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Reliability Analysis of Rock Slope with Planar Failure Using the Direct Coupling Approach

Gian Franco Napa-García
Department of Geotechnical Engineering EESC-USP, São Carlos SP, Brazil, gnapa@sc.usp.br

André Teófilo Beck
Department of Structural Engineering EESC-USP, São Carlos SP, Brazil, atbeck@sc.usp.br

Tarcisio Barreto Celestino
Department of Geotechnical Engineering EESC-USP, São Carlos SP, and Themag Engenharia, São Paulo SP, Brazil, tbcelest@usp.br

SUMMARY: Risks involved in engineering projects need to be quantified objectively. Therefore, a keystone to this process is the quantification of the probabilities of failure of the single and combined failure modes. In geotechnical engineering, the most common technique used to calculate the probability of failure of an engineering system is the Monte Carlo simulation MCS. This technique leads to the exact answer when the number of realizations tends to infinite. This condition can be a handicap. To overcome this handicap, the direct coupling surge as novel technique that can lead with complex scenarios in an efficient way. Thus, this paper presents an application of the direct coupling method to estimate the probabilities of failure of a rock slope analyzed by Hoek (2007). Hoek (2007) performed MCS to obtain the probability of failure of that slope. The direct coupling was implemented in a Mathematica 9.0 version of the StRAnD. Also, the analytical formulation of the problem was written in a Mathematica notebook and coupled to StRAnD. The probability of failure was approximated using FORM and SORM. The results obtained by using SORM are very close to those obtained by Hoek using MCS. MCS performed in this study showed that both FORM and SORM approximations are good estimators of the probability of failure. This similarity between FORM and SORM suggest that the limit state surface was reasonably linear. Our results show that precise estimates of the probability of failure can be obtained by using a small number of callings of the performance with FORM or SORM approximations. With this, the used of advanced reliability techniques can be used to quantify the risk involved in engineering project in a more efficient fashion by the use of the direct coupling approach.

KEYWORDS: Reliability, probability of failure, slope, planar failure, Direct coupling.

1 INTRODUCTION

Hoek (2007) explored the problem of estimating the probability of failure of the Sau Mau Ping slope. In that case, some characteristic were considered as deterministic values and some others as random variables. Between the deterministic values, he considered the slope height, top plane inclination and overall slope of the face; also, some physical properties like specific weight of the rock mass and water. The orientation of the joint family was also considered as deterministic, i.e., with constant orientation.

The geometrical random variables treated by Hoek (2007) were the gap between the tension crack and the slope crown, b, and the water filling percentage, z_w/z, in terms of the tension crack depth, z. b and z area geometrically dependents and then it is just necessary to consider one of them. The mechanical random variables were the joint cohesion, c, joint friction angle, ϕ , and the pseudo-static acceleration of an earthquake, α. He performed Monte Carlo simulation to approximate the probability of failure. He obtained a probability of failure of 6.4% for 5000 samples.

In this paper we used an efficient reliability technique to approximate the probability of failure of the same problem presented by Hoek but using much less computational effort than in Monte Carlo simulation.

2 RELIABILITY TECHNIQUE

2.1 Reliability concepts

Let us group the random or uncertain parameters of the problem in the vector $\mathbf{X}$ of random variables RV. The performance of a specific failure mode is described by its performance function $g(\mathbf{X})$ which divides the sampling space in:

$$\begin{aligned} \Omega_s &= \{\mathbf{X} \mid g(\mathbf{X}) > 0\} \quad \text{safety domain} \\ \Omega_f &= \{\mathbf{X} \mid g(\mathbf{X}) \le 0\} \quad \text{failure domain} \end{aligned} \tag{1}$$

When the performance function is null, i.e., $g(\mathbf{X}) = 0$, it is called the limit state surface LSS. Hence, the probability of failure P_f is the quantification of the chance of $\mathbf{X}$ to adopt values inside Ω_f. Mathematically:

$$P_f = \int_{\Omega_f} f_{\mathrm{X}}(x)\mathrm{d}x \tag{2}$$

where $f_{\mathrm{X}}(x)$ is the joint probability density function of $\mathbf{X}$.

2.1.1 FORM

The first order reliability method FORM is the most used reliability method in structural safety. It consists in basically two tasks: performing a searching of the design point DP, and approximating the probability of failure around this point by a hyperplane. The design point is the most probable vector included in Ω_f. The searching of the DP is usually performed in the Gaussian space $\mathbf{Y}$, i.e., the space where all RV are uncorrelated and normally distributed. There, the DP is $y*$ and it is found by mean of the optimization process of the following restrained system:

$$\begin{aligned} \text{find:} \quad & y^* \\ \text{which minimizes:} \quad & \beta = \sqrt{y^T.y} \\ \text{subjected to:} \quad & g(y) = 0 \end{aligned} \tag{3}$$

where β is the reliability index of the failure mode.

This problem can be solved by optimization algorithms such as HLRF (Hasofer and Lind, 1974; Rackwitz and Fiessler, 1978) or its improved version iHLRF (Liu and Der Kiureghian, 1986). Once the DP is found, the first order approximation is determined as:

$$P_{f-FORM} = \Phi(-\beta) \tag{4}$$

where $\Phi(.)$ is the Gaussian univariate cumulative probability density function.

2.1.2 SORM

The second order reliability method SORM is essentially the same of FORM until finding the DP. The difference lies in the approximation of the probability of failure at the DP. The SORM approximation is made by means of a quadratic hypersurface. This quadratic approximation was formerly suggested by Breitung (1984) and then improved by Tvedt (1983), Koyluoglu and Nielsen (1994), Cai andElishakoff (1994) and Zha and Ono (1999), among others.

Here, the second order approximation of the probability was performed by means of the Breitung´s formulation (Breitung, 1984).

$$P_{f-SORM} = \Phi(-\beta)\cdot\prod_{i=1}^{n-1}\frac{1}{\sqrt{1+\beta\cdot k_i}} \quad (5)$$

where k_i is the ith principal curvature of the second order hypersurface approximation of the limit state; n is the number of RV of the problem; and β is the reliability index in the design point.

2.2 Direct coupling

The direct coupling approach DC consists in using the discrete response of a numerical model to approximate the gradient of the performance function to perform the searching process of the DP. The gradient can be approximated in space $\mathbf{X}$ and later transformed to $\mathbf{Y}$ by means of a Nataf´s transformation. A forward finite differences scheme can be used to approximate every component of the gradient as follows:

$$\frac{\partial g}{\partial \mathbf{X}_i} \approx \frac{g\left(\mathbf{X}_1,\mathbf{X}_2,...,\mathbf{X}_i+h_i,...,\mathbf{X}_n\right)-g\left(\mathbf{X}\right)}{h_i} \quad (6)$$

where h_i is the ith step size component. The step size can be adopted as a constant value or an adaptive quantity.

Here, it was used an initial step size being a fraction h of the vector of standard deviation of the RV such that $h_i = h.\sigma_{\mathbf{X}_i}$. For example, if it were used a step size fraction of 0.1, the step size of every RV would be 0.1 times the standard deviation of the corresponding RV.

The efficiency of the direct coupling comes from the small number of evaluations of the deterministic model to obtain accurate approximations of the probability of failure. It depends on the number of iterations NI necessary to find the DP and the number of random variables NRV considered in the problem. For example, a FORM approximation needs $NI\left(NRV+1\right)$ model callings. A SORM approximation needs NRV^2+NRV additional callings.

3 RELIABILITY FORMULATION

3.1 Slope: planar failure mode

We considered a failure mode corresponding to the planar failure of a slope with a tension crack located at the top of the slope with no interaction between blocks. The consideration of the other possibilities as the crack located at the face and the interaction between blocks was stated and analyzed in Jimenez-Rodriguez et al. (2006).

The problem of planar failure of a rock slope was faced mathematically following the formulation by Hoek and Bray (1981). This formulation was also used by Hoek (2007). The mechanical formulation of this failure mode is given by equations 7 to 13.

It can be verified that the formulation for the Factor of Safety assumes a very complicated form when all the subequations are substituted into the equation 7. Hence, the analytical treatment of the problem would become tedious.

$$FS = \frac{cA+N\tan\phi}{W\left(\sin\psi_p+\alpha\cos\psi_p\right)+V\cos\psi_p} \quad (7)$$

$$N = W\left(\cos\psi_p-\alpha\sin\psi_p\right)-U-V\sin\psi_p \quad (8)$$

$$b_{\max} = H\left(\cot\psi_p-\cot\psi_f\right) \quad (9)$$

$$z = \frac{z_{\max}}{b_{\max}}\left(b_{\max}-b\right) \quad (10)$$

$$A = \frac{H-z}{\sin\psi_p} \quad (11)$$

$$W = \frac{\gamma_r H^2}{2}\left(\left(1-\left(\frac{z}{H}\right)^2\right)\cot\psi_p-\psi_f\right) \quad (12)$$

$$U = \frac{\gamma_w z_w A}{2} \quad (13)$$

$$V = \frac{\gamma_w z_w^2}{2} \quad (14)$$

To perform the reliability analyses, we stated the following performance functions:

$$g(\mathbf{X}) = FS(\mathbf{X}) - 1 \quad (15)$$

The phenomenological formulation was written in a Mathematica notebook file to perform deterministic analyses. Next, the deterministic response is used to perform probabilistic analysis by means of the direct coupling approach. This reliability technique is implemented in the reliability software StRAnD (Beck, 2011) and also in its Mathematica 9.0 version. The DC uses discrete deterministic responses to approximate the gradient of the performance function $g(\mathbf{X})$.

4 EXAMPLE: HOEK (2007)

We analyzed the same problem treated by Hoek (2007) and introduced above (see Figure 1). The properties of deterministic and random variables were the same of those used by Hoek (2007). The deterministic and random variables RV are summarized in Table 1 and Table 2. The random variables considered were the joint friction angle, ϕ; cohesion of the joint, c; the distance from crest to tension crack, b; the depth of water in tension crack, z_w; and the pseudo-static acceleration coefficient, α.

Hoek performed Monte Carlo simulation to estimate the probability of failure of the slope. The simulation process used 5000 random samples for every RV. He found that the probability of failure was 6.4%.

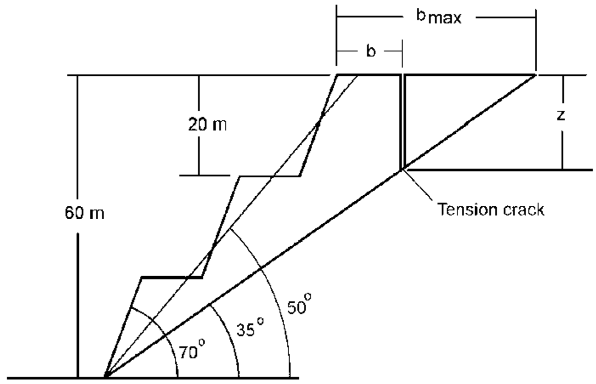

Figure 1. Scheme of Sau Mau Ping slope (Hoek, 2007)

Table 1. Deterministic values

Overall slope height	$H = 60$
Overall slope angle	$\psi_f = 50°$
Upper slope inclination	horizontal
Bench width	$b_{max} = H(\cot\psi_p - \cot\psi_f)$
Unit weight of rock	$\gamma_r = 26$ kN/m^3
Unit weight of water	$\gamma_w = 10$ kN/m^3
Joint inclination	$\psi_p = 35°$

Table 2. Random variables parameters

RV	Mean value	Parameter	Distribution
ϕ [deg]	$\mu_\phi = 35°$	$\sigma_\phi = 5$; 15°-70°	Truncated Normal
c [kPa]	$\mu_c = 100$	$\sigma_c = 20$; 0-250	Truncated Normal
b [m]	$\mu_b = 15.3$	$\sigma_b = 4$	Truncated Normal
z_w	$\mu_{z_w} = z/2$	min=0, max=z	Truncated Exponential
α	$\mu_\alpha = 0.08$	min=0, max=2α	Truncated Exponential

$$* \, z_{max} = H\left(1 - \frac{\tan\psi_p}{\tan\psi_f}\right)$$

4.1 Results and discussion

The deterministic analysis showed that the analyzed slope presented a factor of safety of FS=1.33. On the other hand, the probabilistic analysis was performed by means of the direct coupling approach. The searching process for the design point was performed using the HLRF optimization algorithm. The finite difference step size was fixed in $h=10^{-4}$ for every RV. This value was chosen because it exhibited a stable numerical behavior. It was verified that HLRF method used 4 iterations to satisfy the tolerance in the reliability index β (10^{-3}). The iterative procedure for the searching process of the design point is summarized in Table 3.

Table 3. HLRF iterative procedure to find the DP

z [m]	c [kPa]	ϕ [deg]	z_w/z [-]	α [g]	g(X)	β
14	10	35.0	0.34	0.055	0.3305	0.26627
14.45	8.72	30.61	0.50	0.086	0.0275	1.45375
14.66	8.56	31.01	0.57	0.090	-0.0008	1.56191
14.77	8.61	31.20	0.59	0.089	-0.0001	1.55649
14.81	8.62	31.25	0.60	0.089	-1E-05	1.55581

The final β value was found to be 1.5558. This β value yields a FORM approximation value of the probability of failure of 5.99%. On the other hand, the Breitung´s SORM approximation yielded a value of 6.31% for the probability of failure.

At this point, it is good to remember that Hoek found a value of 6.4% for the probability of failure for the same slope using MCS with 5000 realizations. It is observed that the SORM approximation produced a value quite similar to that obtained by the MCS performed by Hoek (2007).

Additionally, it was performed a MCS using 20000 samples to evaluate the validity of the approximations. It was found that the estimator of the mean value of the probability was 6.12%. Moreover, the 95% confidence interval was bounded between 5.78% and 6.45%. From these results, it can be inferred that both FORM and SORM approximations can be used as estimators of the probability of failure once they both lie in the 95% confidence interval. Similarly, the Hoek´s approximation of the probability of failure lies in this interval. Figure 2 shows the convergence plot of the MCS including the 95% confidence bounds for the reliability index β.

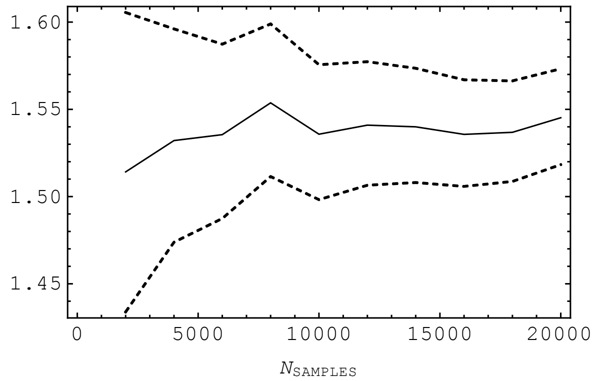

Figure 2. Monte Carlo simulation performed in this study.

MCS provides more information that only the mean value and also that number of realizations influence on the approximations. It also can be seen in Figure 2 that the confidence interval for 5000 realizations was larger than that for 20000 realizations. This evidence the need of performing convergence analyses for the precision of the MCS.

For this example, SORM approximation presented a better approximation to the Hoek´s MCS approximation of the probability of failure than that made by FORM. On the other hand, the MCS performed in this study was correctly approximated by both FORM and SORM. In fact, the FORM approximation presented the smallest error. Finally, owing that FORM and SORM approximations were acceptable, FORM approximation can be considered as sufficient for this problem.

5 CONCLUSIONS

This paper presented a probabilistic study of the stability of a slope with planar failure. The reliability technique used was the direct coupling. The DC uses deterministic responses of the geotechnical model to approximate the probability of failure. To study the technique, the Sau Mau Ping slope was analyzed and its information was taken from Hoek (2007). Here, it was found FORM and SORM approximations presented satisfactory approximations to the MCS approximation. Hence, FORM approximation appears to be sufficient for this type of geotechnical problem. Additionally, importance is brought to the necessity of performing convergence analyses of MCS to evaluate the validity of the results. Finally, The direct coupling approach showed to be a very useful numerical technique to perform reliability analyses in geotechnical problems.

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