# Objective Bayesian analysis for the Lomax distribution

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**Abstract**

In this paper, we propose to make Bayesian inferences for the parameters of the Lomax distribution using non-informative priors, namely the (dependent and independent) Jeffreys prior and the reference prior. We assess Bayesian estimation through a Monte Carlo study with 10,000 simulated datasets. In order to evaluate the possible impact of prior specification on estimation, two criteria were considered: the mean relative error and the mean square error. An application on a real dataset illustrates the developed procedures.

*Keywords:* Jeffreys prior, Lomax distribution, Objective Bayesian analysis, Reliability.
*2010 MSC:* 62F15, 62N05

## 1. Introduction

The Lomax distribution [1], also known as the Pareto Type II distribution (or simply Pareto II), is a heavy-tailed probability distribution often used in business, economics, and actuarial modeling. It is essentially a Pareto distribution that has been shifted so that its support begins at zero [2]. The Lomax distribution has been applied in a variety of contexts ranging from modeling the survival times of patients after a heart transplant [3] to the sizes of computer files on servers [4]. Some authors, such as [5], suggest the use of this distribution as an alternative to the exponential distribution when data are heavy-tailed.

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The primary goal of this paper is to consider the Lomax distribution to describe reliability data. In order to achieve that we consider a Bayesian inference for the parameters of the Lomax distribution using objective priors, namely the Jeffreys prior [6] and the reference prior [7]. The obtained priors are improper and may return improper posteriors, which is undesirable. Therefore, we provide sufficient conditions for the obtained posteriors being proper. We prove that Jeffreys prior leads to a proper posterior while the reference prior leads to an improper posterior. We also show how to represent the Lomax distribution in a hierarchical form by augmenting the model with a latent variable, which makes the Bayesian computations easier to implement. This representation would also allow the user to implement inferences using all-purpose Bayesian statistical packages, like WinBUGS [8] or JAGS [9].

In order to evaluate the performance of the Bayes estimators, we present a simulation study to compare the efficiency of the Bayesian approach with the maximum likelihood inference for estimating the model parameters and check for the possible impact of prior specification. Similar studies have been conducted for other distributions [10, 11, 12]. Finally, the proposed methodology is illustrated on a real dataset related to the active repair times (in hours) for an airborne communication transceiver.

The remainder of this paper is organized as follows. In Section 2, we present the Lomax distribution and list some of its properties. In Section 3, we formulate the Bayesian model using non-informative priors. In Section 4, a simulation study is presented. In Section 5, the methodology is illustrated on a real dataset. Some final comments are given in Section 6.

## 2. Model definition

Here, we use the definition that appears, for instance, in [13].

**Definition 2.1.** A continuous random variable $X$ has a Lomax distribution with parameters $\alpha$ and $\beta$ if its probability density function is given by

$$f(x|\beta,\alpha) = \frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)}, \quad x \geq 0, \tag{1}$$

where $\alpha > 0$ and $\beta > 0$ are the shape and scale parameters, respectively.

We refer to this distribution as Lomax $(\beta, \alpha)$. The median is $\beta(2^{1/\alpha}-1)$ and the mode is zero. The hazard function is given by

$$h(x|\beta,\alpha) = \frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-1}, \quad x \geq 0,$$

which is a decreasing function of $x$, thus making this a suitable model for components that age with time. The survival function is given by

$$S(x|\beta,\alpha) = \left(1+\frac{x}{\beta}\right)^{-\alpha}, \quad x \geq 0.$$

We note also that the Lomax distribution can be expressed as a scale mixure of distributions using the following hierarchical form

$$X|\beta,\lambda \sim \text{Exponential}\left(\frac{\lambda}{\beta}\right), \qquad \lambda|\alpha \sim \text{Gamma}(\alpha,1).$$

This result dates back to [14], where mixtures of exponential and Weibull densities are discussed, and will allow complete conditional distributions to be obtained in closed form and easy to sample from. More recently, [15] also provided a scale mixture representation of Pareto-like densities in the context of shrinkage priors in Bayesian analysis. The result follows from writing the joint density of $X$ and $\lambda$ as

$$f(x|\beta,\lambda)f(\lambda|\alpha) = \frac{1}{\beta\Gamma(\alpha)}\lambda^{\alpha}\exp\left\{-\lambda\left(1+\frac{x}{\beta}\right)\right\}.$$

So, the marginal density of $X$ is given by

$$\begin{aligned} f(x|\beta,\alpha) &= \frac{1}{\beta\Gamma(\alpha)}\int_0^{\infty}\lambda^{\alpha}\exp\left\{-\lambda\left(1+\frac{x}{\beta}\right)\right\}d\lambda \\ &= \frac{1}{\beta\Gamma(\alpha)}\Gamma(\alpha+1)\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} = \frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} \end{aligned}$$

and we can conclude that $X \sim \text{Lomax}(\beta,\alpha)$.

Using this mixture representation, it is not difficult to see that the unconditional mean and variance of $X$ are given by

$$\begin{aligned} E(X) &= \beta E[\lambda^{-1}] = \frac{\beta}{\alpha-1}, \quad \alpha>1, \\ Var(X) &= \beta^2\left\{E\left[\lambda^{-1}\right]^2 + Var\left[\lambda^{-1}\right]\right\} = \frac{\alpha\beta^2}{(\alpha-1)^2(\alpha-2)}, \quad \alpha>2, \end{aligned}$$

since $\lambda^{-1} \sim \text{IG}(\alpha,1)$, where $\text{IG}(a,b)$ denotes the Inverse Gamma distribution with parameters $a$ and $b$, mean $b/(a-1)$, $a>1$, and variance $b^2/(a-1)^2(a-2)$, $a>2$.

Now, suppose that $\mathbf{X} = (X_1,\ldots,X_n)'$ is a random sample of size $n$ from the Lomax distribution (1). Since the $X_i$s are conditionally independent given $\alpha$ and $\beta$, the mixing parameters $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_n)'$ are also a priori independent as a consequence. In fact, this is a data-augmentation scheme which also facilitates posterior computation. The complete conditional distribution of $\boldsymbol{\lambda}$ using the hierarchical form above is given by

$$\begin{aligned} f(\boldsymbol{\lambda}|\mathbf{x},\beta,\alpha) &\propto f(\mathbf{x}|\beta,\boldsymbol{\lambda})\ f(\boldsymbol{\lambda}|\alpha) \\ &\propto \prod_{i=1}^{n}\lambda_i\exp\{-\lambda_i x_i/\beta\}\prod_{i=1}^{n}\lambda_i^{\alpha-1}\exp\{-\lambda_i\} \\ &= \prod_{i=1}^{n}\lambda_i^{\alpha}\exp\left\{-\lambda_i\left(1+\frac{x_i}{\beta}\right)\right\}, \end{aligned}$$

in which case, $[\lambda_1, \ldots, \lambda_n | \mathbf{x}, \beta, \alpha] = \prod_{i=1}^{n} [\lambda_i | \mathbf{x}, \beta, \alpha]$. Finally, the complete conditional distribution of each $\lambda_i$ follows as

$$\lambda_i | \mathbf{x}, \boldsymbol{\lambda}_{-i}, \beta, \alpha \sim \text{Gamma}\left(\alpha + 1, 1 + \frac{x_i}{\beta}\right),$$

where $\boldsymbol{\lambda}_{-i}$ represents the vector $\boldsymbol{\lambda}$ without the $i$-th element.

Again, using the hierarchical form, we obtain the complete conditional distributions of $\alpha$ and $\beta$ as

$$f(\alpha | \mathbf{x}, \boldsymbol{\lambda}, \beta) \propto f(\boldsymbol{\lambda} | \alpha)\ \pi(\beta, \alpha) \propto [\Gamma(\alpha)]^{-n} \left(\prod_{i=1}^{n} \lambda_i\right)^{\alpha - 1} \pi(\beta, \alpha), \quad (2)$$

$$f(\beta | \mathbf{x}, \boldsymbol{\lambda}, \alpha) \propto f(\mathbf{x} | \beta, \boldsymbol{\lambda}) \pi(\beta, \alpha) \propto \beta^{-n} \exp\left\{-\frac{1}{\beta} \sum_{i=1}^{n} \lambda_i x_i\right\} \pi(\beta, \alpha). \quad (3)$$

We note that using this representation of the Lomax distribution, each observation $X_i$ is associated with one mixing parameter $\lambda_i$, whose posterior mean or median can be used to identify a possible outlier. A potential outlier would be indicated, for example, if $\lambda_i$ is estimated substantially small, since this would imply that the exponential distribution has an inflated variance [16].

## 3. Prior specification

We now complete the Bayesian model by specifying a prior distribution for $\alpha$ and $\beta$. We consider non-informative priors on these parameters and verify the existence of their posterior distribution.

A commonly used objective prior in Bayesian analysis is Jeffreys prior [6], which is defined as

$$\pi_{\texttt{J}}(\beta, \alpha) \propto |I(\beta, \alpha)|^{1/2},$$

where $I(\cdot)$ stands for the Fisher information matrix. This is given by

$$I(\beta, \alpha) = n \begin{bmatrix} \dfrac{\alpha}{\beta^2(\alpha + 2)} & -\dfrac{1}{\beta(\alpha + 1)} \\ -\dfrac{1}{\beta(\alpha + 1)} & \dfrac{1}{\alpha^2} \end{bmatrix},$$

from which we obtain

$$\pi_{\texttt{J}}(\beta, \alpha) \propto \frac{1}{\beta(\alpha + 1)\alpha^{1/2}(\alpha + 2)^{1/2}}, \quad \beta, \alpha > 0. \quad (4)$$

Considering independence between the parameters, the Jeffreys joint prior for $(\beta, \alpha)$ is given by

$$\pi_{\texttt{IJ}}(\beta, \alpha) \propto \pi(\beta)\pi(\alpha) = \frac{1}{\beta\alpha}, \quad \beta, \alpha > 0. \quad (5)$$

It is worth noting that the independent Jeffreys prior can also be obtained as a particular reference prior. For the derivation of this reference prior, see Supplemental Material.

**Proposition 3.1.** *For any sample size, the posterior distribution under the Jeffreys prior (4) is proper.*

*Proof.* Under Jeffreys prior, the joint posterior density of $\beta$ and $\alpha$ is given by

$$\pi(\beta,\alpha|\mathbf{x}) \propto \frac{\alpha^{n-1/2}\beta^{-(n+1)}}{(\alpha+1)(\alpha+2)^{1/2}}\prod_{i=1}^{n}\left(1+\frac{x_i}{\beta}\right)^{-(\alpha+1)}.$$

We next show that the integral of this expression is finite for any sample size $n$.

First, we solve

$$\int_0^\infty \beta^{-(n+1)}\prod_{i=1}^{n}\left(1+\frac{x_i}{\beta}\right)^{-(\alpha+1)} d\beta. \tag{6}$$

Consider $y=\min\{x_1,\ldots,x_n\}$. Then, it follows that

$$\left(1+\frac{x_i}{\beta}\right)^{\alpha+1} \quad \geq \quad \left(1+\frac{y}{\beta}\right)^{\alpha+1}$$

for all $\alpha \geq 0$, $i=1,\ldots,n$, and therefore

$$\prod_{i=1}^{n}\left(1+\frac{x_i}{\beta}\right)^{-(\alpha+1)} \quad \leq \quad \left(1+\frac{y}{\beta}\right)^{-n(\alpha+1)}.$$

Thus,

$$\int_0^\infty \beta^{-(n+1)}\prod_{i=1}^{n}\left(1+\frac{x_i}{\beta}\right)^{-(\alpha+1)} d\beta \;\leq \int_0^\infty \beta^{-(n+1)}\left(1+\frac{y}{\beta}\right)^{-n(\alpha+1)} d\beta.$$

Now considering the change of variable $u=\phi(\beta)=\frac{y}{\beta}$, $du=|\phi'(\beta)|d\beta=\frac{y}{\beta^2}d\beta$ with $\phi((0,\infty])=(0,\infty]$ by the change of variables formula we have

$$\begin{aligned}\int_0^\infty \beta^{-(n+1)}\left(1+\frac{y}{\beta}\right)^{-n(\alpha+1)} d\beta &= \frac{1}{y^n}\int_0^\infty \frac{u^{n-1}}{(1+u)^{n\alpha+n}}du = \frac{1}{y^n}B(n,n\alpha)\\ &= \frac{1}{y^n}\frac{\Gamma(n)\Gamma(n\alpha)}{\Gamma(n\alpha+n)} = \frac{(n-1)!}{y^n}\frac{1}{\prod_{j=0}^{n-1}(n\alpha+j)}\end{aligned}$$

for all $\alpha>0$, where $B(u,v)$ stands for the beta function, for $u>0$ and $v>0$.

Then,

$$\int_0^\infty \int_0^\infty \frac{\alpha^{n-1/2}\beta^{-(n+1)}}{(\alpha+1)(\alpha+2)^{1/2}} \prod_{i=1}^{n} \left(1+\frac{x_i}{\beta}\right)^{-(\alpha+1)} d\beta d\alpha$$

$$= \frac{(n-1)!}{y^n} \int_0^\infty \frac{\alpha^{n-1/2}}{(\alpha+1)(\alpha+2)^{1/2}} \frac{1}{\prod_{j=0}^{n-1}(n\alpha+j)} d\alpha$$

$$= \frac{(n-1)!}{y^n} \left( \int_0^1 f(\alpha)d\alpha + \int_1^\infty f(\alpha)d\alpha \right),$$

where $f(\alpha) = \dfrac{\alpha^{n-1/2}}{(\alpha+1)(\alpha+2)^{1/2}} \dfrac{1}{\prod_{j=0}^{n-1}(n\alpha+j)}$.

Now, notice that, for $j \geq 1$ and $\alpha > 0$ we have that $(n\alpha + j) \geq 1$ since $n > 0$. Therefore, for all $\alpha > 0$ it follows that

$$\frac{1}{\prod_{j=0}^{n-1}(n\alpha+j)} \leq \frac{1}{n\alpha}.$$

By the same argument it follows that, for $\alpha > 0$, $\frac{1}{(\alpha+1)} \leq 1$ and $\frac{1}{(\alpha+2)^{1/2}} < \frac{1}{\sqrt{2}}$. Combining all these inequalities we have that, since $n \geq 1$,

$$\int_0^1 f(\alpha)d\alpha \leq \frac{1}{n\sqrt{2}} \int_0^1 \alpha^{n-3/2} d\alpha = \frac{1}{n(n-1/2)\sqrt{2}}.$$

On the other hand, we have that, for $j \geq 0$, $(n\alpha + j) \geq n\alpha$. Therefore, for all $\alpha > 0$ it follows that

$$\frac{1}{\prod_{j=0}^{n-1}(n\alpha+j)} \leq \frac{1}{n^n}\frac{1}{\alpha^n}.$$

By the same argument it follows that, for $\alpha > 0$, $\frac{1}{\alpha+1} \leq \frac{1}{\alpha}$ and $\frac{1}{(\alpha+2)^{1/2}} \leq \frac{1}{\alpha^{1/2}}$. Combining all these inequalities we have that

$$\int_1^\infty f(\alpha)d\alpha \leq \frac{1}{n^n} \int_1^\infty \alpha^{-2} d\alpha = \frac{1}{n^n}.$$

Therefore,

$$\frac{(n-1)!}{y^n} \left( \int_0^1 f(\alpha)d\alpha + \int_1^\infty f(\alpha)d\alpha \right) \leq \frac{(n-1)!}{y^n} \left( \frac{1}{n(n-1/2)\sqrt{2}} + \frac{1}{n^n} \right) < \infty$$

and we can conclude that the posterior distribution using Jeffreys prior is proper for $n \geq 1$. □

**Proposition 3.2.** *For any sample size, the posterior distribution under the independent Jeffreys/reference prior (5) is improper.*

*Proof.* Using an independent Jeffreys/reference prior, the joint posterior density of $\beta$ and $\alpha$ is given by

$$\pi(\beta,\alpha|\mathbf{x}) \propto \alpha^{n-1}\beta^{-(n+1)}\prod_{i=1}^{n}\left(1+\frac{x_i}{\beta}\right)^{-(\alpha+1)}.$$

We next verify whether the integral of this expression is finite.

Note that, by the analogous arguments and the same change of variables used in Jeffreys prior, we have that, for $w=\max\{x_1,\ldots,x_n\}$,

$$\begin{aligned}\int_0^\infty \beta^{-(n+1)}\prod_{i=1}^{n}\left(1+\frac{x_i}{\beta}\right)^{-(\alpha+1)} d\beta &\geq \int_0^\infty \beta^{-(n+1)}\prod_{i=1}^{n}\left(1+\frac{w}{\beta}\right)^{-n(\alpha+1)} d\beta\\ &= \frac{1}{w^n}\frac{\Gamma(n)\Gamma(n\alpha)}{\Gamma(n\alpha+n)} = \frac{(n-1)!}{w^n}\frac{1}{\prod_{j=0}^{n-1}(n\alpha+j)}\end{aligned}$$

and thus

$$\begin{aligned}\int_0^\infty\int_0^\infty \alpha^{n-1}\beta^{-(n+1)}\prod_{i=1}^{n}\left(1+\frac{x_i}{\beta}\right)^{-(\alpha+1)} d\beta d\alpha &\geq \frac{(n-1)!}{w^n}\int_0^\infty \frac{\alpha^{n-1}}{\prod_{j=0}^{n-1}(n\alpha+j)}d\alpha\\ &\geq \frac{(n-1)!}{w^n}\int_1^\infty \frac{\alpha^{n-1}}{\prod_{j=0}^{n-1}(n\alpha+j)}d\alpha.\end{aligned}$$

Now, for $\alpha\geq 1$ and $j<n$ we have that $n\alpha+j<n\alpha-\alpha+n=n(\alpha+1)\leq n2\alpha$ since $\alpha+1\leq 2\alpha$ for $\alpha\geq 1$. Therefore,

$$\frac{1}{\prod_{j=0}^{n-1}(n\alpha+j)} \geq \frac{1}{(2n)^n}\frac{1}{\alpha^n} \tag{7}$$

for all $\alpha\geq 1$ and thus

$$\frac{(n-1)!}{w^n}\int_1^\infty \frac{\alpha^{n-1}}{\prod_{j=0}^{n-1}(n\alpha+j)}d\alpha \geq \frac{(n-1)!}{(2wn)^n}\int_1^\infty \alpha^{-1}d\alpha=\infty.$$

Hence, the posterior distribution using reference prior is improper for $n\geq 1$. □

Let us recall the hierarchical form for the Lomax distribution, presented in Section 2, and derive alternative posterior characterizations. Then, substituting $\pi(\beta,\alpha)$ in the expressions for the complete conditional densities (2)-(3), we obtain

$$f(\alpha|\mathbf{x},\boldsymbol{\lambda},\beta) \propto \frac{1}{(\alpha+1)\alpha^{1/2}(\alpha+2)^{1/2}[\Gamma(\alpha)]^n}\left(\prod_{i=1}^{n}\lambda_i\right)^{\alpha-1}$$

for the dependent Jeffreys prior and

$$f(\alpha|\mathbf{x},\boldsymbol{\lambda},\beta) \propto \frac{1}{\alpha[\Gamma(\alpha)]^n}\left(\prod_{i=1}^{n}\lambda_i\right)^{\alpha-1}$$

for the independence case. So, the complete conditional distribution of $\alpha$ is not of standard form and a Metropolis-Hastings algorithm [17] is used for sampling its values. Likewise, the complete conditional density of $\beta$ is given by

$$f(\beta|\mathbf{x}, \boldsymbol{\lambda}, \alpha) \propto \beta^{-(n+1)} \exp\left\{-\frac{1}{\beta}\sum_{i=1}^{n}\lambda_i x_i\right\},$$

then it follows that

$$\beta|\mathbf{x}, \boldsymbol{\lambda}, \alpha \sim \mathrm{IG}\left(n, \sum_{i=1}^{n}\lambda_i x_i\right).$$

## 4. Simulation study

In this section, we perform a Monte Carlo study to evaluate the methodology described in the previous section. We generated $N = 10,000$ replications of samples of sizes $n = (20, 30, \ldots, 250)$ from the Lomax distribution, considering parameter values $(\beta = 2, \alpha = 1.5)$ and $(\beta = 3, \alpha = 0.5)$. The model was then estimated using the posterior obtained from Jeffreys priors. We used the Metropolis-Hastings algorithm implemented in software R to simulate two chains of values from the posterior distribution. A total of 5,500 iterations with

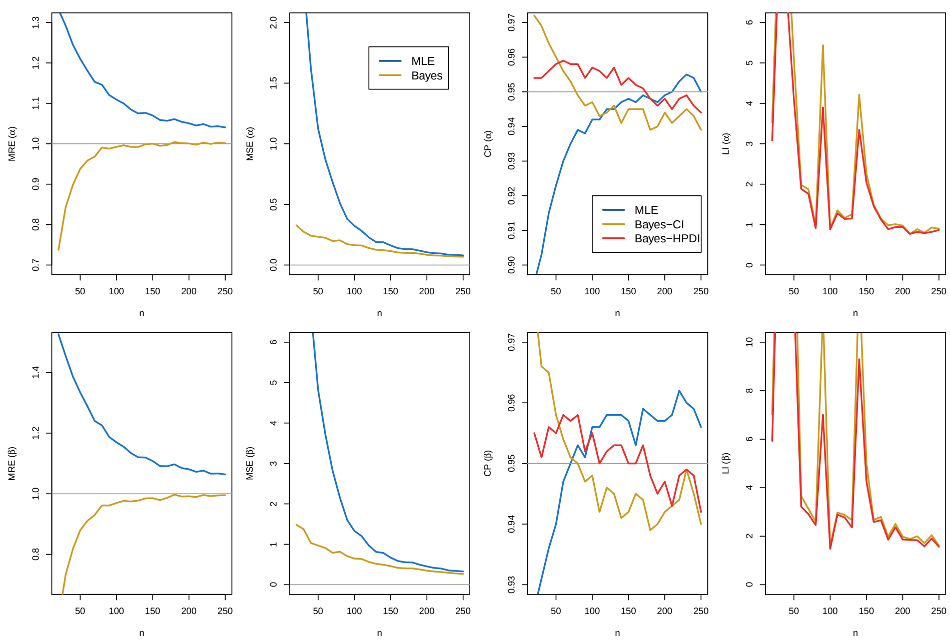


Figure 1: MRE, MSE, CP and LI for the estimates of $\beta = 2$ and $\alpha = 1.5$, for $N = 10,000$ simulated samples of size $n$, and using the MLE and the Bayes estimators.

jumps of 5 and a burn-in of 500 were performed for each chain, thus leading to a final sample of $1,000$ values for each chain. Also, the Geweke convergence diagnostic provided in the R package CODA was used to monitor convergence of the two parallel chains.

Let $\hat{\theta}^{(j)}$ be the estimate of parameter $\theta$ for the $j$-th replication, $j = 1, \ldots, N$. These are the parameter posterior modes calculated from the $10,000$ simulated values for each replication. In order to evaluate the estimation method, two criteria were considered: the mean relative error (MRE) and the mean square error (MSE), which are defined as

$$\mathrm{MRE} = \frac{1}{N}\sum_{j=1}^{N}\frac{\hat{\theta}^{(j)}}{\theta} \quad \text{and} \quad \mathrm{MSE} = \frac{1}{N}\sum_{j=1}^{N}\left(\hat{\theta}^{(j)} - \theta\right)^2.$$

The estimated coverage probability (CP) of the 95% asymptotic confidence intervals, the 95% credibility intervals (CI) and the 95% highest posterior density intervals (HPDI) is also presented. Additionally, we show the length of the obtained Bayesian intervals (LI).

Under this approach, the best estimators will show MRE closer to one and MSE closer to zero. From the classical approach we obtained the CP based on the asymptotic confidence intervals, while from the Bayesian method we obtained it from both 95% CI and 95% HPDI, where the latter were computed using the R package HDInterval. For a large number of experiments considering a 95% credibility level, the frequencies of intervals that covered the true values of $\theta$ should be closer to 95%.

The results from the simulated experiment appear in Figures 1-2. We note that the Bayesian approach returned superior results when compared to the classical maximum likelihood estimation (MLE) method. Concerning accuracy, we obtained adequate results for $\beta$ and $\alpha$ regarding MRE and MSE using the Jeffreys prior. The length of the HPDI showed to be smaller than the obtained by the CI. Moreover, the produced HPDI returned better CP, especially for small sample sizes. Overall, the results obtained for the sample sizes and parameter values considered allow us to recommend the Bayesian approach to make inferences on the parameters of the Lomax distribution.

## 5. Application

In order to illustrate the methodology proposed in this paper, we consider a dataset related to the active repair times (in hours) for an airborne communication transceiver. The equipment was only observed during the time of its active operation. This dataset was first described by Von Alven [18] and available in Table 1.

Figure 3 shows the survival function fitted by different probability distributions (Weibull, Gamma, and Lomax). It can be observed that the Lomax distribution returned a good fit for the considered dataset.

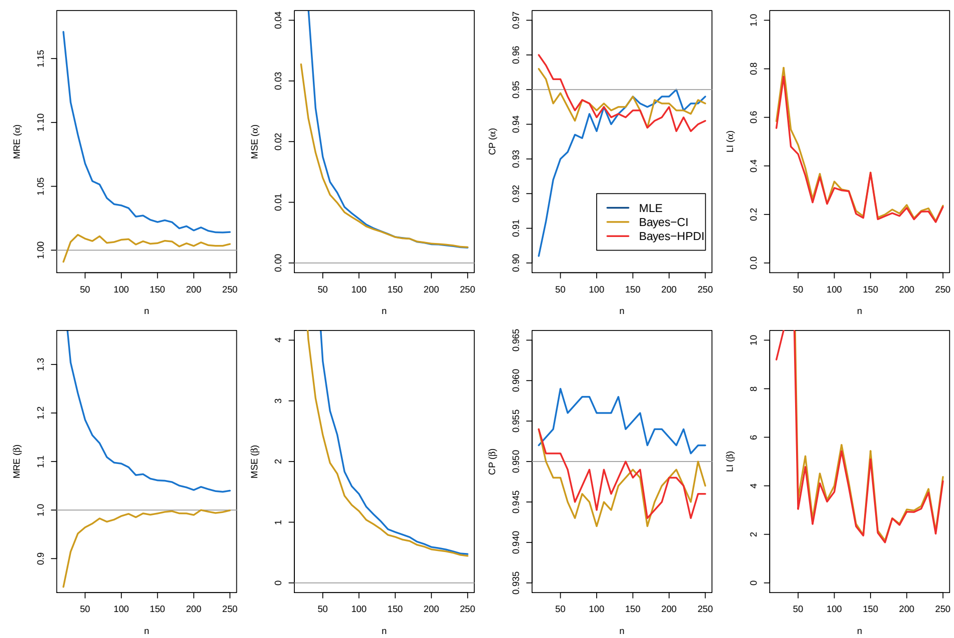


Figure 2: MRE, MSE, CP and LI for the estimates of $\beta = 3$ and $\alpha = 0.5$, for $N = 10,000$ simulated samples of size $n$, and using the MLE and the Bayes estimators.

Table 1: Dataset related to repair times for an airborne communication transceiver ([18]).

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.2 | 0.3 | 0.5 | 0.5 | 0.5 | 0.5 | 0.6 | 0.6 | 0.7 | 0.7 | 0.7 | 0.8 |
| 0.8 | 1.0 | 1.0 | 1.0 | 1.0 | 1.1 | 1.3 | 1.5 | 1.5 | 1.5 | 1.5 | 2.0 |
| 2.0 | 2.2 | 2.5 | 2.7 | 3.0 | 3.0 | 3.3 | 3.3 | 4.0 | 4.0 | 4.5 | 4.7 |
| 5.0 | 5.4 | 5.4 | 7.0 | 7.5 | 8.8 | 9.0 | 10.3 | 22.0 | 24.5 | | |

In order to discriminate among the candidate models, we consider the results of AIC [19] and AICc [20] for these models. As it can be seen in Table 2, the Lomax distribution showed the best fit, with the lowest AIC and AICc values.

The figure available in the Supplemental Material contains the trace, autocorrelation and density plots of the marginal posterior samples for $\alpha$ and $\beta$ generated from Markov chain Monte Carlo (MCMC) methods. In this case, we observed that the chains converged to the target marginal posterior distributions, which is also confirmed through the Geweke's diagnostic criterion [21]. The $z$-scores for $\alpha$ and $\beta$ are, respectively, 0.694 and 0.907, which are smaller than 1.98 (assuming a significance level of 5%). The results of the simulated chains can be assumed to be samples of the marginal posterior distributions for the parameters of the Lomax distribution.

The Bayes estimates of the parameters $\alpha$ and $\beta$ of the Lomax distribution are presented in Table 3.

Therefore, through the proposed methodology, the data related to the ac-

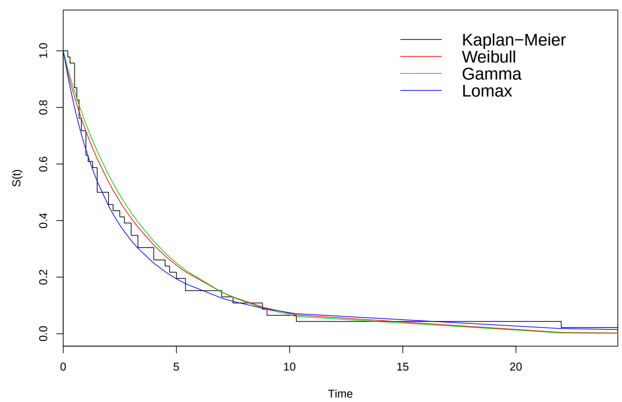


Figure 3: The fitted survival functions superimposed to the empirical survival function, considering the dataset related to repair times for an airborne communication transceiver.

Table 2: The AIC and AICc values for all fitted distributions, considering the airborne communication transceiver data.

| | Weibull | Gamma | Lomax |
|---|---|---|---|
| AIC | 212.94 | 213.86 | **210.70** |
| AICc | 213.22 | 214.14 | **210.98** |

Table 3: Maximum a posteriori (MAP) estimates, standard deviations (SD), 95% CI and 95% HPDI for the parameters $\alpha$ and $\beta$ of the Lomax distribution, considering the airborne communication transceiver data.

| Parameter | MAP | SD | 95% CI | 95% HPDI |
|---|---|---|---|---|
| $\alpha$ | 2.3939 | 2.1164 | (1.2264; 9.1911) | (0.9769; 8.0083) |
| $\beta$ | 5.0882 | 7.0625 | (2.3238; 29.1653) | (1.2850; 24.1780) |

tive repair times (in hours) for an airborne communication transceiver can be well-described by the Lomax distribution, considering the objective Bayesian inference to obtain the parameter estimates of $\alpha$ and $\beta$.

## 6. Concluding remarks

In this paper, we considered the Bayesian method to estimate the parameters of a Lomax distribution under two non-informative prior specifications. We showed that the Jeffreys prior returned a proper posterior, while the reference prior returned an improper posterior and should not be used for the Lomax model. We also obtained a scale mixture representation of the Lomax distribution, in which the complete conditional distribution of the scale parameter is of known closed form and easy to sample. As a by-product, this representation allows for the mixing parameters to be used to identify possible outliers.

The obtained results indicated that the Bayesian approach outperforms the MLE in terms of smaller bias and MSE under the Jeffreys prior specification. In fact, for samples of size 50, we obtained Bayes estimates almost without bias, i.e., with MRE closer to one. Moreover, we observed that the marginal densities are achieved without high computational cost using the MCMC techniques. On the other hand, the densities of the posterior distributions of $\alpha$ and $\beta$ have a very heavy right tail, which contributes to the intervals being wide. In this case, the posterior mean is not a good choice, which leads us to consider the posterior mode as the Bayes estimator. Although the intervals are wide, the simulation results suggested that one typically only needs a few data (at least about 30 observations) to obtain accurate credibility intervals (using the HPDI) for the parameters of the Lomax distribution, while the MLE-based approach does not return good coverage probabilities, especially for small samples.

There are many further possible extensions of this paper. For instance, the presence of censoring, covariates and long-term survivals are quite common in practical situations. Our approach should thus be investigated further in these settings.

**Acknowledgements**

The authors are grateful to the Editor and the two reviewers for their helpful and useful comments that improved the manuscript. Pedro L. Ramos is grateful to the São Paulo State Research Foundation (FAPESP Proc. 2017/25971-0).

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