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A NEW CHARACTERIZATION OF THE CLIFFORD TORUS

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ABSTRACT. In this paper we introduce the notion of contact angle. We deduce formulas to Laplacian and curvature for a minimal surface in S^3 and give a characterization of the Clifford Torus as the only minimal surface in S^3 with contact angle between 0 and $\frac{\pi}{2}$. Finally, we construct an example of minimal surface in S^3 with non constant contact angle.

1. INTRODUCTION

In [1] Chern and Wolfson have developed basic formulas and some results on minimal surfaces using the Kähler angle. In this work, we will construct results on minimal surfaces by the contact angle.
The contact angle is the angle between the canonic contact distribution and the tangent space of the surface (see 2). Using Gauss and Codazzi equations, we deduce formulas to Laplacian and Gaussian Curvature to S and proof the following two theorems:

Theorem 1 *The Clifford Torus is the only minimal surface in* S^3 *with contact angle* $0 \leq \beta < \frac{\pi}{2}$ *(or* $-\frac{\pi}{2} < \beta \leq 0$*).*

Theorem 2 *The Clifford Torus is the only minimal surface in* S^3 *with constant contact angle .*

At the last section, we give two examples of minimal surfaces in S^3. At the first one, we determine that the contact angle β of the Clifford Torus is $\beta = 0$ and the second one we determine that the contact angle of the totally geodesic sphere is $\beta = arc\cos(x_2)$, and therefore, non constant (see 4).

2. CONTACT ANGLE OF MINIMAL SURFACES IN S^3

Consider a surface S immersed in S^3. The contact angle is the angle between the canonic contact distribution Δ and the tangent plane of S. The contact distribution is defined by $\Delta = TS^3 \cap J(TS^3)$, where

J is the complex structure of C^2.
In S^3 consider the frame (f_1, f_2, f_3) , such that $f_3 = iz$. Then (f_1, f_2) is a frame of Δ.
The covariant derivative is given by:

$$\begin{aligned} Df_1 &= w_1^2 f_2 + w^2 f_3 \\ Df_2 &= w_2^1 f_1 - w^1 f_3 \\ Df_3 &= -w^2 f_1 + w^1 f_2 \end{aligned} \tag{1}$$

where (w^1, w^2, w^3) is dual frame of (f_1, f_2, f_3).
Note that $Df_3 = J$ at the distribution Δ.
Consider e_1 unitary vector field in $TS \cap \Delta$, where Δ is the contact distribution.
Then the contact angle is defined by:

$$\begin{aligned} e_1 &= f_1 \\ e_2 &= \sin(\beta)\, f_2 + \cos(\beta)\, f_3 \\ e_3 &= -\cos(\beta)\, f_2 + \sin(\beta)\, f_3 \end{aligned} \tag{2}$$

where β is the angle between f_3 and $e_2, (e_1, e_2)$ are tangents to S and e_3 is normal to S

3. EQUATIONS FOR THE CURVATURE AND LAPLACIAN

Consider $(\theta^1, \theta^2, \theta^3)$ dual frame of (e_1, e_2, e_3)

$$\begin{aligned} \theta^1 &= w^1 \\ \theta^2 &= \sin(\beta)\, w^2 + \cos(\beta)\, w^3 \\ \theta^3 &= -\cos(\beta)\, w^2 + \sin(\beta)\, w^3 \end{aligned} \tag{3}$$

At the surface S, we have $\theta^3 = 0$, then we obtain the equation:

$$sin(\beta)\, w^3 = cos(\beta) w^2 \tag{4}$$

From (3), we determine:

$$\begin{aligned} d\theta^1 &+ \sin(\beta)(w_2^1 - \cos(\beta)\theta^2) \wedge \theta^2 = 0 \\ d\theta^2 &+ \sin(\beta)(w_1^2 + \cos(\beta)\theta^2) \wedge \theta^1 = 0 \\ d\theta^3 &= d\beta \wedge \theta^2 - \cos(\beta) w_1^2 \wedge\ w^1 + (1 + \sin^2(\beta))\theta^1 \wedge \theta^2 \end{aligned} \tag{5}$$

Therefore the conexion form of S is given by:

$$\theta_2^1 = \sin(\beta)(w_2^1 - \cos(\beta)\theta^2) \tag{6}$$

Computing De_3 at the basis (e_1, e_2), we obtain coeficients of fundamental second form:

$$De_3 = \theta_1^3 e_1 + \theta_2^3 e_2. \tag{7}$$

where:

$$\begin{aligned} \theta_1^3 &= \cos(\beta)w_1^2 + \sin(\beta)^2\theta^2 \\ \theta_2^3 &= -(d\beta + \theta^1) \end{aligned} \tag{8}$$

Using the equation $d\theta^3 = 0$ (simmetry), we have:

$$w_2^1(e_2) = -\frac{\beta_1}{\cos\beta} - \frac{(1+\sin^2\beta)}{\cos\beta} \tag{9}$$

where: $d\beta(e_1) = \beta_1$.
Using the condition of minimallity, we have:

$$\theta_1^3 \wedge \theta^2 + \theta_2^3 \wedge \theta^1 = 0 \tag{10}$$

then we obtain:

$$w_2^1(e_1) = \frac{\beta_2}{\cos(\beta)} \tag{11}$$

where: $d\beta(e_2) = \beta_2$.
Using (6), (9), and (11), we can get:

$$\begin{aligned} \theta_2^1 &= \tan(\beta)(\beta_2\theta^1 - (\beta_1 + 2)\theta^2) \\ \theta_1^3 &= \beta_2\theta^1 - (\beta_1 + 1)\theta^2 \\ \theta_2^3 &= -(\beta_1 + 1)\theta^1 - \beta_2\theta^2 \end{aligned} \tag{12}$$

From Gauss equation:

$$d\theta_1^2 = \theta^1 \wedge \theta^2 + \theta_1^3 \wedge \theta_2^3 \tag{13}$$

We obtain that:

$$d\theta_2^1 = (|\nabla(\beta)|^2 + 2\beta_1)\ (\theta^2 \wedge \theta^1) \tag{14}$$

where:

$$K = 1 - |\nabla(\beta) + e_1|^2 \tag{15}$$

Computing the differential of θ_2^1:

$$\begin{aligned} d\theta_2^1 &= \sec^2(\beta)(|\nabla\beta|^2 + 2\beta_1)(\theta^2 \wedge \theta^1) \\ &+ (\tan(\beta)\Delta(\beta) + 2\tan^2(\beta)(\beta_1 + 2))(\theta^2 \wedge \theta^1) \end{aligned} \tag{16}$$

Combining (16) and (14), we have a new formula for the laplacian:

$$\Delta(\beta) = -\tan(\beta)((\beta_1 + 2)^2 + \beta_2^2) \tag{17}$$

Or equivalentelly, we can write:

$$\Delta(\beta) = -\tan(\beta)|\nabla(\beta) + 2e_1|^2 \tag{18}$$

Codazzi equations are:

$$d\theta_1^3 + \theta_2^3 \wedge \theta_1^2 = 0 \tag{19}$$

$$d\theta_2^3 + \theta_1^3 \wedge \theta_2^1 = 0 \tag{20}$$

The first equation give the same equation that (18) and the second equation is automatically verified.

4. PROOF OF THEOREMS

Using (18), we obtain that for $0 \leq \beta < \frac{\pi}{2}$ $(-\frac{\pi}{2} < \beta < 0)$, we have $\Delta(\beta) \leq 0$ $(\Delta(\beta) \geq 0)$, using the Hopf's Lemma, we have $\beta =$ constant, therefore $K = 0$ and complete the proof of theorem 1.

If $\beta =$ constant, we deduce using the Gauss equation that $K = 0$, consequentelly, the minimal surface is the Clifford Torus and proof the theorem 2.

Note that if $\beta_1 \geq 0$ at the Gauss equation, then $K \leq 0$, consequentelly, $S \subset S^3$ is the Clifford Torus

5. EXAMPLES

5.1. Contact Angle of Clifford Torus in S^3.

Consider the torus in S^3 defined by:

$$T^2 = \{(z_1, z_2) \in C^2 / z_1\bar{z}_1 = \frac{1}{2}, z_2\bar{z}_2 = \frac{1}{2}\}$$

We consider the immersion:

$$f(u_1, u_2) = \frac{\sqrt{2}}{2}(e^{iu_1}, e^{iu_2})$$

$T(T^2)$ is generate by $\frac{\partial}{\partial u_1}$ and $\frac{\partial}{\partial u_2}$ it is means that:

$$a\frac{\partial}{\partial u_1} + b\frac{\partial}{\partial u_2} = \lambda z^{\perp}$$

where $z^{\perp} = (-\bar{z}_2, \bar{z}_1)$

Using the condition above and the fact that $|\lambda| = 1$, we obtain:

$$\lambda = ie^{i(u_1+u_2)}$$

The unitary vector fields are:

$$\begin{cases} e_1 = ie^{i(u_1+u_2)}z^{\perp} \\ e_2 = iz \\ e_3 = e^{i\alpha}iz^{\perp} \end{cases}$$

The contact angle is the angle between e_2 and f_3 ,

$$\begin{aligned} cos(\beta) &= \langle e_2, f_3 \rangle \\ &= 1 \end{aligned}$$

Therefore, the contact angle is:

$$\beta = 0$$

The fundamental second form at the basis (e_1, e_2) is:

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

5.2. Minimal surface in S^3 with non constant contact angle. Consider the surface described by:

$$\begin{cases} z_2 - \overline{z}_2 & = & 0 \\ (x_1)^2 + (y_1)^2 + (x_2)^2 + (y_2)^2 & = & 1 \end{cases}$$

We see that the unitary fields are:

$$\begin{cases} e_1 = \frac{1}{\sqrt{1-x_2^2}}(-x_1x_2, -y_1x_2, 1 - x_2^2, 0) \\ \\ e_2 = \frac{1}{\sqrt{x_1^2+y_1^2}}(y_1, -x_1, 0, 0) \\ \\ e_3 = (0, 0, 0, 1) \end{cases}$$

The contact angle is the angle between e_2 and f_3,

$$\begin{aligned} cos(\beta) &= \langle e_2, f_3 \rangle \\ &= x_2 \end{aligned}$$

Therefore, the contact angle is:

$$\beta = arc\,cos\,(x_2)$$

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