minors under the coproduct are given by*

$$\Delta(t^{a_1\cdots a_r}_{b_1\cdots b_r}(u)) = \sum_{c_1<\cdots<c_r} t^{a_1\cdots a_r}_{c_1\cdots c_r}(u) \otimes t^{c_1\cdots c_r}_{b_1\cdots b_r}(u) \ , \tag{17}$$

summed over all subsets of indices $\{c_1, \ldots, c_r\}$ *from* $\{1, \ldots, n\}$.

For $m \geq 1$ introduce the series $a_m(u), b_m(u)$ and $c_m(u)$ by

$$a_m(u) = t^{1\cdots m}_{1\cdots m}(u),\ b_m(u) = t^{1\cdots m}_{1\cdots m-1,m+1}(u),\ c_m(u) = t^{1\cdots m-1,m+1}_{1\cdots m}(u).$$

The coefficients of these series generate the algebra $\mathrm{Y}(n)$, they are called the *Drinfeld generators*.

Definition 5.2. Let V be a $Y(n)$-module. A nonzero $v \in V$ is called singular if:

(i) v is a weight vector (with respect to all $t_{ii}(u)$);
(ii) $b_m(u)v = 0$ for any $m \geq 1$.

Let $E_{ij}, i, j = 1, \ldots, n$ denote the standard basis elements of the Lie algebra $\mathfrak{gl}_n$. We have a natural embedding

$$U(\mathfrak{gl}_n) \to \mathrm{Y}(n)\ ,\ E_{ij} \mapsto t^{(1)}_{ij}.$$

Moreover, for any $a \in \mathbb{C}$ the mapping

$$\varphi_a : t_{ij}(u) \mapsto \delta_{ij} + \frac{E_{ij}}{u-a} \tag{18}$$

defines an algebra epimorphism from $\mathrm{Y}(n)$ to the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$ so that any $\mathfrak{gl}_n$-module can be extended to a $\mathrm{Y}(n)$-module via (18). Consider the simple $\mathfrak{gl}_n$-module $L(\lambda)$ with highest weight $\lambda = (\lambda_1, \ \ldots, \ \lambda_n)$ with respect to the upper triangular Borel subalgebra generated by E_{ij}, $i < j$. The corresponding $\mathrm{Y}(n)$-module is denoted by $L_a(\lambda)$, and we call it the *evaluation module*. We keep the notation $L(\lambda)$ for the module $L_a(\lambda)$ with $a = 0$. The coproduct Δ defined by (16) allows one to consider the tensor products $L_{a_1}(\lambda^{(1)}) \otimes L_{a_2}(\lambda^{(2)}) \otimes \cdots \otimes L_{a_l}(\lambda^{(l)})$ as $\mathrm{Y}(n)$-modules.

Let L be a $\mathfrak{gl}_n$-module with finite dimensional weight subspaces,

$$L = \bigoplus_{\mu} L_\mu, \ \dim L_\mu < \infty.$$

Then we define the restricted dual to L by

$$L^* = \bigoplus_{\mu} L^*_\mu.$$

The elements of L^* are finite linear combinations of the vectors dual to the vectors of any weight basis of L. The space L^* can be equipped with a $\mathfrak{gl}_n$-module structure by

$$(E_{ij} f)(v) = f(-E_{n-i+1,n-j+1} v) \ , \ f \in L^*, \ v \in L.$$

Denote by ω the anti-automorphism of the algebra $\mathrm{Y}(n)$, defined by

$$\omega : t_{ij}(u) \mapsto t_{n-i+1,n-j+1}(-u).$$

Suppose now that the $\mathfrak{gl}_n$ action on L is obtained by the restriction of an action of $\mathrm{Y}(n)$. Then the $\mathfrak{gl}_n$-module structure on L^* can be regarded as the restriction of the $\mathrm{Y}(n)$-module structure defined by

$$(xf)(v) = f(\omega(x)v) \ , \ \text{for } x \in \mathrm{Y}(n) \text{ and } \ f \in L^*, \ v \in L.$$

For any $\lambda = (\lambda_1, \ \ldots \ , \ \lambda_n)$ we set $\tilde{\lambda} = (-\lambda_n, \ \ldots, \ -\lambda_1)$. Then we have

Proposition 5.3. *[20] Let L be the tensor product $L(\lambda^{(1)}) \otimes L(\lambda^{(2)}) \otimes \cdots \otimes L(\lambda^{(l)})$. Then the $\mathrm{Y}(n)$-module L^* is isomorphic to the tensor product module*

$$L(\tilde{\lambda}^{(1)}) \otimes L(\tilde{\lambda}^{(2)}) \otimes \ldots \otimes L(\tilde{\lambda}^{(l)}) \ .$$

Proposition 5.4. *Suppose that the $\mathrm{Y}(n)$-module*

$$L(\lambda^{(1)}) \otimes L(\lambda^{(2)}) \otimes \cdots \otimes L(\lambda^{(l)}) \tag{19}$$

is simple. Then any permutation of the tensor factors gives an isomorphic $\mathrm{Y}(n)$-module.

Proof. Denote the tensor product by L. Note that L is a $\mathrm{Y}(n)$-module with highest weight $(\lambda_1(u), \ \ldots, \ \lambda_n(u))$. Consider a module L' obtained by a certain permutation of the tensor factors in (19). The tensor product ζ' of the highest weight vectors of the modules $L(\lambda^{(i)})$ is a singular vector in L' whose weight is the same as the highest weight of L. This implies that ζ' generates a highest weight submodule in L' such that its simple quotient is isomorphic to L. However, L and L' have the same formal character as $\mathfrak{gl}_n$-modules which implies that L and L' are isomorphic. □

Let $\lambda(u) = (\lambda_1(u), \ldots, \lambda_n(u))$, where $\lambda_i(u) = \prod_{s=1}^{t} (u + \lambda_i^{(s)})$, $i = 1, \ldots, n$. Set $\lambda^{(i)} = (\lambda_1^{(i)}, \ldots, \lambda_n^{(i)})$ and consider a simple highest weight $\mathfrak{gl}_n$-module $L(\lambda^{(i)})$.

6. Simplicity of tensor product

In this section we discuss the simplicity of tensor product of highest weight relation modules for the Yangians.

Definition 6.1. Let $\lambda^{(i)} = (\lambda_1^{(i)}, \ \ldots, \ \lambda_n^{(i)})$, $i = 1, \ldots, l$ be n-tuples of complex numbers. We will call the set $\{\lambda^{(1)}, \ \ldots, \ \lambda^{(l)}\}$ *generic* if for each pair of indices $1 \le i < j \le l$ we have $\lambda_s^{(i)} - \lambda_t^{(j)} \notin \mathbb{Z}$, $s, t = 1, \ldots, n$. We will say that $\lambda(u)$ is *integral* if it is not generic.

Denote by $L(\lambda^{(i)})$ the simple $\mathfrak{gl}_n$-module with highest weight $\lambda^{(i)}$, $i = 1, \ldots, l$. Our first result is the simplicity of tensor product $L(\lambda^{(1)}) \otimes L(\lambda^{(2)}) \otimes \cdots \otimes L(\lambda^{(l)})$ in the generic case. Recall that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is good if $\lambda_i - \lambda_j \notin \mathbb{Z}$ or $\lambda_i - \lambda_j > i - j$ for any $1 \le i < j \le n$.

Theorem 6.2. *Let $\{\lambda^{(1)}, \ \ldots, \ \lambda^{(l)}\}$ be a generic set with good $\lambda^{(i)}$, $i = 1, \ldots, l$. Then the $\mathrm{Y}(n)$-module*

$$L(\lambda^{(1)}) \otimes L(\lambda^{(2)}) \otimes \cdots \otimes L(\lambda^{(l)})$$

is simple.

We will establish sufficient conditions of simplicity of the $\mathrm{Y}(n)$-module $L(\lambda) \otimes L(\mu)$ with good integral λ and μ. This extends the result of [21] to some infinite dimensional highest weight modules, though unlike in [21] we can not show the necessity of these conditions for the simplicity of the tensor product, neither can we prove it for any number of tensor factors.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ be n-tuples of complex numbers and consider simple highest weight $\mathfrak{gl}_n$-modules $L(\lambda)$ and $L(\mu)$. Set also

$$l_i := \lambda_i - i + 1, \ m_i := \mu_i - i + 1, \ i = 1, \cdots, n.$$

For any pair of indices $i < j$ the set $\{l_j, l_{j-1}, \ldots, l_i\}$ is the union of pairwise disjoint sets $\{l_{i_{11}}, \ldots, l_{i_{1m_1}}\}, \{l_{i_{21}}, \ldots, l_{i_{2m_2}}\}, \ldots, \{l_{i_{t1}}, \ldots, l_{i_{tm_t}}\}$ such that $l_{i_{ra}} - l_{i_{sb}} \notin \mathbb{Z}$ for any $r \neq s$, $l_{i_{ra}} - l_{i_{rb}} \in \mathbb{Z}$ and $i_{ra} > i_{rb}$ for $a < b$.

We shall denote

$$\lfloor l_{i_{r1}}, l_{i_{rm_r}} \rfloor^- := \begin{cases} \{l_{i_{r1}}, l_{i_{r1}} + 1, \ldots, l_{i_{rm_r}}\} \setminus \{l_{i_{r1}}, l_{i_{r2}}, \ldots, l_{i_{rm_r}}\}, i_{rm_1} = j \\ \{l_{i_{rm_1}} + z \mid z \in \mathbb{Z}_{\le 0}\} \setminus \{l_{i_{r1}}, l_{i_{r2}}, \ldots, l_{i_{rm_r}}\}, i_{rm_1} \neq j \end{cases}$$

and

$$\lfloor l_{i_{r1}}, l_{i_{rm_r}} \rfloor^+ := \begin{cases} \{l_{i_{r1}}, l_{i_{r1}} + 1, \ldots, l_{i_{rm_r}}\} \setminus \{l_{i_{r1}}, l_{i_{r2}}, \ldots, l_{i_{rm_r}}\}, i_{rm_1} = i \\ \{l_{i_{rm_r}} + z \mid z \in \mathbb{Z}_{\ge 0}\} \setminus \{l_{i_{r1}}, l_{i_{r2}}, \ldots, l_{i_{rm_r}}\}, i_{rm_r} \neq i \end{cases}$$

$$\langle l_j, l_i \rangle^- = \bigcup_{r=1}^{t} \lfloor l_{i_{r1}}, l_{i_{rm_r}} \rfloor^-$$

$$\langle l_j, l_i \rangle^+ = \bigcup_{r=1}^{t} \lfloor l_{i_{r1}}, l_{i_{rm_r}} \rfloor^+$$

Theorem 6.3. *Let λ and μ be good integral $\mathfrak{gl}_n$-highest weights. Suppose that for each pair of indices $1 \le i < j \le n$ we have*

$$m_j \notin \langle l_j, l_i \rangle^-, m_i \notin \langle l_j, l_i \rangle^+ \ \text{or} \ l_j \notin \langle m_j, m_i \rangle^-, l_i \notin \langle m_j, m_i \rangle^+. \tag{20}$$

Then the $\mathrm{Y}(n)$-module $L(\lambda) \otimes L(\mu)$ is simple.

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In the following we prove Theorem 6.2 and Theorem 6.3. The proofs closely follow the proof of Theorem 3.1 in [21] for finite dimensional modules. We include the details for completeness.

6.1. Integral case

We start with the proof of Theorem 6.3. Assume that $L(\lambda)\otimes L(\mu)$ is not simple as $\mathrm{Y}(n)$-module. Let ξ and ξ' denote the highest weight vectors of the $\mathfrak{gl}_n$-modules $L(\lambda)$ and $L(\mu)$, respectively. Consider a nonzero $\mathrm{Y}(n)$-submodule N of $L(\lambda)\otimes L(\mu)$. Then N must contain a nonzero singular vector ζ. We will show by induction on n that $\zeta\in\mathbb{C}\cdot\xi\otimes\xi'$. Since λ is good then $L(\lambda)$ is a relation $\mathfrak{gl}_n$-module by [12], Proposition 5.7. We denote by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{gl}_n$ consisting of diagonal matrices. We identify an element $w\in\mathfrak{h}^*$ with the n-tuple consisting of values of w on the standard basis of $\mathfrak{h}$.

Consider the Gelfand-Tsetlin basis $\mathcal{B}(\lambda)$ of $L(\lambda)$. The tableau corresponding to the element ξ is of the form $[R]=(r_{ij})$ with $r_{ij}=\lambda_j-j+1$ for all $i=1,\dots,n$, $j=1,\dots,i$.

The element ζ can be written uniquely as a finite sum:

$$\zeta=\sum_{[L]\in\mathcal{B}(\lambda)}[L]\otimes m_L, \tag{21}$$

where $m_L\in L(\mu)$.

Viewing $L(\lambda)\otimes L(\mu)$ as a $\mathfrak{gl}_n$-module we immediately see that ζ is a weight $\mathfrak{gl}_n$-singular vector, that is $E_{ij}\zeta=0$ for all $i<j$. Moreover, all elements $[L]\otimes m_L$ in (21) have the same $\mathfrak{gl}_n$-weight.

If $[L]=(l_{ij})$, then the weight $w(L)$ of $[L]$ is a sequence

$$\left\{\sum_{i=1}^{k}l_{ki}-\sum_{i=1}^{k-1}l_{k-1,i}+k-1,\quad k=1,\dots,n\right\}.$$

Given two weights $w,w'\in\mathfrak{h}^*$, we shall write $w\preceq w'$ if $w'-w$ is a $\mathrm{Z}_{\geq 0}$-linear combination of the simple roots of $\mathfrak{gl}_n$. This defines a partial order on the set of weights of $\mathfrak{gl}_n$.

Denote by $supp\,\zeta$ the set of tableaux $[L]\in\mathcal{B}(\lambda)$ for which $m_L\neq 0$ in (21). Let $[L^0]$ be a minimal element in $supp\,\zeta$ with respect to the partial ordering on the weights $w(L)$'s.

Since $t^{1\cdots m}_{1\ldots m-1,m+1}(u)\zeta=0$, we have

$$\sum_{c_1<\cdots<c_m}\sum_{L}t^{1\cdots m}_{c_1\ldots c_m}(u)[L]\otimes t^{1\cdots m}_{1\ldots m-1,m+1}(u)m_L= \tag{22}$$

$$=t^{1\cdots m}_{1\ldots m}(u)[L^0]\otimes t^{1\cdots m}_{1\ldots m-1,m+1}(u)m_{L^0}+\ldots=0.$$

Hence, $t^{1\cdots m}_{1\ldots m-1,m+1}(u)m_{L^0}=0$ for all m. Thus m_{L^0} is a highest vector of $L(\mu)$ and we conclude that m_{L^0} is a scalar multiple of ξ'. This immediately implies that L^0 is determined uniquely. For any $L\in supp\,\zeta$ we have $w(L)\succeq w(L^0)$. If $[L]=(l_{ij})\in supp\,\xi$ and $[L^0]=(l^0_{ij})$ then we have $l_{ij}-l^0_{ij}\in\mathrm{Z}_{\geq 0}$ for $1\leq j\leq i\leq n-1$.

Permuting $L(\lambda)$ and $L(\mu)$ if necessary and applying Proposition 5.4, we assume that $m_n\notin\langle l_n,l_1\rangle^-, m_1\notin\langle l_n,l_1\rangle^+$.

Lemma 6.4. *The $(n-1)$-th row of $[L^0]$ is $(l^0_1,\ \dots,\ l^0_{n-1})$, where $l^0_i=\lambda_i-i+1$.*

Proof. Suppose the contrary. Then for each j with $l^0_{n-1,j}\neq l^0_j$ there exists a minimal $r(j)$ such that $[L'(r(j))]=[L^0+\delta_{n-1,j_1}+\cdots+\delta_{n-r(j),j_{r(j)}}]$ is a Gelfand-Tsetlin tableau of $L(\lambda)$ with $j_1=j$. Choose j such that $r(j)$ is minimal and denote it by r. Also set $L'=L'(r)$.

Since ζ is a singular vector, we have

$$t^{1,\cdots,n-r}_{1,\cdots,n-r-1,n}(u)\zeta = 0$$

and hence, by (17) we have

$$\sum_{c_1<\cdots<c_{n-r}} \sum_{L} t^{1\cdots n-r}_{c_1\ldots c_{n-r}}(u)[L] \otimes t^{c_1\cdots c_{n-r}}_{1\cdots n-r-1,n}(u) m_L = 0. \tag{23}$$

Following the proof of Lemma 3.5 in [21] we look at the coefficient of $[L'] \otimes m_{L^0}$ in the expansion of the left hand side. It comes from the following two summands in (23):

$$t^{1\cdots n-r}_{1\ldots n-r-1,n}(u)[L^0] \otimes t^{1\ldots n-r-1,n}_{1\cdots n-r-1,n}(u) m_{L^0} \tag{24}$$

and

$$t^{1\cdots n-r}_{1\ldots n-r}(u)[L'] \otimes t^{1\ldots n-r}_{1\cdots n-r-1,n}(u) m_{L'}, \tag{25}$$

if $[L'] \in supp\, \zeta$.

Consider (24) first. Due to the minimality of r we have $E_{in}[L^0] = 0$ for $n-r < i \leq n-1$. Hence

$$E_{n-r,n}[L^0] = (-1)^{r-1} E_{n-1,n} E_{n-2,n-1} \cdots E_{n-r,n-r+1}[L^0].$$

Therefore the expansion of $E_{n-r,n}[L^0]$ contains a term $a[L']$ with $a \neq 0$.

It will be convenient to use polynomial quantum minors defined by:

$$T^{j_1\cdots j_m}_{i_1\cdots i_m}(u) = u(u-1)\cdots(u-m+1) t^{j_1\cdots j_m}_{i_1\cdots i_m}(u).$$

Then the coefficient of $[L']$ in $T^{1\cdots n-r}_{1\ldots n-r-1,n}(u)[L^0]$ equals

$$a(u + l^0_{n-r,1}) \cdots \bigwedge_{i_r} \cdots (u + l^0_{n-r,n-r})\ .$$

On the other hand,

$$T^{1\ldots n-r-1,n}_{1\cdots n-r-1,n}(u) m_{L^0} = (u+m_1)\cdots(u+m_{n-r-1})(u+m_n+r) m_{L^0}.$$

Hence,

$$\begin{gathered} T^{1\cdots n-r}_{1\ldots n-r-1,n}(u)[L^0] \otimes T^{1\ldots n-r-1,n}_{1\cdots n-r-1,n}(u) m_{L^0} = \\ a(u + l^0_{n-r,1}) \cdots \bigwedge_{i_r} \cdots (u + l^0_{n-r,n-r})(u+m_1)\cdots(u+m_{n-r-1})(u+m_n+r)([L'] \otimes m_{L^0}). \end{gathered} \tag{26}$$

Consider now (25). We have

$$T^{1\cdots n-r}_{1\cdots n-r}(u)[L'] = (u + l^0_{n-r,1}) \cdots (u + l^0_{n-r,i_r} + 1) \cdots (u + l^0_{n-r,n-r})[L'].$$

Let $[L_\mu]$ be the highest weight tableau of $L(\mu)$ in the Gelfand-Tsetlin realization of $L(\mu)$. Then m_{L^0} is a multiple of $[L_\mu]$. Comparing the weights of $[L^0] \otimes m_{L^0}$ and $[L'] \otimes m_{L'}$ we see that $m_{L'}$ is a multiple of the tableau $[L_{\mu,r}] := [L_\mu - \delta_{n-1,j_1} - \cdots - \delta_{n-r,j_r}]$. Since $(n-r,j) \geq (n-r-1,j)$ for $j = 1,\ldots,n-r-1$ and

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the $(n-r-1)$-th row of each patter is $(\mu_1, \ldots, \mu_{n-r-1})$, we have that $j_r = n-r$ and the $(n-r)$-th row of $[L_{\mu,r}]$ is $(m_1, \ldots, m_{n-r-1}, m_{n-r}-1)$.

Therefore $E_{n-r,n}m_{L'}$ is a scalar multiple of m_{L^0}. If $[L']$ is not in $supp\,\zeta$ then $m_{L'} = 0$. In both cases we have that $E_{n-r,n}m_{L'} = bm_{L^0}$ for some constant b, and so

$$T^{1\cdots n-r}_{1\ldots n-r-1,n}(u)m_{L'} = b\cdot(u+m_1)\cdots(u+m_{n-r-1})m_{L^0}.$$

We have

$$T^{1\cdots n-r}_{1\cdots n-r}(u)[L'] \otimes T^{1\cdots n-r}_{1\ldots n-r-1,n}(u)m_{L'} =$$
$$b\cdot(u+m_1)\cdots(u+m_{n-r-1})(u+l^0_{n-r,1})\cdots(u+l^0_{n-r,i_r}+1)\cdots(u+l^0_{n-r,n-r})([L']\otimes m_{L'}).$$

Combining these results we obtain

$$a(u+m_n+r) + b\cdot(u+l^0_{n-r,i_r}+1) = 0.$$

In particular, we have $b = -a \neq 0$ and $m_n = l^0_{n-r,i_r} - r + 1$. By the minimality of r we have $l^0_{n-s,i_s} = l^0_{n-s+1,i_{s-1}} + 1$ and $m_n = l^0_{n-1,i}$.

By the definition of $[L^0]$ we have $l_i - l^0_{n-1,i} \geq 0$ and $l^0_{n-1,i} - l_k \geq 0$, where k is the minimal index such that $k > i$ and $\lambda_i - \lambda_k \in \mathbb{Z}_{\geq 0}$. This implies $l_i - m_n \in Z_{>0}$ and $m_n - l_k \in Z_{>0}$. Thus $m_n \in \langle l_n, l_1\rangle^-$, which is a contradiction. This completes the proof of the lemma. □

Lemma 6.4 implies that all tableaux $[L] \in supp\,\zeta$ belong to the $\mathfrak{gl}_{n-1}$-submodule $L(\lambda_-)$ of $L(\lambda)$ generated by ξ. Note that the module $L(\lambda_-)$ is simple with the highest weight $\lambda_- = (\lambda_1, \ldots, \lambda_{n-1})$ by [12], Proposition 5.3. We have $E_{nn}[L] = \lambda_n[L]$ for all $[L] \in supp\,\zeta$. Moreover,

$$w(L) + w(m_L) = w(L^0) + \mu.$$

Hence, $E_{nn}m_L = \mu_n m_L$ and the $(n-1)$-th row of each tableau m_L coincides with $\mu := (\mu_1, \ldots, \mu_{n-1})$. We see that each m_L belongs to the $\mathfrak{gl}_{n-1}$-submodule $L(\mu_-)$ generated by ξ', which is simple highest weight $\mathfrak{gl}_{n-1}$-module. Therefore, $\zeta \in L(\lambda_-) \otimes L(\mu_-)$.

The $\mathrm{Y}(n-1)$-module structure on $L(\lambda_-) \otimes L(\mu_-)$ coincides with the one obtained by restriction from $\mathrm{Y}(n)$ to the subalgebra generated by the $t_{ij}(u)$ with $1 \leq i, j \leq n-1$ by (16) and (15). The vector ζ is singular for $\mathrm{Y}(n-1)$ (it is annihilated by $b_1(u), \ldots, b_{n-2}(u)$). By the assumption of the theorem, for each pair (i, j) such that $1 \leq i < j \leq n-1$ the condition (20) is satisfied. Therefore $L(\lambda_-) \otimes L(\mu_-)$ is simple $\mathrm{Y}(n-1)$-module by the induction hypothesis. Hence, ζ is a scalar multiple of $\xi \otimes \xi'$.

It remains to show that $L(\lambda) \otimes L(\mu)$ is generated by $\xi \otimes \xi'$. Suppose that $\xi \otimes \xi'$ generates a proper submodule N in $\mathcal{L} = L(\lambda) \otimes L(\mu)$. Set

$$\tilde{N} = \{f \in L^* \mid f(v) = 0 \text{ for all } v \in N\}.$$

Then $\tilde{N}$ is a nonzero (since $N \neq \mathcal{L}$) submodule of $\mathcal{L}^*$. By Proposition 5.3 and above argument, $\tilde{N}$ contains a singular vector ζ. As it was shown above ζ is a scalar multiple of $\xi^* \otimes \xi'^*$ of the highest weight vectors of $L(\lambda)^*$ and $L(\mu)^*$ respectively. On the other hand, $\xi^* \otimes \xi'^* \notin \tilde{N}$ giving a contradiction. Hence, $\xi \otimes \xi'$ generates $L(\lambda) \otimes L(\mu)$. Since all singular elements of the highest weight $\mathrm{Y}(n)$-module $L(\lambda) \otimes L(\mu)$ belong to $\mathbb{C}\cdot(\xi \otimes \xi')$, the module $L(\lambda) \otimes L(\mu)$ is simple. This completes the proof of Theorem 6.3.

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6.2. Generic highest weight modules

Now we prove Theorem 6.2 by induction on l. The case $l = 2$ is a consequence of Theorem 6.3, since the conditions of Theorem 6.3 trivially follow from the conditions of Theorem 6.2.

We assume now that $l > 2$ and denote by K the tensor product $L(\lambda^{(2)}) \otimes \cdots \otimes L(\lambda^{(l)})$. Suppose that K is simple highest weight $\mathrm{Y}(n)$-module. We will show that $\mathcal{L} = L(\lambda^{(1)}) \otimes K$ is simple. Then Theorem 6.2 follows by induction.

The proof of simplicity of $\mathcal{L}$ is similar to the proof of Theorem 6.3. Suppose N is a nonzero $\mathrm{Y}(n)$-submodule of $\mathcal{L}$. Then N must contain a singular vector ζ:

$$\zeta = \sum_{[L]} [L] \otimes m_L, \tag{27}$$

summed over finitely many Gelfand-Tsetlin tableaux $[L]$ of $L(\lambda^{(1)})$, where $m_L \in K$.

Following the proof of Theorem 6.3 we choose a minimal element $[L^0]$ of the set of tableaux $[L]$ occurring in (27) with respect to the partial ordering on the weights $w(\Lambda)$. As before $[L^0]$ is determined uniquely, m_{L^0} is a scalar multiple of ξ' and for any $[L]$ that occurs in (27) $w(L) \succeq w(L^0)$. Moreover, for each entry l_{ij} of $[L]$ occurring in (27) we have $l_{ij} - l^0_{ij} \in \mathrm{Z}_{\geq 0}$, for $1 \leq j \leq i \leq n-1$.

We also have an analog of Lemma 6.4

Lemma 6.5. *The $(n-1)$-th row of $[L^0]$ is $(l^0_1, \ \dots, \ l^0_{n-1})$, where $l^0_i = \lambda_i - i + 1$.*

Proof. Choose $[L']$ as in the proof of Lemma 6.4. Since ζ is a singular vector, we have

$$0 = T^{1,\cdots,n-r}_{1,\cdots,n-r-1,n}(u)\zeta =$$
$$= \sum_{c_1<\cdots<c_{n-r}} \sum_{L} T^{1\cdots n-r}_{c_1\dots c_{n-r}}(u)[L] \otimes T^{c_1\cdots c_{n-r}}_{1\cdots n-r-1,n}(u) m_L.$$

The coefficient of $[L'] \otimes m_{L^0}$ in the expansion of the left hand side of (21) is the following

$$(u + l^0_{n-r,1}) \cdots \bigwedge_{i_r} \cdots (u + l^0_{n-r,n-r}) \prod_{i=2}^{k} (u + m^{(i)}_1) \cdots (u + m^{(i)}_{n-r-1}) \times$$
$$(a \prod_{i=2}^{k} (u + m^{(i)}_n + r) + g(u)(u + l^0_{n-r,i_r} + 1)) = 0,$$

where $a \neq 0$ and $g(u)$ is a certain polynomial in u.

Put $u = -l^0_{n-r,i_r} - 1$. Since a is nonzero, we get $m^{(j)}_n = l^0_{n-r,i_r} - r + 1 = l^0_{n-1,i}$ for some $2 \leq j \leq k$. Thus $\lambda^{(j)}_n - \lambda^{(1)}_i \in \mathbb{Z}$ which is a contradiction. $\square$

It remains to show that $\xi \otimes \xi'$ generates $\mathcal{L}$. The argument is the same as in the proof of Theorem 6.3. This completes the proof of Theorem 6.2.

Remark 6.6. We can combine Theorem 6.2 and Theorem 6.3 and obtain simplicity of the tensor product

$$L(\lambda) \otimes L(\mu) \otimes L(\nu_1) \otimes \dots \otimes L(\nu_s),$$

where λ and μ satisfy the conditions of Theorem 6.3, $\nu_1, \dots, \nu_s$ satisfy the conditions of Theorem 6.2 and $\nu^j_i - \lambda_k \notin \mathbb{Z}$, $\nu^j_i - \mu_k \notin \mathbb{Z}$ for all possible i, j, k.

Please cite this article in press as: V. Futorny et al., Gelfand-Tsetlin representations of finite W-algebras, J. Pure Appl. Algebra (2019), https://doi.org/10.1016/j.jpaa.2019.106226

7. Proof of Theorem 3.16

Let $\mathcal{C}$ be a pre-admissible set of relations and $[L]$ a tableau satisfying $\mathcal{C}$. Assume that $\mathcal{C}$ is a union of indecomposable sets from $\mathfrak{F}$. We will show that for any defining relation $g=0$ in $W(\pi)$ and $[l]\in\mathcal{B}_{\mathcal{C}}([L])$ holds $g[l]=0$. Recall that the action of generators of $W(\pi)$ in $V_{\mathcal{C}}([L])$ is given by (12), where $d_r^{(t)}[l]=d_r^{(t)}(l)[l]$ and the action of $d_r'^{(t)}$ on $[l]$ is a multiplication by a scalar which is polynomial in l. Also recall that the vector $[l\pm\delta_{ri}^{(k)}]$ is zero if it does not satisfy $\mathcal{C}$.

Set

$$e_{r,k,i}^{(t)}(l)=\begin{cases}-\dfrac{\prod\limits_{j,t}(l_{r+1,j}^{(t)}-l_{r,i}^{(k)})}{\prod\limits_{(j,t)\neq(i,k)}(l_{r,j}^{(t)}-l_{r,i}^{(k)})}b_{r,k,i}^{(t)}(l), & \text{if } [l]\in\mathcal{B}_{\mathcal{C}}([L])\\ 0, & \text{if } [l]\notin\mathcal{B}_{\mathcal{C}}([L]),\end{cases}$$

$$f_{r,k,i}^{(t)}(l)=\begin{cases}\dfrac{\prod\limits_{j,t}(l_{r-1,j}^{(t)}-l_{r,i}^{(k)})}{\prod\limits_{(j,t)\neq(i,k)}(l_{r,j}^{(t)}-l_{r,i}^{(k)})}c_{r,k,i}^{(t)}(l), & \text{if } [l]\in\mathcal{B}_{\mathcal{C}}([L])\\ 0, & \text{if } [l]\notin\mathcal{B}_{\mathcal{C}}([L]),\end{cases}$$

$$\Phi(l,z_1,\dots,z_m)=\begin{cases}1, & \text{if } [l+z_1+\ldots+z_t]\in\mathcal{B}_{\mathcal{C}}([L]) \text{ for any } 1\le t\le m\\ 0, & \text{otherwise.}\end{cases}$$

Note that $e_{r,k,i}^{(t)}(l)$ and $f_{r,k,i}^{(t)}(l)$ are rational functions in the components of $[l]$ and $e_{r,k,i}^{(p_{r+1}-p_r+1)}(l)=-\dfrac{\prod\limits_{j,t}(l_{r+1,j}^{(t)}-l_{r,i}^{(k)})}{\prod\limits_{(j,t)\neq(i,k)}(l_{r,j}^{(t)}-l_{r,i}^{(k)})}$, $f_{r,k,i}^{(1)}(l)=\dfrac{\prod\limits_{j,t}(l_{r-1,j}^{(t)}-l_{r,i}^{(k)})}{\prod\limits_{(j,t)\neq(i,k)}(l_{r,j}^{(t)}-l_{r,i}^{(k)})}$.

Now the action of generators can be written as follows:

$$d_r^{(t)}[l]=d_r^{(t)}(l)[l], \tag{28}$$

$$e_r^{(t)}[l]=\sum_{i,k}\Phi(l,\delta_{ri}^{(k)})e_{r,k,i}^{(t)}(l)\,[l+\delta_{ri}^{(k)}], \tag{29}$$

$$f_r^{(t)}[l]=\sum_{i,k}\Phi(l,-\delta_{ri}^{(k)})f_{r,k,i}^{(t)}(l)\,[l-\delta_{ri}^{(k)}]. \tag{30}$$

We proceed with the verification of defining relations.

1.

$$[d_i^{(r)},d_j^{(s)}][l]=0.$$

The statement is obvious.

2.

$$[e_i^{(r)},f_j^{(s)}][l]=-\delta_{ij}\sum_{t=0}^{r+s-1}d_i'^{(t)}\,d_{i+1}^{(r+s-t-1)}[l]. \tag{31}$$

The tableaux that appear in the equation (31) are of the form $[l+\delta_{i,u_1}^{(k_1)}-\delta_{j,u_2}^{(k_2)}]$. Assume $[l+\delta_{i,u_1}^{(k_1)}-\delta_{j,u_2}^{(k_2)}]\in\mathcal{B}_{\mathcal{C}}([l])$ and $|i-j|>1$. Under these conditions $[l+\delta_{i,u_1}^{(k_1)}],[l-\delta_{j,u_2}^{(k_2)}]\in\mathcal{B}_{\mathcal{C}}([l])$. Let $[v]$ be a tableau with $\mathbb{Z}$-independent entries. Then we have $[e_i^{(r)},f_j^{(s)}][v]=0$. Therefore the coefficient of $[l+\delta_{i,u_1}^{(k_1)}-\delta_{j,u_2}^{(k_2)}]$ on both sides of (31) is equal.

Suppose now that $|i-j|=1$ and there is no relation between (k_1,i,u_1) and (k_2,j,u_2). Similarly to the case $|i-j|>1$, let $[v]$ be a tableau with $\mathbb{Z}$-independent entries. By comparing the coefficients of

$[l+\delta^{(k_1)}_{i,u_1}-\delta^{(k_2)}_{j,u_2}]$ and $[v+\delta^{(k_1)}_{i,u_1}-\delta^{(k_2)}_{j,u_2}]$ we conclude that the coefficient of $[l+\delta^{(k_1)}_{i,u_1}-\delta^{(k_2)}_{j,u_2}]$ on both sides of (31) is equal.

Suppose $|i-j|=1$ and there is a relation between (k_1,i,u_1) and (k_2,j,u_2). We denote by $\mathcal{C}'$ the set that consists of this relation. Let $[v]$ be a tableau such that $v^{(k_m)}_{i,u_m}=l^{(k_m)}_{i,u_m}$, $m=1,2$ and all other entries are $\mathbb{Z}$-independent. By Example 3.9 $V_{\mathcal{C}'}([v])$ is a $W(\pi)$-module. Thus $[e^{(r)}_i,f^{(s)}_j][v]=0$. Since $[l+z]\in\mathcal{B}_{\mathcal{C}}([l])$ if and only if $[v+z]\in\mathcal{B}_{\mathcal{C}'}([v])$ where $z=\delta^{(k_1)}_{i,u_1},-\delta^{(k_2)}_{j,u_2}$ or $\delta^{(k_1)}_{i,u_1}-\delta^{(k_2)}_{j,u_2}$. Therefore the coefficient of $[l+\delta^{(k_1)}_{i,u_1}-\delta^{(k_2)}_{j,u_2}]$ on both sides of (31) is equal.

Suppose $i=j$ and $(k_1,u_1)\neq(k_2,u_2)$. Then there is no relation between (k_1,i,u_1) and (k_2,i,u_2). Similarly to the case $|i-j|>1$, we prove that the coefficient of $[l+\delta^{(k_1)}_{i,u_1}-\delta^{(k_2)}_{j,u_2}]$ on both sides of (31) are equal.

Suppose $i=j$ and $(k_1,u_1)=(k_2,u_2)=(k,u)$. Let $[v]$ be a tableau with $\mathbb{Z}$-independent entries. Then $[e^{(r)}_i,f^{(s)}_j][v]=-\delta_{ij}\sum_{t=0}^{r+s-1}d_i'^{(t)}\,d_{i+1}^{(r+s-t-1)}[v]$.

The coefficient of $[l]$ on the left hand side is as follows:

$$\sum_{k,u}e^{(r)}_{i,k,u}(v-\delta^{(k)}_{iu})f^{(s)}_{i,k,u}(v)-\sum_{k,u}f^{(s)}_{i,k,u}(v+\delta^{(k)}_{iu})e^{(r)}_{i,k,u}(v).$$

We denote the coefficient of $[v]$ on the right hand side by $h(v)$.

Since $\Phi(l,-\delta^{(k)}_{iu},\delta^{(k)}_{iu})=\Phi(l,-\delta^{(k)}_{iu})$ and $\Phi(l,\delta^{(k)}_{iu},-\delta^{(k)}_{iu})=\Phi(l,\delta^{(k)}_{iu})$, the coefficient of $[l]$ in $[e^{(r)}_i,f^{(s)}_j][l]$ is

$$\sum_{k,u}\Phi(l,-\delta^{(k)}_{iu})e^{(r)}_{i,k,u}(l-\delta^{(k)}_{iu})f^{(s)}_{i,k,u}(l)-\sum_{k,u}\Phi(l,\delta^{(k)}_{iu})f^{(s)}_{i,k,u}(l+\delta^{(k)}_{iu})e^{(r)}_{i,k,u}(l).$$

The coefficient of $[l]$ in $-\delta_{ij}\sum_{t=0}^{r+s-1}d_i'^{(t)}\,d_{i+1}^{(r+s-t-1)}[l]$ is $h(l)$. We have

$$\begin{aligned}&\sum_{k,u}\Phi(l,-\delta^{(k)}_{iu})e^{(r)}_{i,k,u}(l-\delta^{(k)}_{iu})f^{(s)}_{i,k,u}(l)-\sum_{k,u}\Phi(l,\delta^{(k)}_{iu})f^{(s)}_{i,k,u}(l+\delta^{(k)}_{iu})e^{(r)}_{i,k,u}(l)\\ &=\sum_{k,u,\Phi(l,-\delta^{(k)}_{iu})=1}e^{(r)}_{i,k,u}(l-\delta^{(k)}_{iu})f^{(s)}_{i,k,u}(l)-\sum_{k,u,\Phi(l,\delta^{(k)}_{iu})=1}f^{(s)}_{i,k,u}(l+\delta^{(k)}_{iu})e^{(r)}_{i,k,u}(l)\\ &=\lim_{v\to l}\left(\sum_{k,u,\Phi(l,-\delta^{(k)}_{iu})=1}e^{(r)}_{i,k,u}(v-\delta^{(k)}_{iu})f^{(s)}_{i,k,u}(v)-\sum_{k,u,\Phi(l,\delta^{(k)}_{iu})=1}f^{(s)}_{i,k,u}(v+\delta^{(k)}_{iu})e^{(r)}_{i,k,u}(v)\right).\end{aligned}$$

In order to show

$$\sum_{k,u}\Phi(l,-\delta^{(k)}_{iu})e^{(r)}_{i,k,u}(l-\delta^{(k)}_{iu})f^{(s)}_{i,k,u}(l)-\sum_{k,u}\Phi(l,\delta^{(k)}_{iu})f^{(s)}_{i,k,u}(l+\delta^{(k)}_{iu})e^{(r)}_{i,k,u}(l)=h(l)$$

it is sufficient to prove that

$$\lim_{v\to l}\left(\sum_{k,u,\Phi(l,-\delta^{(k)}_{iu})=0}e^{(r)}_{i,k,u}(v-\delta^{(k)}_{iu})f^{(s)}_{i,k,u}(v)-\sum_{k,u,\Phi(l,\delta^{(k)}_{iu})=0}f^{(s)}_{i,k,u}(v+\delta^{(k)}_{iu})e^{(r)}_{i,k,u}(v)\right)=0, \tag{32}$$

computing we have

$$\lim_{v\to l}\left(\sum_{k,u,\Phi(l,-\delta^{(k)}_{iu})\neq0}e^{(r)}_{i,k,u}(v-\delta^{(k)}_{iu})f^{(s)}_{i,k,u}(v)-\sum_{k,u,\Phi(l,\delta^{(k)}_{iu})\neq0}f^{(s)}_{i,k,u}(v+\delta^{(k)}_{iu})e^{(r)}_{i,k,u}(v)\right)$$

$$= \sum_{k,u} \Phi(l, -\delta_{iu}^{(k)}) e_{i,k,u}^{(r)}(l - \delta_{iu}^{(k)}) f_{i,k,u}^{(s)}(l) - \sum_{k,u} \Phi(l, \delta_{iu}^{(k)}) f_{i,k,u}^{(s)}(l + \delta_{iu}^{(k)}) e_{i,k,u}^{(r)}(l),$$

$$\lim_{v\to l} \left(\sum_{k,u} e_{i,k,u}^{(r)}(v - \delta_{iu}^{(k)}) f_{i,k,u}^{(s)}(v) - \sum_{k,u} f_{i,k,u}^{(s)}(v + \delta_{iu}^{(k)}) e_{i,k,u}^{(r)}(v) \right)$$
$$= \sum_{k,u} \Phi(l, -\delta_{iu}^{(k)}) e_{i,k,u}^{(r)}(l - \delta_{iu}^{(k)}) f_{i,k,u}^{(s)}(l) - \sum_{k,u} \Phi(l, \delta_{iu}^{(k)}) f_{i,k,u}^{(s)}(l + \delta_{iu}^{(k)}) e_{i,k,u}^{(r)}(l).$$

On the other hand,

$$\left(\sum_{k,u} e_{i,k,u}^{(r)}(v - \delta_{iu}^{(k)}) f_{i,k,u}^{(s)}(v) - \sum_{k,u} f_{i,k,u}^{(s)}(v + \delta_{iu}^{(k)}) e_{i,k,u}^{(r)}(v) \right) = h(v)$$

where $h(v)$ is a polynomial in v. Then the limit is $h(l)$.

Thus we have

$$\sum_{k,u} \Phi(l, -\delta_{iu}^{(k)}) e_{i,k,u}^{(r)}(l - \delta_{iu}^{(k)}) f_{i,k,u}^{(s)}(l) - \sum_{k,u} \Phi(l, \delta_{iu}^{(k)}) f_{i,k,u}^{(s)}(l + \delta_{iu}^{(k)}) e_{i,k,u}^{(r)}(l) = h(l).$$

The following statements can be verified by direct computation:

(i) If $\Phi(l, -\delta_{iu}^{(k)}) = 0$ and $l_{iu}^{k} - l_{iu'}^{k'} \neq 1$ for any $(k', u') \neq (k, u)$, then

$$\lim_{v\to l} e_{i,k,u}^{(r)}(v - \delta_{iu}^{(k)}) f_{i,k,u}^{(s)}(v) = 0.$$

(ii) If $\Phi(l, \delta_{iu}^{(k)}) = 0$ and $l_{iu'}^{k'} - l_{iu}^{k} \neq 1$ for any $(k', u') \neq (k, u)$, then

$$\lim_{v\to l} f_{i,k,u}^{(s)}(v + \delta_{iu}^{(k)}) e_{i,k,u}^{(r)}(v) = 0.$$

(iii) If $l_{iu}^{k} - l_{iu'}^{k'} = 1$, then $\Phi(l, -\delta_{iu}^{(k)}) = \Phi(l, \delta_{iu'}^{(k')}) = 0$ and

$$\lim_{v\to l} \left(e_{i,k,u}^{(r)}(v - \delta_{iu}^{(k)}) f_{i,k,u}^{(s)}(v) - f_{i,k',u'}^{(s)}(v + \delta_{iu}^{(k)}) e_{i,k',u'}^{(r)}(v) \right) = 0.$$

Therefore (32) holds and we complete the proof.

3.

$$[d_i^{(r)}, e_j^{(s)}][l] = (\delta_{ij} - \delta_{i,j+1}) \sum_{t=0}^{r-1} d_i^{(t)}\, e_j^{(r+s-t-1)}[l], \tag{33}$$

$$[d_i^{(r)}, f_j^{(s)}][l] = (\delta_{i,j+1} - \delta_{ij}) \sum_{t=0}^{r-1} f_j^{(r+s-t-1)}\, d_i^{(t)}[l]. \tag{34}$$

To prove that for every $[l + \delta_{j,t}^{(k)}] \in \mathcal{B}_{\mathcal{C}}([l])$, the coefficients on both sides of (33) are equal, consider a tableau $[v]$ with $\mathbb{Z}$-independent entries. For $[v]$ the coefficients on both sides of (33) are equal. Taking the limit $v \to l$ we obtain the statement.

The Relation (34) can be proved by the same argument.

4.

$$[e_i^{(r)}, e_i^{(s+1)}][l] - [e_i^{(r+1)}, e_i^{(s)}][l] = e_i^{(r)} e_i^{(s)}[l] + e_i^{(s)} e_i^{(r)}[l], \tag{35}$$

$$[f_i^{(r+1)}, f_i^{(s)}][l] - [f_i^{(r)}, f_i^{(s+1)}][l] = f_i^{(r)} f_i^{(s)}[l] + f_i^{(s)} f_i^{(r)}[l]. \tag{36}$$

The tableaux which appear in the Equation (35) are of the form $[l+2\delta_{i,s}^{(k)}]$ and $[l+\delta_{i,s}^{(k)}+\delta_{i,t}^{(r)}]$, $(k,s) \neq (r,t)$. In the following we show the for any such tableau in $\mathcal{B}_{\mathcal{C}}([l])$ the coefficients on both sides of (35) are equal. It is easy to see that when $[l+2\delta_{i,s}^{(k)}] \in \mathcal{B}_{\mathcal{C}}([l])$ then $[l+\delta_{i,s}^{(k)}] \in \mathcal{B}_{\mathcal{C}}([l])$. Hence the corresponding value of Φ is 1 and the coefficients on both sides are equal. Similarly, if $[l+\delta_{i,s}^{(k)}+\delta_{i,t}^{(r)}] \in \mathcal{B}_{\mathcal{C}}([l])$ then $[l+\delta_{i,s}^{(k)}], [l+\delta_{i,t}^{(r)}] \in \mathcal{B}_{\mathcal{C}}([l])$. Thus the coefficients of $[l+\delta_{i,s}^{(k)}+\delta_{i,t}^{(r)}]$, $(k,s) \neq (r,t)$ on both sides of (35) are equal.

Consider a tableau $[v]$ with $\mathbb{Z}$-independent entries. For $[v]$ the coefficients on both sides of (35) are equal. Taking the limit $v \to l$ we obtain Equation (35).

The Relation (36) can be proved using the same arguments.

5.

$$[e_i^{(r)}, e_{i+1}^{(s+1)}][l] - [e_i^{(r+1)}, e_{i+1}^{(s)}][l] = -e_i^{(r)} e_{i+1}^{(s)}[l], \tag{37}$$

$$[f_i^{(r+1)}, f_{i+1}^{(s)}][l] - [f_i^{(r)}, f_{i+1}^{(s+1)}][l] = -f_{i+1}^{(s)} f_i^{(r)}[l]. \tag{38}$$

The tableaux which appear in Equation (37) are of the form $[l+\delta_{i,s}^{(k)}+\delta_{i+1,t}^{(r)}]$. Let $[l+\delta_{i,s}^{(k)}+\delta_{i+1,t}^{(r)}] \in \mathcal{B}_{\mathcal{C}}([l])$. If there is no relation between (k,i,s) and $(r,i+1,t)$, then $[l+\delta_{i,s}^{(k)}], [l+\delta_{i+1,t}^{(r)}] \in \mathcal{B}_{\mathcal{C}}([l])$. By the argument on the proof of (7) we have the same coefficients of $[l+\delta_{i,s}^{(k)}+\delta_{i+1,t}^{(r)}]$ on both sides of (37).

Assume $\mathcal{C}' = \{(r,i+1,t) \geq (k,i,s)\} \subset \mathcal{C}$. It is admissible by Example 3.9. Let $[v]$ be a tableau such that $v_{i+1,t}^{(r)} = l_{i+1,t}^{(r)}$, $v_{i,s}^{(k)} = l_{i,s}^{(k)}$ and all other entries are $\mathbb{Z}$-independent. Then $V_{\mathcal{C}'}([v])$ is a $W(\pi)$-module and $[e_i^{(r)}, e_{i+1}^{(s+1)}][v] - [e_i^{(r+1)}, e_{i+1}^{(s)}][v] = -e_i^{(r)} e_{i+1}^{(s)}[v]$. Since $[l+z] \in \mathcal{B}_{\mathcal{C}}([l])$ if and only if $[v+z] \in \mathcal{B}_{\mathcal{C}'}([v])$ for $z = \delta_{i,s}^{(k)}, \delta_{i+1,t}^{(r)}$, by substituting l for v in the coefficients of $[v+\delta_{i,s}^{(k)}+\delta_{i+1,t}^{(r)}]$ we obtain the coefficients of $[l+\delta_{i,s}^{(k)}+\delta_{i+1,t}^{(r)}]$. Therefore the coefficient of $[l+\delta_{i,s}^{(k)}+\delta_{i+1,t}^{(r)}]$ on both sides of (37) are equal.

Similarly one treats the case when $\{(k,i,s) > (r,i+1,t)\} \subset \mathcal{C}$. This completes the proof of (37). The equality (38) can be proved by the same argument.

6.

$$[e_i^{(r)}, e_j^{(s)}][l] = 0, \qquad \text{if} \quad |i-j| > 1,$$

$$[f_i^{(r)}, f_j^{(s)}][l] = 0, \qquad \text{if} \quad |i-j| > 1.$$

The proof is analogous to the proof of Relations (37) and (38).

7.

$$[e_i^{(r)}, [e_i^{(s)}, e_j^{(t)}]][l] + [e_i^{(s)}, [e_i^{(r)}, e_j^{(t)}]][l] = 0, \qquad \text{if} \quad |i-j| = 1, \tag{39}$$

$$[f_i^{(r)}, [f_i^{(s)}, f_j^{(t)}]][l] + [f_i^{(s)}, [f_i^{(r)}, f_j^{(t)}]][l] = 0, \qquad \text{if} \quad |i-j| = 1. \tag{40}$$

Potential tableaux in the equality (39) are of the form $[l+\delta_{i,u_1}^{(k_1)}+\delta_{i,u_2}^{(k_2)}+\delta_{i,u_3}^{(k_3)}]$ (we want to show that the coefficient of such tableaux is zero). Assume $[l+\delta_{i,u_1}^{(k_1)}+\delta_{i,u_2}^{(k_2)}+\delta_{i,u_3}^{(k_3)}] \in \mathcal{B}_{\mathcal{C}}([l])$. Suppose first that there is no relation between (k_1,i,u_1), (k_2,i,u_2) and (k_3,j,u_3). Let $[v]$ be a tableau with $\mathbb{Z}$-independent entries. Then

$$[e_i^{(r)}, [e_i^{(s)}, e_j^{(t)}]][v] + [e_i^{(s)}, [e_i^{(r)}, e_j^{(t)}]][v] = 0.$$

The coefficient of $[l+\delta^{(k_1)}_{i,u_1}+\delta^{(k_2)}_{i,u_2}+\delta^{(k_3)}_{j,u_3}]$ is obtained by substituting l for v in the coefficient of $[v+\delta^{(k_1)}_{i,u_1}+\delta^{(k_2)}_{i,u_2}+\delta^{(k_3)}_{j,u_3}]$, which is zero.

Suppose $j=i+1$. If $(k_3,j,u_3)\geq(k_1,i,u_1)$ and there is no relation between (k_3,j,u_3) and (k_2,i,u_2), then set $\mathcal{C}'=\{(k_3,j,u_3)\geq(k_1,i,u_1)\}$. Consider a tableau $[v]$ such that $v^{(k_m)}_{i_m,u_m}=l^{(k_m)}_{i_m,u_m}$ for $m=1,2,3$, $i_1=i_2=i, i_3=i+1$ and all other entries are $\mathbb{Z}$-independent. Then $[l+z]\in\mathcal{B}_{\mathcal{C}}([l])$ if and only if $[v+z]\in\mathcal{B}_{\mathcal{C}'}([v])$, where z is $\delta^{(k_m)}_{i_m,u_m}$ or $\delta^{(k_{m_1})}_{i_{m_1},u_{m_1}}+\delta^{(k_{m_2})}_{i_{m_2},u_{m_2}}$, $m,m_1,m_2=1,2,3$.

Since $[e_i^{(r)},[e_i^{(s)},e_j^{(t)}]][v]+[e_i^{(s)},[e_i^{(r)},e_j^{(t)}]][v]=0$, we have that the coefficient of $[l+\delta^{(k_1)}_{i,u_1}+\delta^{(k_2)}_{i,u_2}+\delta^{(k_3)}_{j,u_3}]$ is zero.

If there exists a relation between (k_3,j,u_3) and (k_1,i,u_1), then by the same argument one can show that the coefficient of $[l+\delta^{(k_1)}_{i,u_1}+\delta^{(k_2)}_{i,u_2}+\delta^{(k_3)}_{j,u_3}]$ is zero.

If $(k_1,i,u_1)>(k_3,j,u_3)$ and $(k_3,j,u_3)\geq(k_2,i,u_2)$, then there exists $(k_4,i-1,u_4)$ such that $(k_1,i,u_1)\geq(k_4,i-1,u_4)$ and $(k_4,i-1,u_4)\geq(k_2,i,u_2)$. Let $\mathcal{C}'=\{(k_1,i,u_1)>(k_3,j,u_3),(k_3,j,u_3)\geq(k_2,i,u_2),(k_1,i,u_1)\geq(k_4,i-1,u_4),(k_4,i-1,u_4)>(k_2,i,u_2)\}$ and $[v]$ a tableau such that $v^{(k_m)}_{i_m,u_m}=l^{(k_m)}_{i_m,u_m}$ for $m=1,2,3,4$, $i_1=i_2=i, i_3=i+1, i_4=i-1$ and all other entries are $\mathbb{Z}$-independent. Then $[l+z]\in\mathcal{B}_{\mathcal{C}}([l])$ if and only if $[v+z]\in\mathcal{B}_{\mathcal{C}'}([v])$, where z is $\delta^{(k_m)}_{i_m,u_m}$ or $\delta^{(k_{m_1})}_{i_{m_1},u_{m_1}}+\delta^{(k_{m_2})}_{i_{m_2},u_{m_2}}$, $m,m_1,m_2=1,2,3$.

Since $[e_i^{(r)},[e_i^{(s)},e_j^{(t)}]][v]+[e_i^{(s)},[e_i^{(r)},e_j^{(t)}]][v]=0$, we have that the coefficient of $[l+\delta^{(k_1)}_{i,u_1}+\delta^{(k_2)}_{i,u_2}+\delta^{(k_3)}_{j,u_3}]$ is zero.

The case $j=i-1$ is treated similarly. This completes the proof of (39). The second equality can be proved in the same way. We complete the proof of the sufficiency of conditions in Theorem 3.16.

Suppose there exists an adjoining triple $(k,i,j),(r,i,t)$ which does not satisfy the condition (1). Then applying the RR-method to $\mathcal{C}$, after finitely many steps, we will obtain a set of relations from Example 3.10 which is not admissible. Thus $\mathcal{C}$ is not admissible.

Acknowledgement

V.F. is supported in part by the CNPq grant (304467/2017-0) and by the FAPESP grant (2014/09310-5). L. E. R. is supported by the FAPESP grant (2018/17955-7). J. Z. is supported by the FAPESP grant (2015/05927-0).

References

[1] T. Arakawa, Representation theory of W-algebras, Invent. Math. 169 (2007) 219–320.
[2] T. Arakawa, Introduction to W-algebras and their representation theory, in: F. Callegaro, G. Carnovale, F. Caselli, C. De Concini, A. De Sole (Eds.), Perspectives in Lie Theory, in: Springer INdAM Series, vol. 19, Springer, Cham, 2017.
[3] P. Bouwknegt, K.J. Schoutens, W symmetry in CFT, Phys. Rep. 223 (1993) 183.
[4] J. Brundan, A. Kleshchev, Shifted Yangians and finite W-algebras, Adv. Math. 200 (2006) 136–195.
[5] J. Brundan, A. Kleshchev, Representations of shifted Yangians and finite W-algebras, in: Memoirs AMS, 2008, 107 pp..
[6] B. Feigin, E. Frenkel, Quantization of Drinfeld-Sokolov reduction, Phys. Lett. B 246 (1990) 75–81.
[7] V. Futorny, D. Grantcharov, L.E. Ramirez, P. Zadunaisky, Gelfand-Tsetlin theory for rational Galois algebras, preprint, available online at, https://arxiv.org/abs/1801.09316.
[8] V. Futorny, A. Molev, S. Ovsienko, Harish-Chandra modules for Yangians, Represent. Theory 9 (2005) 426–454.
[9] V. Futorny, A. Molev, S. Ovsienko, Gelfand-Tsetlin bases for representations of finite W-algebras and shifted Yangians, in: Lie Theory and Its Applications in Physics VII, 2008, Heron Press, Varna, 2007, pp. 352–363.
[10] V. Futorny, A. Molev, S. Ovsienko, Gelfand-Kirillov conjecture and Gelfand-Tsetlin modules for finite W-algebras, Adv. Math. 223 (2010) 773–796.
[11] V. Futorny, S. Ovsienko, Galois orders in skew monoid rings, J. Algebra 324 (2010) 598–630.
[12] V. Futorny, L.E. Ramirez, J. Zhang, Combinatorial construction of Gelfand-Tsetlin modules for $\mathfrak{gl}_n$, Adv. Math. 343 (2019) 681–711.
[13] I. Gelfand, M. Tsetlin, Finite-dimensional representations of the group of unimodular matrices, Dokl. Akad. Nauk SSSR 71 (1950) 825–828.
[14] I. Gelfand, M. Graev, Finite-dimensional irreducible representations of the unitary and complete linear group and special functions associated with them, Izv. Ros. Akad. Nauk., Ser. Mat. 29 (6) (1965) 1329–1356.
[15] V. Kac, M. Wakimoto, Quantum reduction and representation theory of superconformal algebras, Adv. Math. 185 (2004) 400–458, Corrigendum: Adv. Math. 193 (2005) 453–455.

[16] V. Kac, S.S. Roan, M. Wakimoto, Quantum reduction for affine superalgebras, Commun. Math. Phys. 241 (2003) 307–342.

[17] B. Kostant, On Whittaker vectors and representation theory, Invent. Math. 48 (1978) 101–184.

[18] F. Lemire, J. Patera, Formal analytic continuation of Gelfand's finite dimensional representations of $\mathfrak{gl}(n, \mathbb{C})$, J. Math. Phys. 20 (5) (1979) 820–829.

[19] T.E. Lynch, Generalized Whittaker Vectors and Representation Theory, PhD thesis, MIT, 1979.

[20] A. Molev, Yangians and Classical Lie Algebras, Mathematical Surveys and Monographs, vol. 143, American Mathematical Society, Providence, RI, 2007.

[21] A. Molev, Irreducibility criterion for tensor products of Yangian evaluation modules, Duke Math. J. 112 (2002) 307–341.

[22] M. Nazarov, V. Tarasov, Representations of Yangians with Gelfand-Zetlin bases, J. Reine Angew. Math. 496 (1998) 181–212.

[23] S. Ovsienko, Finiteness statements for Gelfand–Tsetlin modules, in: Algebraic Structures and Their Applications, Math. Inst., Kiev, 2002.

[24] A. Premet, Special transverse slices and their enveloping algebras, Adv. Math. 170 (2002) 1–55.

[25] E. Ragoucy, P. Sorba, Yangian realizations from finite W-algebras, Commun. Math. Phys. 203 (1999) 551–576.

[26] A. De Sole, A. Kac, Finite vs affine W-algebras, Jpn. J. Math. 1 (2006) 137–261.

Please cite this article in press as: V. Futorny et al., Gelfand-Tsetlin representations of finite W-algebras, J. Pure Appl. Algebra (2019), https://doi.org/10.1016/j.jpaa.2019.106226