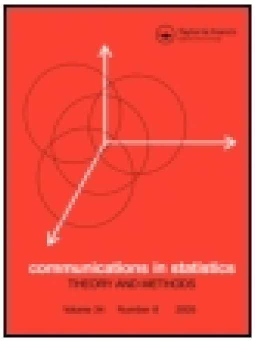

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A new class of slash-elliptical distributions

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Abstract

In this paper, we consider a generalization of the modified slash distribution. We define the new family through the quotient between an elliptically distributed random variable and the power of an exponential random variable with parameter equals to 2, both independent. We use the same idea to extend the model for the multivariate case and study general important properties from the resultant family. We perform inference by the method of moments and maximum likelihood. We present a simulation study which indicates satisfactory parameter recovery by using the estimation approaches. Illustrations reveals that it has potential for doing well in real problems.

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Heavy-tails distributions; kurtosis; moments; slash-elliptical distribution

1. Introduction

The canonical slash distribution was originated from the ratio between two independent random variables, one is the standard normal and other, an uniform (0, 1) distribution. This distribution presents heavier tails than the normal distribution, that is, it has a larger kurtosis coefficient. Properties of this family are discussed in Rogers and Tukey (1972), Mosteller and Tukey (1977) and Johnson, Kotz, and Balakrishnan (1995). Maximum likelihood estimators for location and scale parameters are discussed in Kafadar (1982). Wang and Genton (2006) described multivariate symmetrical and skew-multivariate extensions of the slash-distribution. Another asymmetric version of this family is discussed in the work of Arslan (2008). Arslan and Genc (2009) discussed a symmetric extension of the multivariate slash distribution and Genc (2007) discussed a symmetric generalization of the slash distribution. Furthermore, Gómez, Quintana, and Torres (2007) and Gómez and Venegas (2008) introduced the slash-elliptical (SE) family of distributions and Gómez, Olivares-Pacheco, and Bolfarine (2009) utilize the SE family to extend the Birnbaum-Saunders distribution. This methodology is used in Olmos et al. (2012, 2014) to extend the half-normal distribution. Recently Cavalcanti and Cysneiros (2017) developed non linear regression models with error distribution having the slash-elliptical family.

In this context, Reyes, Gómez, Bolfarine (2013) obtained a modification of the standard class of slash distributions, denominated standard modified slash and it is described as follows. We say that X has a modified slash (MS) distribution with parameter $q > 0$ (which we denote

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$X \sim MS(0,1,q)$) if it can be expressed as

$$X = \frac{Z}{U^{\frac{1}{q}}} \tag{1}$$

where $Z \sim N(0,1)$ is independent from $U \sim \text{Exp}(2)$. The density function of X has tails heavier than ones of the standard slash distribution and, therefore, higher kurtosis. The standard normal distribution is obtained as a limiting case for $q \to \infty$. The density function of X is given by

$$f_X(x;q) = \frac{2}{\sqrt{2\pi}} \int_0^\infty v^{\frac{1}{q}} e^{-\frac{1}{2}x^2 v^{\frac{2}{q}} - 2v} \, \mathrm{d}v$$

For $q = 1$, we obtain the canonical modified slash distribution with density function

$$f_X(x) = \begin{cases} \frac{2}{x^2}\left[\frac{1}{\sqrt{2\pi}} - \frac{2e^{\frac{2}{x^2}}}{|x|}\Phi\left(-\frac{2}{|x|}\right)\right] & \text{if} \quad x \neq 0 \\ (8\pi)^{-\frac{1}{2}} & \text{if} \quad x = 0 \end{cases}$$

For details, see Reyes, Gómez, Bolfarine (2013).

From Equation (1) it is obtained that

$$Z = U^{\frac{1}{q}} X \sim N(0,1) \tag{2}$$

The class of distributions that we will study is based on a generalization of Equation (2), considering that the distribution of Z is a more general class than the normal distribution. In this case, we consider the class of elliptical distributions, whose theory and properties can be seen in the works of Kelker (1970) and Cambanis, Huang, and Simons (1981). A general compilation from this theory can be founded in Fang, Kotz, and Ng (1990). Specifically, a k-dimensional random variable (r.v.) $\boldsymbol{Z}$ has elliptical distribution with location parameter $\boldsymbol{\mu} \in \mathbb{R}^k$ and positive definite scale matrix $\boldsymbol{\Sigma}$ (we denote $\boldsymbol{Z} \sim EL_k(\boldsymbol{\mu}, \boldsymbol{\Sigma}; g)$) if the density function of $\boldsymbol{Z}$ can be written as

$$f_{\boldsymbol{Z}}(\boldsymbol{z}) = |\boldsymbol{\Sigma}|^{-\frac{1}{2}} g\left((\boldsymbol{z} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{z} - \boldsymbol{\mu})\right), \quad \boldsymbol{z} \in \mathbb{R}^k$$

where $g(\cdot)$ is a function mapping from the non negative real numbers to the non negative real numbers satisfying the condition

$$\int_0^\infty u^{-\frac{k}{2}} g(u) \, \mathrm{d}u < \infty$$

For the univariate case ($k = 1$), the density function of $Z \sim EL(\mu, \sigma; g)$ it is reduced to

$$f_Z(z) = \frac{1}{\sigma} g\left(\left(\frac{z-\mu}{\sigma}\right)^2\right), \quad z, \mu \in \mathbb{R}, \ \sigma > 0$$

Some particular cases of distributions belonging to this class and their respective function $g(\cdot)$ are presented in Table 1. The new class of distributions that we proposed can be expressed as $\boldsymbol{X} = \boldsymbol{Z}U^{-\frac{1}{q}}$, $q > 0$, where $\boldsymbol{Z} \sim EL(0, 1; g)$ is independent from $U \sim \text{Exp}(2)$.

The elements in this class will be termed modified slash elliptical (MSE) distribution, denoted as $\boldsymbol{X} \sim \text{MSE}(0, 1, q; g)$. The main motivation to construct a new class is to show that a well-known symmetric distribution can be transformed into a new one, providing thus

Table 1. Some distributions belonging to the elliptical class of distributions.

Distribution	$g(u)$
Normal	$(2\pi)^{-\frac{k}{2}}e^{-\frac{u}{2}}$
Cauchy	$\frac{\Gamma\left[\frac{1}{2}(k+1)\right]}{\pi^{\frac{1}{2}(k+1)}}(1+u)^{-\frac{1}{2}(k+1)}$
T-student	$\frac{\Gamma\left[\frac{1}{2}(\nu+k)\right]}{\Gamma\left[\frac{\nu}{2}\right](\nu\pi)^{\frac{k}{2}}}(1+\frac{u}{\nu})^{-\frac{1}{2}(\nu+k)}$

growth in the kurtosis coefficient. The model is then able to fit data with substantial amount of extreme observations. The new distribution main features are its mathematical tractability, flexibility, and applicability.

The paper is organized as follows. In Section 2, we present the MSE family of distribution in the univariate case. Moments and, in particular, the kurtosis coefficient are studied. Moments and maximum likelihood estimation are considered and results of a real-data application is reported. The main conclusion is that the new proposal can be a viable alternative to competing models. In Section 3, we propose a multivariate extension of the MSE family of distributions. Properties like moments and maximum likelihood and moments estimation are studied. Results of a real-bivariate data application are reported with a satisfactory performance of the MSE model. A small-scale simulation study is also considered. Main conclusions are presented in Section 4.

2. Univariate MSE distribution

We provide the general stochastic representation of the MSE family, deriving its more important properties such as the density function and moments. Let $Y \sim \text{MSE}(\mu, \sigma, q; g)$. Therefore, Y is a r.v. that can be stochastically represented as

$$Y = \sigma \frac{W}{U^{\frac{1}{q}}} + \mu \tag{3}$$

where $W \sim EL(0, 1; g)$ is independent from $U \sim \text{Exp}(2)$, with $q > 0$, $\mu \in \mathbb{R}$ and $\sigma > 0$.

Proposition 1. *Let* $Y \sim \text{MSE}(0, 1, q; g)$. *The density function of* Y *is given by*

$$f_Y(y; 0, 1, q) = \begin{cases} 2\int_0^{\infty} t^{\frac{1}{q}} e^{-2t} g(y^2 t^{\frac{2}{q}})\, dt & \text{if } y \neq 0 \\ 2^{-\frac{1}{q}} g(0)\Gamma\left(\frac{q+1}{q}\right) & \text{if } y = 0 \end{cases}$$

Proof. From Equation (3), using the independence of W and U and based on the Jacobian of the appropriate transformation $Y = \frac{W}{U^{\frac{1}{q}}}$ and $T = U$, it is obtained the joint density, namely

$$f_{(Y,T)}(y, t) = 2e^{-2t} t^{\frac{1}{q}} g(y^2 t^{\frac{2}{q}}) \quad y \in \mathbb{R},\ t > 0$$

Then, integrating with respect to t, the marginal density of Y is obtained and given by

$$f_Y(y) = 2\int_0^{\infty} t^{\frac{1}{q}} e^{-2t} g\left(y^2 t^{\frac{2}{q}}\right) dt$$

For $y = 0$, the result is immediate. □

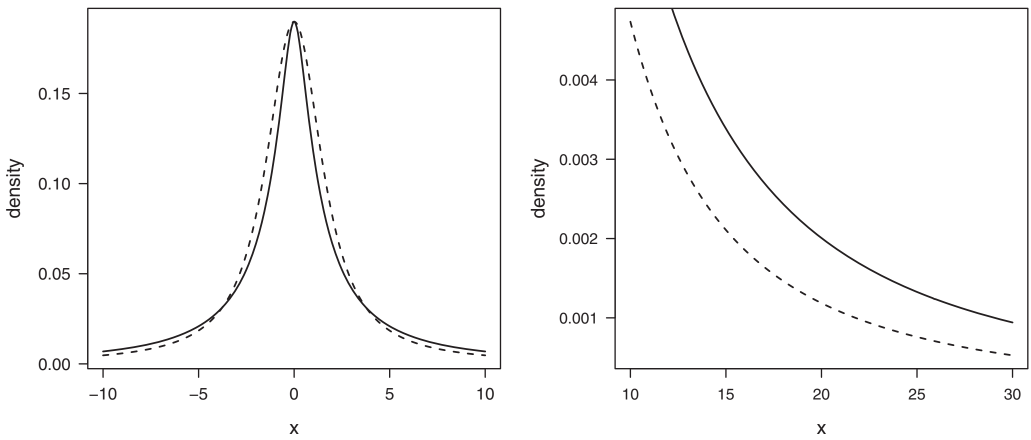

Figure 1. Density functions for MSt distribution (solid line) and St distribution (dashed line), with $\nu = 5$.

Corollary 1. *The density of $Y \sim \mathrm{MSE}(\mu, \sigma, q; g)$ is given by*

$$f_Y(y;\mu,\sigma,q) = \begin{cases} \dfrac{2}{\sigma}\displaystyle\int_0^\infty t^{\frac{1}{q}} e^{-2t} g\left(\left(\frac{y-\mu}{\sigma}\right)^2 t^{\frac{2}{q}}\right) \mathrm{d}t & \text{if } y \neq \mu \\ \dfrac{1}{\sigma} 2^{-\frac{1}{q}} g(0)\Gamma\left(\dfrac{q+1}{q}\right) & \text{if } y = \mu \end{cases} \tag{4}$$

Proof. The proof is immediate considering that the transformation in Equation (3) define a location-scale family of distributions. □

Corollary 2. *The canonical* MSE, *i.e.* $Y \sim \mathrm{MSE}(0, 1, 1; g)$, *has density function given by*

$$f_Y(y;0,1,1) = \begin{cases} 2\displaystyle\int_0^\infty t e^{-2t} g(y^2 t^2)\, \mathrm{d}t & \text{if } y \neq 0 \\ \frac{1}{2} g(0) & \text{if } y = 0 \end{cases} \tag{5}$$

Proof. Replacing $q = 1$ in Proposition 1 the result is obtained. □

Figure 1 shows the graphic of modified slash-t(MSt) distribution compared with the slash-t(St) for the canonical case and 5 degrees of freedom.

2.1. Properties of the MSE distribution

Proposition 2. *Let $Y|U = u \sim EL(0, u^{-\frac{1}{q}}; g)$ and $U \sim \mathrm{Exp}(2)$. Then $Y \sim \mathrm{MSE}(0, 1, q; g)$.*

Proof. We start by considering

$$f_Y(y;q) = \int_0^\infty f_{Y|U}(y|u) f_U(u)\, \mathrm{d}u = \int_0^\infty 2u^{\frac{1}{q}} e^{-2u} g(y^2 u^{\frac{2}{q}})\, \mathrm{d}u$$

In other words, $Y \sim \mathrm{MSE}(0, 1, q; g)$. □

Remark 1. Proposition 2 shows that the modified slash-elliptical distribution can be represented as a particular scale mixture from the elliptical class of distribution and the exponential distribution with parameter equals to 2.

2.2. Moments and kurtosis

We observe that the odd moments of the MSE are zero for $\mu = 0$ and $\sigma = 1$. This is clear using the stochastic representation in Equation (3), because the odd moments of the canonic elliptical distribution are zero. The following proposition shows the computation for the moments in the MSE distribution.

Proposition 3. *Let $X \sim \mathrm{MSE}(0, 1, q; g)$ and $Y \sim \mathrm{MSE}(\mu, \sigma, q; g)$. Then, for $r = 1, 2, 3, \ldots$ and $q > r$ we have that*

$$\mu_r = E(X^r) = \begin{cases} 2^{\frac{r}{q}}\Gamma\left(\frac{q-r}{q}\right) a_{r/2} & \text{if } r \text{ is even} \\ 0 & \text{if } r \text{ is odd} \end{cases} \tag{6}$$

and

$$E(Y^r) = \sum_{k=0}^{r} \binom{r}{k} \sigma^k \mu^{r-k} \mu_k \tag{7}$$

where $a_{r/2} = \int_{-\infty}^{\infty} w^r g(w^2)\, dw$.

Proof. Using the stochastic representation in Equation (3) for the case $\mu = 0$ and $\sigma = 1$ and considering W and U independent, we have that

$$\mu_r = E(X^r) = E\left[\left(\frac{W}{U^{\frac{1}{q}}}\right)^r\right] = E(W^r)E(U^{-\frac{r}{q}}).$$

It is straightforward to show that $E[W^r] = \int_{-\infty}^{\infty} w^r g(w^2)\, dw$. Also, $E[U^{-\frac{r}{q}}] = 2^{\frac{r}{q}}\Gamma\left(\frac{q-r}{q}\right)$, $q > r$ (see Reyes, Gómez, and Bolfarine 2013).

The second result is obtained applying the expectation for the r-th power in the representation given in Equation (3). □

Corollary 3. *Let $Y \sim \mathrm{MSE}(\mu, \sigma, q; g)$, then we have that*

$$E(Y) = \mu \quad \text{and} \quad \mathrm{Var}(Y) = \sigma^2 2^{\frac{2}{q}} \Gamma\left(\frac{q-2}{q}\right) a_1, \quad q > 2$$

Proposition 4. *Let $Y \sim \mathrm{MSE}(\mu, \sigma, q; g)$. Then, the kurtosis coefficient of Y is*

$$\beta_2 = \frac{\Gamma\left(\frac{q-4}{q}\right)}{\Gamma\left(\frac{q-2}{q}\right)^2} \frac{a_2}{a_1^2}, \quad q > 4 \tag{8}$$

Table 2. Some values for the standardized kurtosis Coefficient SE and MSE class.

q	β_2^{SE}	β_2^{MSE}
5	1.80κ	2.07κ
6	1.33κ	1.46κ
7	1.19κ	1.27κ
8	1.13κ	1.18κ
9	1.09κ	1.13κ
10	1.07κ	1.10κ

Table 3. Tails comparisond for different canonical distributions of SE and MSE class.

Distribution	$P(Y > 3)$	$P(Y > 5)$	$P(Y > 10)$	$P(Y > 20)$	$P(Y > 30)$
Normal	0.0013	0.0000	0.0000	0.0000	0.0000
S	0.1351	0.0806	0.0401	0.0200	0.0133
MS	0.1867	0.1277	0.0710	0.0375	0.0255
Cauchy	0.1024	0.0628	0.0317	0.0159	0.0106
SC	0.1558	0.0955	0.0478	0.0238	0.0159
MSC	0.2027	0.1422	0.0814	0.0438	0.0300
t_5	0.0150	0.0021	0.0001	0.0000	0.0000
St_5	0.1554	0.0952	0.0477	0.0238	0.0158
MSt_5	0.2021	0.1418	0.0811	0.0437	0.0299

Proof. Using the standardized kurtosis coefficient it is obtained that

$$\beta_2 = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_1^2\mu_2 - 3\mu_1^4}{(\mu_2 - \mu_1^2)^2}$$

Replacing with the moments given in Proposition 3 the result is obtained. □

Where β_2^{SE} and β_2^{MSE} kurtosis coefficient SE and MSE class, respectively, $\kappa = \frac{a_2}{a_1^2}$, $q > 4$ and $a_i < \infty$, for $i = 1, 2$.

2.3. Tails comparison of SE and MSE distributions

In this section, we perform a brief comparison illustrating that the tails of the MSE distribution are heavier than the ones for the SE distribution.

For this, we consider the canonical version ($q = 1$) of SE and MSE distributions considering $g(\cdot)$ to be Normal, Cauchy, and t-Student distribution with 5 degrees of freedom. Table 3 shows $P(Y > y)$ for different values of y in the mentioned distributions. It is evident that the MSE distribution has tails much heavier than the SE distribution.

Remark 2. Tables 2 and 3 illustrates the fact that the modified slash elliptical distributions have heavier tails than the tails of the slash elliptical distribution.

2.4. Inference

2.4.1. Moment estimators

In the following proposition, we present the moments estimators for μ, σ, and q in the MSE distribution.

Proposition 5. *Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from* $\text{MSE}(\mu; \sigma, q; g)$. *The moments estimators of $\theta = (\mu, \sigma, q)$ are given by*

$$\hat{\mu}_M = \bar{Y},\ \hat{\sigma}_M = \frac{2^{-\frac{1}{\hat{q}_M}} S}{\left(\Gamma\left(\frac{\hat{q}_M - 2}{\hat{q}_M}\right) a_1\right)^{\frac{1}{2}}},\ \gamma_2 = \frac{\Gamma\left(\frac{\hat{q}_M-4}{\hat{q}_M}\right) a_2}{\Gamma\left(\frac{\hat{q}_M-2}{\hat{q}_M}\right)^2 a_1^2}, \quad \hat{q}_M > 4$$

where $\bar{Y}$, S, and γ_2 are the mean, standard deviation and kurtosis coefficient obtained from the sample, respectively.

Proof. Using Corollary 3 and Proposition 4 we have that

$$\mu = \bar{Y} \tag{9}$$

$$\sigma^2 = \frac{2^{-\frac{2}{q}} \text{Var}(Y)}{\Gamma\left(\frac{q-2}{q}\right) a_1}, \quad q > 2 \tag{10}$$

$$\beta_2 = \frac{\Gamma\left(\frac{q-4}{q}\right)}{\Gamma\left(\frac{q-2}{q}\right)^2} \frac{a_2}{a_1^2}, \quad q > 4 \tag{11}$$

Replacing $E(Y)$, $\text{Var}(Y)$, and β_2 by the sample versions and solving Equations (9), (10), and (11), the result is obtained. □

Remark 3. The non linear equation for q presented in Equation (11) can be solved numerically using, for instance, the software R (R Core Team 2016) through the function `try`.

2.4.2. Maximum likelihood estimation

We discuss now maximum likelihood estimation. Given a random sample $Y_1, \ldots, Y_n$ from the distribution of $\text{MSE}(\mu, \sigma, q; g)$, the log-likelihood function can be written as

$$l(\mu, \sigma, q) = n \log(2) - n \log(\sigma) + \sum_{i=1}^{n} \log G(y_i) \tag{12}$$

where $G(y_i) = G(y_i; \mu, \sigma, q) = \int_0^\infty t^{\frac{1}{q}} e^{-2t} g\left[(\frac{y_i-\mu}{\sigma})^2 t^{\frac{2}{q}}\right] \mathrm{d}t$ and hence the maximum likelihood equations are given by

$$\sum_{i=1}^{n} \frac{G_1(y_i)}{G(y_i)} = 0 \tag{13}$$

$$\sum_{i=1}^{n} \frac{G_2(y_i)}{G(y_i)} = \frac{n}{\sigma} \tag{14}$$

$$\sum_{i=1}^{n} \frac{G_3(y_i)}{G(y_i)} = 0 \tag{15}$$

where, $G_1(y_i) = \frac{\partial}{\partial \mu} G(y_i)$, $G_2(y_i) = \frac{\partial}{\partial \sigma} G(y_i)$, $G_3(y_i) = \frac{\partial}{\partial q} G(y_i)$. The expressions for $G_1(y_i)$, $G_2(y_i)$, and $G_3(y_i)$ are given by

$$G_1(y_i) = -\frac{2}{\sigma} \int_0^\infty t^{\frac{2}{q}} e^{-2t} g'(z_i^2) z_i \mathrm{d}t$$

Table 4. Parameter estimates for models St with ν known.

ν_0	μ	σ	q	Log-likelihood
1	181.8862	20.2566	16.4971	−3971.637
2	181.5428	17.7580	2.0292	−3955.936
3	181.4309	17.7009	1.6380	−3959.241
4	181.4024	17.4758	1.4764	−3962.502
5	181.4007	17.2727	1.3905	−3964.727
10	181.4192	16.8721	1.2539	−3968.965
20	181.4268	16.7793	1.2063	−3970.687
30	181.4284	16.7747	1.1935	−3971.191

$$G_2(y_i) = -\frac{2}{\sigma}\int_0^\infty t^{\frac{1}{q}} e^{-2t} g'(z_i^2) z_i^2 \, \mathrm{d}t$$

$$G_3(y_i) = -\frac{1}{q^2}\int_0^\infty t^{\frac{1}{q}} e^{-2t} \log(t) \left[g(z_i^2) + 2z_i^2 g'(z_i^2)\right] \mathrm{d}t$$

where $z_i = \left(\frac{y_i-\mu}{\sigma}\right) t^{\frac{1}{q}}$. Solutions for Equations (13–15) can be obtained using numerical procedures such as the Newton–Raphson procedure. Equivalently, to obtain the maximum likelihood estimates, we can maximize the directly log-likelihood function in Equation (12) using extant software. For instance, in R Core Team (2016), this can be implemented via the `optim` function. Initial values for the algorithm can be obtained based on the moments estimators of Proposition 5.

2.5. *Application 1*

We consider here data coming from an entomology experiment concerning ants. A total of $n = 730$ ants were placed singly in the center of an arena. The measurements correspond to the initial direction in which they moved in relation to a visual stimulus at an angle of 180° from the zero direction, rounded to the nearest 10°. The data were first presented in Jander (1957) and SenGupta and Pal (2001). Tables 4 and 5 shows maximum likelihood estimates St and later analyzed in Batschelet (1981) and SenGupta and Pal (2001) and MSt models under the assumption of known degrees of freedom. The maximum likelihood estimates are those maximizing the likelihood function letting ν vary over a set of positive integers. This approach lead to the value $\nu_0 = 2$ for the St model, and for the MSt model, $\nu_0 = 3$.

Table 6 and Figure 2 shows the maximum likelihood estimates and the histogram of these data, respectively, including estimated densities under a regular St model and a MSt model. The criteria used for comparisons is the AIC (Akaike 1974) indicating that the MSt model presents the best fit for the data set under study. Besides the better fit, the use of MSt model also provides a second advantage for this particular problem. In both models, St and MSt, the variance depend on a_1, i.e., the second moment for the basal model (in this case, t distribution). Existence of a_1 is guaranteed only for $\nu > 2$. Then, for the St model the estimated variance of Y is infinite while that of MSt is finite, allowing, among other things, to construct confidence interval to predict future observations.

Using results from Section 2.4.1, moment estimates can be computed considering (fixing) $\nu = 10$, leading to $\widehat{\mu}_M = 176.438$, $\widehat{\sigma}_M = 47.136$, and $\widehat{q}_M = 8.604$, which were used as initial estimates for the maximum likelihood approach.

Table 5. Parameter estimates for model MSt with ν known.

ν_0	μ	σ	q	Log-likelihood
1	181.8871	20.07151	18.1702	−3971.656
2	181.6650	16.5384	2.2915	−3952.611
3	181.6080	16.5400	1.9169	−3952.488
4	181.5884	16.5551	1.7781	−3953.471
5	181.5957	16.5527	1.7052	−3954.288
10	181.5940	16.5822	1.5866	−3956.106
20	181.6002	16.6406	1.5413	−3956.974
30	181.6117	16.6087	1.5224	−3957.232

Table 6. Maximum likelihood estimates and corresponding standard error (SE) for St and MSt models.

Parameter	St (SE)	MSt (SE)
μ	181.5674 (1.2318)	181.6320 (1.2074)
σ	17.7279 (1.1994)	16.5481 (1.4661)
q	2.1056 (0.07974)	2.06848 (0.3039)
ν	1.8934 (0.2324)	2.4531 (0.5561)
AIC	7919.872	7913.222

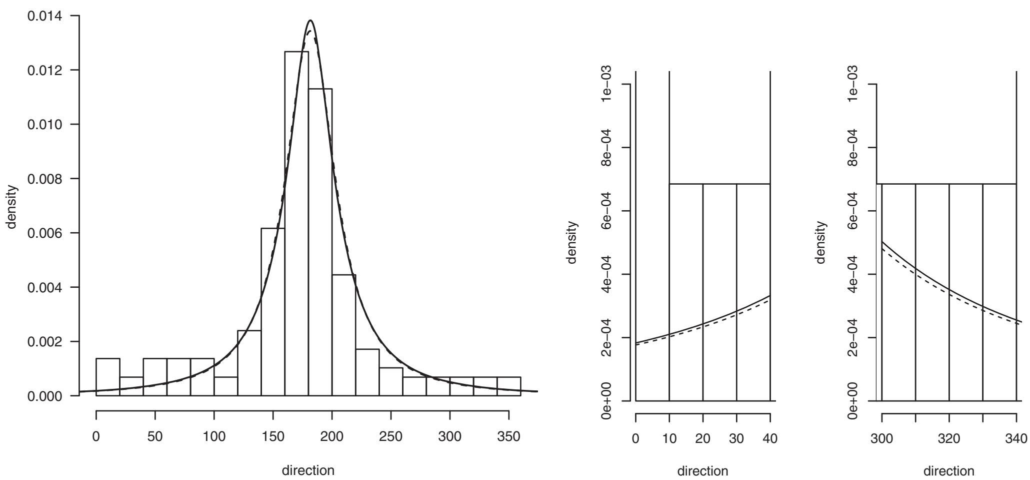

Figure 2. Histogram of initial for 730 ants. Overlaid on top is the MSt density with parameters estimated via ML (solid line) and estimated via ML the St density (dashed line). Right panel depicts tail comparisons for densities MSt and St.

Remark 4. Tables 4 and 5 illustrates the fact that the log-likelihood function is maximum for ν_0 equals 2 and 3 for models St and MSt, respectively. Using theses results we perform maximum likelihood estimation for unknown ν with the estimates given in Table 6 bellow.

Remark 5. Considering Table 6 and using Akaike (1974) criterion, we may conclude that the MSt model fits data better than model St. The profile likelihood for the parameters is illustrated in Figures 3–6 revealing that it has a unique mode (and maximum).

3. The multivariate MSE distribution

We consider a multivariate extension of the slash-elliptical class. We say that $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_k)^\top$ has multivariate-modified slash-elliptical (MMSE) distribution with

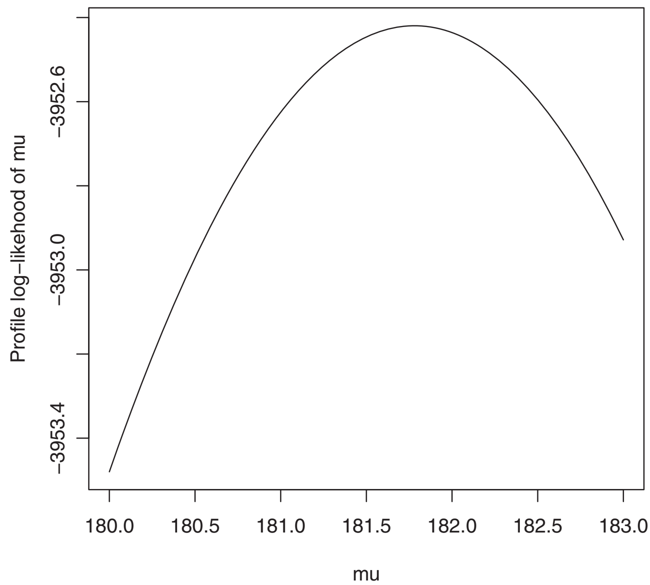

Figure 3. The profile log-likelihood function of μ for the ants data set.

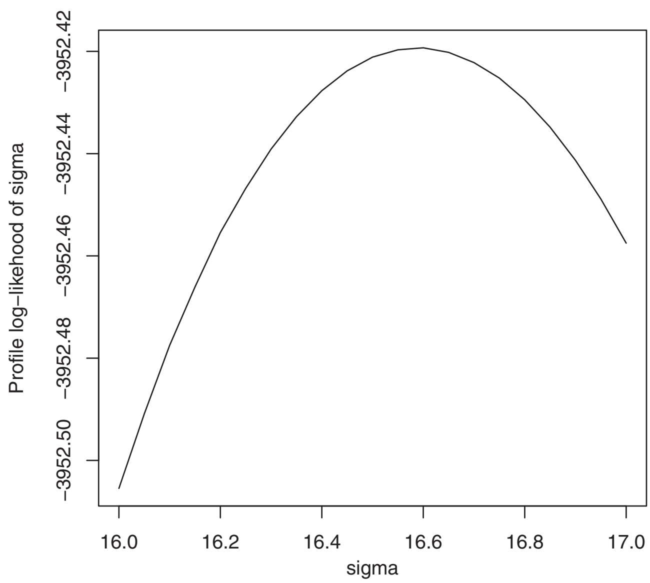

Figure 4. The profile log-likelihood function of σ for the ants data set.

location vector $\boldsymbol{\mu}$, positive definite-scale matrix $\boldsymbol{\Sigma}$ and kurtosis parameter q if it can be represented as

$$\boldsymbol{Y} = \boldsymbol{\Sigma}^{\frac{1}{2}} \frac{\boldsymbol{X}}{U^{\frac{1}{q}}} + \boldsymbol{\mu} \tag{16}$$

where $\boldsymbol{X} \sim EL_k(\mathbf{0}, \mathbf{I}_k; g)$ is independent from $U \sim \exp(2)$. We denote $\boldsymbol{Y} \sim \text{MMSE}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, q; g)$. For the matrix $\boldsymbol{\Sigma} = \left(s_{jj'}\right)_{j=1,\ldots,k}^{j'=1,\ldots,k}$, we assume the parametrization

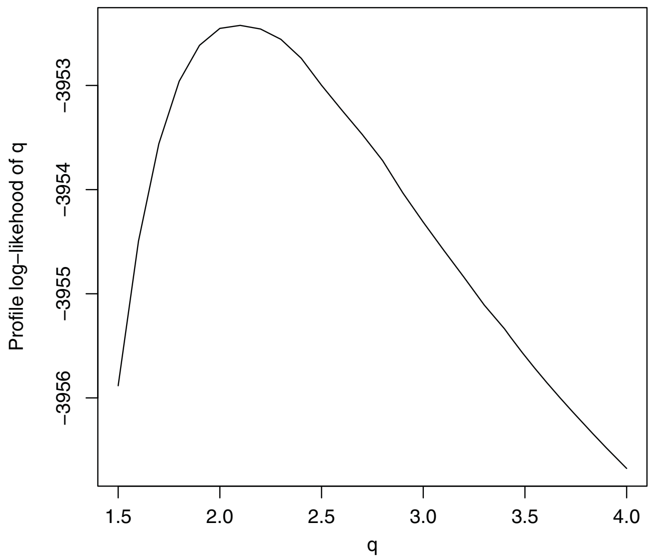

Figure 5. The profile log-likelihood function of q for the ants data set.

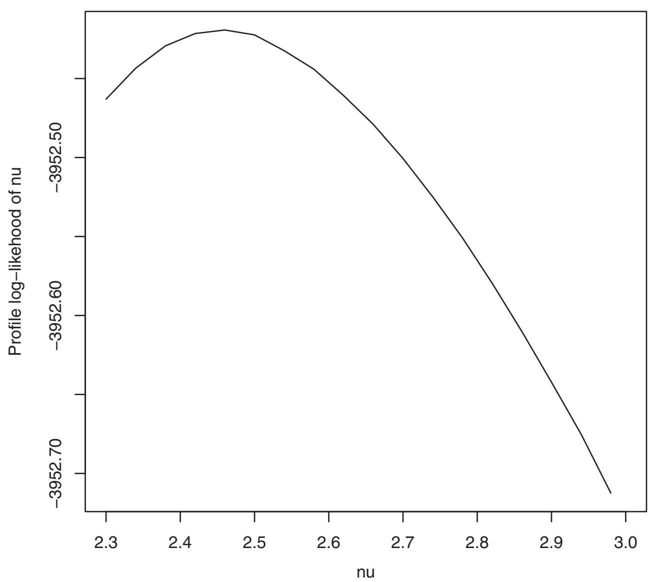

Figure 6. The profile log-likelihood function of ν for the ants data set.

$$\left(s_{jj'}\right)_{j=1,\ldots,k}^{j'=1,\ldots,k} = \begin{cases} \sigma_j^2 & \text{if } j = j' \\ \rho_{jj'}\sigma_j\sigma_{j'} & \text{if } j \neq j' \end{cases}$$

where $\sigma_1^2, \ldots, \sigma_k^2 > 0$ and $\rho_{12}, \rho_{13}, \ldots, \rho_{(k-1)k} \in (-1, 1)$.

The density function of $\boldsymbol{Y}$ is given in the following Proposition.

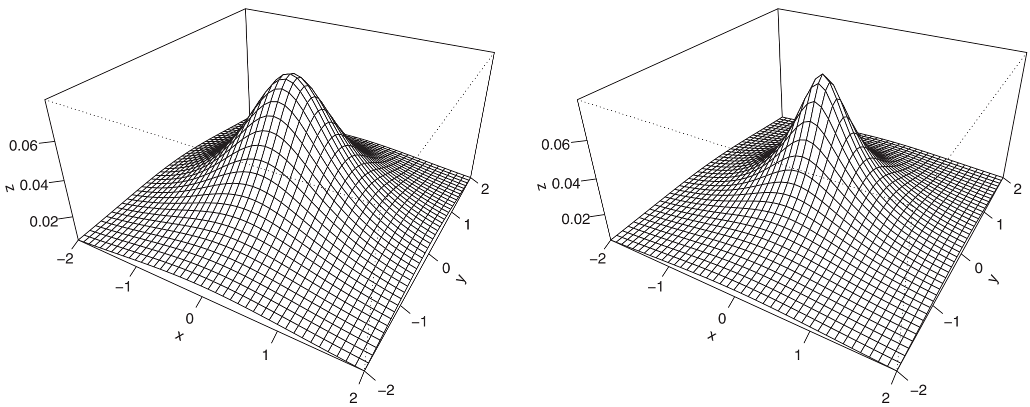

Figure 7. Density function for the BMS and BMSC (right).

Proposition 6. *Let* $\mathbf{Y} \sim \text{MMSE}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, q; g)$. *Then, the density function of* $\mathbf{Y}$ *is*

$$f(\mathbf{y}) = 2|\boldsymbol{\Sigma}|^{-\frac{1}{2}} \int_0^\infty t^{\frac{k}{q}} e^{-2t} g[(\mathbf{y}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu}) t^{\frac{2}{q}}]\, \mathrm{d}t$$

Proof. From Equation (16) and using the transformation

$$\boldsymbol{Y} = \boldsymbol{\Sigma}^{\frac{1}{2}} \frac{\boldsymbol{X}}{U^{\frac{1}{q}}} + \boldsymbol{\mu}$$
$$T = U$$

it is straightforward that the joint density of $(\boldsymbol{Y}, T)$ based on the Jacobian is

$$f_{(\boldsymbol{Y},T)}(\mathbf{y}, t) = 2|\boldsymbol{\Sigma}|^{-\frac{1}{2}} t^{\frac{k}{q}} e^{-2t} g[(\mathbf{y}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu}) t^{\frac{2}{q}}]$$

Integrating in relation to t the marginal density of $\boldsymbol{Y}$ is obtained. □

Figure 7 presents the graphics of bivariate density for the canonical bivariate modified slash (BMS) and canonical bivariate modified slash Cauchy (BMSC) distributions.

Proposition 7. *Let* $\mathbf{Y} \sim \text{MMSE}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, q; g)$ *hence, it follows that*

$$E(\boldsymbol{Y}) = \boldsymbol{\mu} \quad \textit{and} \quad \text{Var}(\boldsymbol{Y}) = 2^{2/q}\Gamma\left(\frac{q-2}{q}\right)\boldsymbol{\Sigma} a_1, \quad q > 2$$

Proof. The proof is similar to that of Proposition 3 considering the representation in Equation (16). □

3.1. Inference

3.1.1. Moment estimators

Proposition 8. *For a fixed* q, *the moment estimators for* $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n \sim \text{MMSE}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, q; g)$ *are given by*

$$\widetilde{\boldsymbol{\mu}} = \overline{\boldsymbol{Y}}$$

$$\widetilde{\boldsymbol{\Sigma}} = \frac{2^{-2/q}\boldsymbol{S}}{a_1\Gamma\left((q-2)/q\right)}, \quad q > 2$$

where $\overline{\boldsymbol{Y}}$ and $\boldsymbol{S}$ are the sample mean and variance.

Proof. Similar to Proposition 5. □

3.1.2. Maximum likelihood estimation

Let $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n$ a random sample from the distribution MMSE$(\boldsymbol{\mu}, \boldsymbol{\Sigma}, q; g)$. The log-likelihood function is given by

$$l(\mu, \boldsymbol{\Sigma}, q) = n\log(2) - \frac{n}{2}\log|\boldsymbol{\Sigma}| + \sum_{i=1}^{n}\log G(\boldsymbol{y}_i)$$

where $G(\boldsymbol{y}_i) = G(\boldsymbol{y}_i; \boldsymbol{\mu}, \boldsymbol{\Sigma}, q) = \int_0^\infty t^{\frac{1}{q}} e^{-2t} g\left[(\boldsymbol{y}_i - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{y}_i - \boldsymbol{\mu})t^{\frac{2}{q}}\right] \mathrm{d}t$. The maximum likelihood equations are given by

$$\sum_{i=1}^{n}\frac{G_1(\boldsymbol{y}_i)}{G(\boldsymbol{y}_i)} = 0 \tag{17}$$

$$\sum_{i=1}^{n}\frac{G_{2j}(\boldsymbol{y}_i)}{G(\boldsymbol{y}_i)} = \frac{n}{2|\boldsymbol{\Sigma}|}\frac{\partial|\boldsymbol{\Sigma}|}{\partial\eta_j} \tag{18}$$

$$\sum_{i=1}^{n}\frac{G_3(\boldsymbol{y}_i)}{G(\boldsymbol{y}_i)} = 0 \tag{19}$$

where $G_1(\boldsymbol{y}_i) = \frac{\partial G(\boldsymbol{y}_i;\boldsymbol{\mu},\boldsymbol{\Sigma},q)}{\partial\boldsymbol{\mu}}$, $G_{2j}(\boldsymbol{y}_i) = \frac{\partial G(\boldsymbol{y}_i;\boldsymbol{\mu},\boldsymbol{\Sigma},q)}{\partial\eta_j}$, $\eta = (\sigma_1^2, \ldots, \sigma_k^2, \rho_{12}, \ldots, \rho_{(k-1)k})$, and $G_3(\boldsymbol{y}_i) = \frac{\partial G(\boldsymbol{y}_i;\boldsymbol{\mu},\boldsymbol{\Sigma},q)}{\partial q}$. Note that

$$G_1(\boldsymbol{y}_i) = -2(\boldsymbol{y}_i - \boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}\int_0^\infty t^{\frac{3}{q}}e^{-2t}g'\left[v_i t^{\frac{2}{q}}\right]\mathrm{d}t$$

$$G_{2j}(\boldsymbol{y}_i) = (\boldsymbol{y}_i - \boldsymbol{\mu})^\top\left(\frac{\partial\boldsymbol{\Sigma}^{-1}}{\partial\eta_j}\right)(\boldsymbol{y}_i - \boldsymbol{\mu})\int_0^\infty t^{\frac{3}{q}}e^{-2t}g'\left[v_i t^{\frac{2}{q}}\right]\mathrm{d}t$$

$$G_3(\boldsymbol{y}_i) = -\frac{1}{q^2}\int_0^\infty t^{\frac{1}{q}}e^{-2t}\log(t)\left\{g\left[v_i t^{\frac{2}{q}}\right] + 2t^{\frac{2}{q}}v_i g'\left[v_i t^{\frac{2}{q}}\right]\right\}\mathrm{d}t$$

where $v_i = (\boldsymbol{y}_i - \boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}_i - \boldsymbol{\mu})$. Equations (17)–(19) can be solved using, for instance, the Newton–Raphson algorithm. As mentioned in Subsection 2.4.2, the moment estimators can be used to initialize the Newton–Raphson algorithm in the maximum likelihood estimation.

3.2. Simulation study

In this section, we present a simulation study for the BMS distribution with location parameter $\boldsymbol{\mu} = (\mu_1, \mu_2)$, scale matrix $\boldsymbol{\Sigma} = \begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}$, and kurtosis parameter q, considering the following combinations.

Table 7. Recovery parameters study for the BMS distribution based on 1000 replicates.

		$n = 70$		$n = 100$		$n = 200$			$n = 70$		$n = 100$		$n = 200$	
Parameter	true	mean	s.d.	mean	s.d.	mean	s.d.	true	mean	s.d.	mean	s.d.	mean	s.d.
μ_1	−1.0	−0.985	0.171	−1.001	0.149	−1.002	0.098	−1.0	−0.986	0.132	−0.990	0.107	−1.004	0.076
μ_2	2.0	1.997	0.145	1.987	0.112	1.991	0.076	2.0	1.995	0.108	1.998	0.083	2.001	0.059
σ_1^2	0.5	0.586	0.241	0.573	0.173	0.520	0.106	0.5	0.529	0.154	0.492	0.126	0.498	0.088
ρ	0.0	0.004	0.143	−0.008	0.126	0.000	0.091	0.0	0.005	0.131	0.000	0.114	0.000	0.081
σ_2^2	0.3	0.341	0.137	0.330	0.101	0.312	0.065	0.3	0.317	0.108	0.304	0.076	0.296	0.052
q	1.0	1.064	0.111	1.057	0.077	1.016	0.051	3.0	3.152	0.838	3.015	0.682	2.975	0.401
μ_1	−1.0	−1.004	0.241	−1.002	0.197	−0.996	0.138	−1.0	−1.011	0.191	−0.997	0.152	−1.002	0.107
μ_2	2.0	2.011	0.214	1.989	0.165	2.001	0.119	2.0	1.988	0.154	1.987	0.127	1.999	0.090
σ_1^2	1.0	1.112	0.392	1.075	0.321	1.046	0.212	1.0	1.057	0.315	1.017	0.262	1.007	0.175
ρ	0.5	0.502	0.129	0.478	0.100	0.498	0.068	0.5	0.487	0.133	0.490	0.087	0.499	0.061
σ_2^2	0.7	0.769	0.312	0.748	0.225	0.728	0.150	0.7	0.746	0.237	0.713	0.183	0.710	0.123
q	1.0	1.054	0.101	1.038	0.074	1.007	0.047	3.0	3.447	1.521	3.219	1.252	3.070	0.408
μ_1	−4.0	−3.973	0.155	−3.969	0.143	−3.973	0.114	−4.0	−3.997	0.119	−3.992	0.107	−3.988	0.076
μ_2	1.0	0.987	0.142	0.986	0.112	0.992	0.107	1.0	1.004	0.109	1.001	0.083	0.998	0.059
σ_1^2	0.5	0.563	0.223	0.557	0.181	0.507	0.121	0.5	0.541	0.176	0.524	0.131	0.500	0.088
ρ	0.0	−0.002	0.147	−0.006	0.127	−0.011	0.104	0.0	−0.002	0.159	0.006	0.114	−0.003	0.082
σ_2^2	0.3	0.341	0.126	0.338	0.109	0.306	0.073	0.3	0.332	0.120	0.312	0.078	0.300	0.053
q	1.0	1.076	0.117	1.059	0.082	1.006	0.058	3.0	3.469	1.101	3.298	0.831	3.021	0.408
μ_1	−4.0	−3.948	0.250	−3.993	0.202	−3.965	0.147	−4.0	−4.008	0.215	−3.996	0.151	−3.994	0.107
μ_2	1.0	1.003	0.219	0.991	0.167	0.995	0.117	1.0	0.991	0.181	0.984	0.127	0.997	0.089
σ_1^2	1.0	1.154	0.441	1.095	0.339	1.056	0.225	1.0	1.051	0.310	1.018	0.251	1.011	0.176
ρ	0.5	0.485	0.127	0.494	0.103	0.497	0.068	0.5	0.498	0.120	0.492	0.086	0.495	0.061
σ_2^2	0.7	0.796	0.324	0.752	0.234	0.736	0.153	0.7	0.762	0.294	0.711	0.176	0.706	0.123
q	1.0	1.056	0.118	1.043	0.080	1.026	0.054	3.0	3.243	0.797	3.156	0.649	3.057	0.404

- Two vectors of means: $(\mu_1, \mu_2) = (-1.0, 2.0)$ and $(\mu_1, \mu_2) = (-4.0, 1.0)$.
- Two scale matrices: $(\sigma_1^2, \sigma_2^2, \rho) = (0.5, 0.3, 0.0)$ and $(\sigma_1^2, \sigma_2^2, \rho) = (1.0, 0.7, 0.5)$.
- Two kurtosis parameter: $q = 1$ and $q = 3$.
- Three sample sizes: $n = 70$, $n = 100$, and $n = 200$.

For each scenario, the data were simulated based on the stochastic representation given in Equation (16) and applied the maximum likelihood procedure explained in Subsection 3.1.2. Codes for this simulation was included as supplementary material. The main focus of the study is to evaluate the recovery of the parameters. Table 7 shows those results. On the other hand, we also consider $\mu_1 = \mu_2 = 0$, $\sigma_1^2 = \sigma_2^2 = 1$, $q = 1$ and different values for ρ and the sample size. Those results are presented in Table 8. Note that the bias is small and decreases as n is increased. On the other hand, the standard error also decreases as n is increasing. Both suggesting, as expected, that the estimators of all parameters are consistent.

3.3. Application 2

In this subsection, we present an application to a real-data set of a bivariate nature. It consists of 86 concentration measurements of the minerals rubidium and strontium made by the Mining Department, University of Atacama, Chile, in soil samples. Table 9 below presents a descriptive summary of the data. Note the high descriptive value of the sample kurtosis for the Rubidium variable. Sample means and variances (var) are also shown.

We fitted bivariate slash (BS) and bivariate MS (BMS) models for this data, considering the parametrization for Sigma mentioned in the previous subsection. Considering $q_0 = 3$,

Table 8. Recovery parameters study for the BMS distribution based on 1000 replicates.

		$n=70$		$n=100$		$n=200$			$n=70$		$n=100$		$n=200$	
Parameter	true	mean	s.d.	mean	s.d.	mean	s.d.	true	mean	s.d.	mean	s.d.	mean	s.d.
μ_1	0	0.017	0.235	0.006	0.196	0.007	0.139	0	0.012	0.241	0.003	0.202	0.000	0.140
μ_2	0	−0.019	0.235	−0.004	0.196	−0.005	0.140	0	−0.001	0.242	0.001	0.202	0.005	0.141
σ_1^2	1	1.033	0.370	1.002	0.300	0.991	0.195	1	1.092	0.406	1.083	0.339	1.039	0.222
ρ	−0.9	−0.886	0.037	−0.889	0.030	−0.892	0.021	0.3	0.298	0.139	0.295	0.117	0.299	0.083
σ_2^2	1	1.040	0.373	1.003	0.298	0.994	0.196	1	1.091	0.403	1.088	0.339	1.038	0.222
q	1	1.056	0.098	1.028	0.074	1.005	0.049	1	1.053	0.094	1.049	0.077	1.013	0.050
μ_1	0	0.000	0.257	−0.013	0.196	0.005	0.139	0	0.001	0.249	−0.003	0.202	−0.005	0.138
μ_2	0	0.002	0.254	0.009	0.196	0.003	0.139	0	0.009	0.246	0.008	0.202	−0.001	0.140
σ_1^2	1	1.101	0.420	1.052	0.310	1.043	0.204	1	1.105	0.400	1.059	0.326	1.038	0.215
ρ	−0.7	−0.695	0.080	−0.695	0.067	−0.697	0.048	0.5	0.499	0.115	0.498	0.097	0.498	0.068
σ_2^2	1	1.106	0.423	1.051	0.310	1.041	0.205	1	1.097	0.399	1.076	0.329	1.032	0.213
q	1	1.055	0.100	1.033	0.073	1.013	0.049	1	1.064	0.094	1.040	0.076	1.014	0.051
μ_1	0	0.006	0.244	−0.005	0.203	0.004	0.141	0	−0.007	0.241	−0.008	0.217	−0.002	0.138
μ_2	0	−0.011	0.243	0.004	0.203	−0.005	0.140	0	−0.012	0.240	−0.002	0.211	−0.003	0.138
σ_1^2	1	1.103	0.402	1.087	0.330	1.033	0.211	1	1.079	0.393	1.074	0.328	1.035	0.213
ρ	−0.5	−0.493	0.117	−0.499	0.097	−0.494	0.070	0.7	0.696	0.080	0.699	0.072	0.700	0.050
σ_2^2	1	1.099	0.399	1.083	0.330	1.040	0.213	1	1.084	0.395	1.068	0.327	1.031	0.213
q	1	1.053	0.094	1.047	0.076	1.008	0.048	1	1.059	0.095	1.038	0.075	1.013	0.052
μ_1	0	−0.009	0.240	0.001	0.199	−0.002	0.139	0	−0.003	0.239	−0.005	0.192	0.001	0.136
μ_2	0	−0.009	0.240	0.007	0.201	−0.003	0.139	0	−0.005	0.237	−0.008	0.193	0.001	0.136
σ_1^2	1	1.079	0.402	1.077	0.338	1.021	0.209	1	1.024	0.382	1.003	0.285	1.003	0.203
ρ	−0.3	−0.297	0.138	−0.300	0.117	−0.296	0.082	0.9	0.887	0.037	0.887	0.030	0.892	0.020
σ_2^2	1	1.083	0.401	1.080	0.340	1.023	0.210	1	1.017	0.378	1.007	0.288	1.006	0.201
q	1	1.049	0.092	1.041	0.079	1.007	0.048	1	1.044	0.095	1.031	0.073	1.012	0.051

Table 9. Descriptive summary for the mineral data.

	mean	var	kurtosis
Rubidium	88.57	3076.74	13.75
Strontium	407.08	63474.99	3.93

Table 10. Results of fitting models BS and BMS for the mineral data.

	BS (SD)	BMS (SD)
μ_1	82.182 (2.952)	82.614 (2.466)
μ_2	406.428 (23.628)	402.338 (24.231)
σ_1^2	553.425 (45.445)	356.155 (30.561)
σ_2^2	32531.333 (44.588)	32597.037 (29.724)
ρ	−0.726 (0.024)	−0.739 (0.023)
q	1.145 (0.060)	1.404 (0.060)
AIC	2014.21	2012.54

moment estimators for remaining parameters are given by $\widetilde{\mu}_1 = 88.57$, $\widetilde{\mu}_2 = 407.08$, $\widetilde{\sigma}_1^2 = 723.51_1$, $\widetilde{\sigma}_2^2 = 14,926.34$ to $\widetilde{\rho} = -0.41$, which can be used as initial values for the maximum likelihood estimates. The maximum likelihood estimates and corresponding standard deviations for both models are given in Table 10. Moreover, according to the AIC criterion (Akaike 1974), the model that provides the best fit is the BMS model. Finally, Figure 8 shows the contour plots for the BMS model in mineral data set.

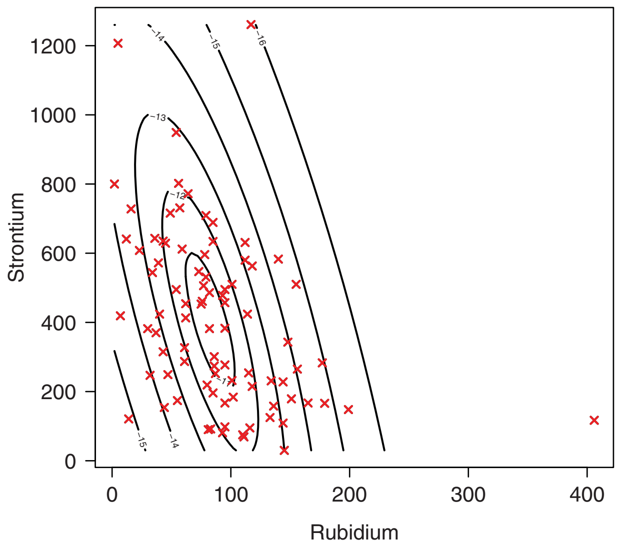

Figure 8. Contour plots for the BMS model in mineral data.

4. Final discussion

In this paper, we introduced a new class of symmetric distributions with heavy tails. The main idea is to extend the work of Reyes, Gómez, and Bolfarine (2013) assuming a more general class of distributions as the elliptical family, instead of the normal distribution. We consider the univariate and multivariate versions of this distribution and explore some properties as moments and kurtosis coefficient. Parameter estimation is discussed based on moments and maximum likelihood estimation. Two real-data set are presented, showing the good performance of our proposition over the ordinary slash-elliptical class of distributions.

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