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ESTIMATION OF TIME VARYING LINEAR SYSTEMS

by

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Estimation of Time Varying Linear Systems

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1 Introduction

In the theory of stationary processes, time-invariant linear systems are of considerable interest. If X_t and Y_t, $t \in \mathbb{Z} = \{0, \pm 1, \cdots\}$, are two stationary processes with zero mean, then for observations $\{X_t, Y_t, 1 \leq t \leq N\}$ we want to estimate the filter weights w_u in the model

$$Y_t = \sum_{u=-\infty}^{\infty} w_u X_{t-u} + \epsilon_t, \tag{1}$$

where ϵ_t is a stationary process with zero mean, orthogonal to X_t. The estimation of w_u involves the spectrum $f_x(\omega)$ of X_t, the cross-spectrum between X_t and Y_t, $f_{xy}(\omega)$ and is implemented estimating first the transfer function $W(\omega) = \sum_u w_u \exp(-i\omega u)$. Estimation of the error spectrum $f_\epsilon(\omega)$ is also of interest. See Brillinger (1975, chapter 8) for details. In practice, the infinite sum in (1) is replaced by a finite one. See Robinson (1979).

In this paper we consider time-varying linear systems of the form (1), where the filter coefficients are functions of time. For this situation, the processes that appear in (1) will be taken as locally stationary, in the sense of Dahlhaus (1997), to be defined in section 2. In our approach, we will use two types of estimators: kernel estimators and estimators based on wavelets.

Wavelets are a contemporary tool, which have found uses in many areas, including signal processing, image coding and compression, turbulence, statistics, numerical analysis, etc. Good mathematical references are Chui (1992) and Daubechies (1992). References for uses of wavelets in statistics are Donoho and Johnstone (1990), Donoho (1993), Nason (1994). For uses in time series analysis see Brillinger (1994a, b), Neumann (1996), von Sachs and Schneider (1996), von Sachs, Nason and Kroisandt (1996), Gao (1997), Neumann and von Sachs (1997) and Chiann and Morettin (1998).

In section 2 we give the basic ideas on locally stationary processes and the concept of evolutionary spectra. Section 3 presents two-dimensional orthonormal wavelet bases. Wavelet estimators and kernel estimators of the evolutionary spectrum will be presented in section 4 and 5, respectively. In section 6, we present time varying linear systems, in section 7 we present some simulations and we conclude with final comments in section 8.

2 Locally stationary processes and evolutionary spectra

Stationary models have always been the main focus of interest in the theoretical treatment of time series analysis. The classical Cramér spectral representation of a stationary stochastic process $\{X_t,\, t \in \mathcal{Z}\}$ is given by

$$X_t = \int_{-\pi}^{\pi} \exp(i\omega t)dZ(\omega) = \int_{-\pi}^{\pi} A(\omega)\exp(i\omega t)d\xi(\omega), \tag{2}$$

where $dZ(\omega)$ and $d\xi(\omega)$ are orthogonal and orthonormal increment processes, respectively.

On the other hand, many phenomena in the applied science show a non-stationary behaviour (e.g. in economics, sound analysis, geophysics), the second order structure of these processes is no longer time-shift invariant but changes over time. Priestley (1981) introduced a time dependence in the amplitude function $A(\omega)$, i.e., he considered processes having a time varying spectral representation

$$X_t = \int_{-\pi}^{\pi} \exp(i\omega t) A_t(\omega) d\xi(\omega), \quad t \in Z, \tag{3}$$

with an orthogonal increment process $\xi(\omega)$ and a time varying transfer function $A_t(\omega)$. But within the approach of Priestley, asymptotic considerations are not possible.

Dahlhaus (1997) defined a general class of nonstationary processes having a time varying spectral representation. In this approach Dahlhaus defines a sequence of doubly indexed processes as follows.

Definition 1: A sequence of stochastic processes $\{X_{t,T},\ t = 1, \cdots, T\}$, is called *locally stationary* if there exists a representation

$$X_{t,T} = \mu(\frac{t}{T}) + \int_{-\pi}^{+\pi} \exp(i\omega t) A(\frac{t}{T}, \omega) d\xi(\omega), \tag{4}$$

where

(i) $\xi(\omega)$ is a stochastic process on $[-\pi, \pi]$ with $\overline{\xi(\omega)} = \xi(-\omega)$, $\mathrm{E}(\xi(\omega)) = 0$, with orthonormal increments, i.e.,

$$\mathrm{Cov}[d\xi(\omega), d\xi(\omega')] = \delta(\omega - \omega')d\omega,$$

and such that

$$\mathrm{Cum}\{d\xi(\omega_1), \cdots, d\xi(\omega_k)\} = \eta(\sum_{j=1}^{k} \omega_j) g_k(\omega_1, \cdots, \omega_{k-1}) d\omega_1 \cdots d\omega_k,$$

where $\mathrm{Cum}\{\cdots\}$ denotes the cumulant of k-th order, $g_1(\omega) = 0$, $g_2(\omega) = 1$, $\mid g_k(\omega_1, \cdots, \omega_{k-1}) \mid \leq \mathrm{const}_k$ for all k and $\eta(\omega) = \sum_{j=-\infty}^{\infty} \delta(\omega + 2\pi j)$ is the Dirac comb;

(ii) $A(u, \omega)$ is a function on $[0, 1] \times [-\pi, \pi]$ which is 2π periodic in ω, with $A(u, -\omega) = \overline{A(u, \omega)}$.

The functions $A(u,\omega)$ and $\mu(u)$ are assumed to be continuous in u, because the smoothness of A in u guarantees that the process has locally a stationary behaviour.

For simplicity, we assume that $\mu(u) = 0$.

Remark: In Dahlhaus (1997), the representation (4) is based on a sequence of functions $A^o_{t,T}(\omega)$ instead of the function $A(u,\omega)$, the difference being that it has to fulfill

$$\sup_{t,\omega} \mid A^o_{t,T}(\omega) - A(\frac{t}{T},\omega) \mid \leq KT^{-1},$$

for some positive constant K. For reasons of notational convenience, we use the representation (4), noting that all results will continue to hold for the broader class.

Now we define for $u \in (0,1)$ and fixed T, the Wigner-Ville spectrum

$$f_T(u,\omega) = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} Cov\{X_{[uT-s/2],T}, X_{[uT+s/2],T}\} \exp(-i\omega s), \qquad (5)$$

where the $X_{t,T}$ is given by (4), with $A(u,\omega) = A(0,\omega)$ for $t < 1$ and $A(u,\omega) = A(1,\omega)$ for $t > T$ and $u = \frac{t}{T}$ is the re-scaled time to the interval $[0,1]$.

Definition 2: The *evolutionary spectrum* of $\{X_{t,T}\}$ given in (4) is defined, for $u \in (0,1)$, by

$$f(u,\omega) = \mid A(u,\omega) \mid^2 . \qquad (6)$$

Dahlhaus (1997, theorem 1.2) shows that under smoothness conditions on A, $f_T(u,\omega)$ tends in squared mean to $f(u,\omega)$.

3 Two-dimensional orthonormal wavelet bases

This section describes the construction of two-dimensional wavelet bases using different one-dimensional wavelets bases.

There are two possibility to build a two-dimensional wavelet basis:

a) extending two one-dimensional MRA to build a two-dimensional wavelet basis, with only one scale j;

b) taking the simple tensor products of one-dimensional wavelets with different scales j_1, j_2 for each dimension.

Let V_j be subspaces of $L^2([0,1])$ satisfying

(a)
$$\cdots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots; \tag{7}$$

(b)
$$\overline{\bigcup}_{j\in Z} V_j = L^2([0,1]), \quad \bigcap_{j\in Z} V_j = \{\emptyset\}; \tag{8}$$

(c) for all $f \in L^2([0,1])$, we have
$$f \in V_j \iff f(2^{-j}\cdot) \in V_0; \tag{9}$$

(d)
$$f \in V_j \iff f(\cdot - 2^{-j}n) \in V_j, \tag{10}$$

for all $n \in Z$.

There exists $\phi \in V_0$ such that
$$\{\phi_{0,k}; k \in Z\} \tag{11}$$

is an orthonormal basis in V_0, where, for all $j, k \in Z$,
$$\phi_{j,k} = 2^{j/2}\phi(2^j x - k). \tag{12}$$

(10) and (11) imply that $\{\phi_{j,k}; k \in Z\}$ constitute an orthonormal basis for V_j for all $j \in Z$. We often call ϕ the scaling function.

The basic idea of multiresolution analysis is that there exists an orthonormal wavelet basis $\{\psi_{j,k}, j, k \in Z\}$ of $L^2([0,1])$,
$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \tag{13}$$

such that for all $f \in L^2([0,1])$,
$$P_{j+1}f = P_j f + \sum_{k\in Z} < f, \psi_{j,k} > \psi_{j,k}, \tag{14}$$

where P_j is the orthogonal projection onto V_j and

$$< f, \psi_{j,k} >= \int f(t)\psi_{j,k}(t)dt. \tag{15}$$

For every $j \in Z$, define the subspaces W_j generated by $\{\psi_{j,k}\}_k$ to be the orthogonal complement of V_j in V_{j+1}. Then we have

$$V_{j+1} = V_j \oplus W_j \tag{16}$$

and

$$W_j \perp W_{j'}, \quad if \quad j \neq j'. \tag{17}$$

It follows that for $j > l$,

$$V_j = V_l \oplus W_l \oplus \cdots \oplus W_{j-1}, \tag{18}$$

where all these subspaces are orthogonal.

Assume that we have an orthonormal basis of compactly supported wavelets of $L^2(0,1])$. For any j, let the subspaces $\mathcal{V}_j = V_j \otimes V_j$, defining a two-dimensional MRA. Then it can be shown that this set of subspaces inherits the properties (a) to (d). So we have

$$L^2([0,1] \times [0,1]) = \overline{\bigcup_{j=l}^{\infty} \mathcal{V}_j} = \overline{\bigcup_{j=l}^{\infty} V_j \otimes V_j},$$

which shows the possibility to build a basis of $L^2([0,1] \times [0,1])$ from tensor products of functions from different one-dimensional bases, $\{\phi_{l,k}, \psi_{j,k}, j \geq l, k\}$.

Also, defining $\Phi_{j,k_1,k_2}(x,y) = \phi_{j,k_1}(x)\phi_{j,k_2}(y)$, we have that the set

$$\{\Phi_{j,k_1,k_2}, k_1, k_2 \in Z\}$$

constitutes an orthonormal basis for $\mathcal{V}_j$.

As in the one-dimensional case, define $\mathcal{W}_j$ to be the orthogonal complement of $\mathcal{V}_j$ in $\mathcal{V}_{j+1}$, so we have

$$\begin{aligned}
\mathcal{V}_{j+1} &= V_{j+1} \otimes V_{j+1} \\
&= (V_j \oplus W_j) \otimes (V_j \oplus W_j) \\
&= V_j \otimes V_j \oplus ((V_j \otimes W_j) \oplus (W_j \otimes V_j) \oplus (W_j \otimes W_j)) \\
&= \mathcal{V}_j \oplus \mathcal{W}_j,
\end{aligned}$$

with

$$\mathcal{W}_j = \overline{span}\{\Psi^m_{j,K}(x,y) : K = (k_1,k_2), m = h,v,d\},$$

and

$$\Psi^h_{j,K}(x,y) = \phi_{j,k_1}(x)\psi_{j,k_2}(y),$$
$$\Psi^v_{j,K}(x,y) = \psi_{j,k_1}(x)\phi_{j,k_2}(y),$$
$$\Psi^d_{j,K}(x,y) = \psi_{j,k_1}(x)\psi_{j,k_2}(y).$$

Note that $\mathcal{W}_j$ is made up of three different wavelets: horizontal, vertical and diagonal.

Now for any j^*, we can write $\mathcal{V}_{j^*}$ into two different ways:

$$\begin{aligned}\mathcal{V}_{j^*} &= V_{j^*} \otimes V_{j^*} = \mathcal{V}_{j^*-1} \oplus \mathcal{W}_{j^*-1} \\ &= \cdots = \mathcal{V}_l \oplus \bigoplus_{j=l}^{j^*-1} \mathcal{W}_j \\ &= V_l \otimes V_l \oplus \bigoplus_{j=l}^{j^*-1} [(V_j \otimes W_j) \oplus (W_j \otimes V_j) \oplus (W_j \otimes W_j)], \end{aligned} \tag{19}$$

or alternatively,

$$\begin{aligned}\mathcal{V}_{j^*} &= (V_l \oplus W_l \oplus \cdots \oplus W_{j^*-1}) \otimes (V_l \oplus W_l \oplus \cdots \oplus W_{j^*-1}) \\ &= V_l \otimes V_l \oplus [\bigoplus_{j=l}^{j^*-1} (W_j \otimes V_l)] \oplus [\bigoplus_{j=l}^{j^*-1} (V_l \otimes W_j)] \oplus [\bigoplus_{j_1,j_2=l}^{j^*-1} (W_{j_1} \otimes W_{j_2})]. \end{aligned} \tag{20}$$

From (19), we obtain a basis $\mathcal{B}_1$ of $L_2([0,1] \times [0,1])$ as

$$\begin{aligned}\mathcal{B}_1 &= \{\phi_{l,k_1}(x)\phi_{l,k_2}(y)\}_{k_1,k_2} \cup \\ & \bigcup_{j \geq l} \{\phi_{j,k_1}(x)\psi_{j,k_2}(y), \psi_{j,k_1}(x)\phi_{j,k_2}(y), \psi_{j,k_1}(x)\psi_{j,k_2}(y)\}_{k_1,k_2}. \end{aligned} \tag{21}$$

So $\mathcal{B}_1$ can be represented by

$$\mathcal{B}_1 = \{\Phi_{l,\mathbf{K}}(x,y), \mathbf{K} = (k_1,k_2)\}_{\mathbf{K}} \cup \{\Psi^m_{j,\mathbf{K}}(x,y), \mathbf{K} = (k_1,k_2), m = h,v,d\}_{j \geq l, \mathbf{K}}. \tag{22}$$

According to (20), another construction is given by

$$\begin{aligned}\mathcal{B}_2 &= \{\phi_{l,k_1}(x)\phi_{l,k_2}(y)\}_{k_1,k_2} \cup (\bigcup_{j_1 \geq l} \{\psi_{j_1,k_1}(x)\phi_{l,k_2}(y)\}_{k_1,k_2}) \\ & \cup (\bigcup_{j_2 \geq l} \{\phi_{l,k_1}(x)\psi_{j_2,k_2}(y)\}_{k_1,k_2}) \cup (\bigcup_{j_1,j_2 \geq l} \{\psi_{j_1,k_1}(x)\psi_{j_2,k_2}(y)\}_{k_1,k_2}). \end{aligned} \tag{23}$$

For notational convenience, we write $\psi_{l-1,k}$ for $\phi_{l,k}$ and we define $\mathcal{U}_I$ the basis functions, where I denote the multiindex $I = (j_1, j_2, k_1, k_2)$, so $\mathcal{B}_2$ can be written as

$$\mathcal{B}_2 = \{\mathcal{U}_I(x, y), I = (j_1, j_2, k_1, k_2)\}, \tag{24}$$

with

$$\mathcal{U}_I(x, y) = \psi_{j_1,k_1}(x)\psi_{j_2,k_2}(y).$$

Note that some of the $\psi_{j,k}$ are father wavelets $\phi_{j+1,k}$.

The decomposition of an $L^2([0,1] \times [0,1])$ function f using $\mathcal{B}_1$ is given by

$$f(x, y) = \sum_{\mathbf{K}} c_{l,\mathbf{K}} \Phi_{l,\mathbf{K}}(x, y) + \sum_{j=l} \sum_{\mathbf{K}} \sum_{m=h,v,d} d^m_{j,\mathbf{K}} \Psi^m_{j,\mathbf{K}}(x, y), \tag{25}$$

where the coefficients are computed as

$$c_{l,\mathbf{K}} = \int_{[0,1]\times[0,1]} f(x, y)\Phi_{l,\mathbf{K}}(x, y)dxdy,$$

$$d^m_{j,\mathbf{K}} = \int_{[0,1]\times[0,1]} f(x, y)\Psi^m_{j,\mathbf{K}}(x, y)dxdy.$$

Alternatively, a decomposition of an $L^2([0,1] \times [0,1])$ function f using $\mathcal{B}_2$ is given by

$$f(x, y) = \sum_I d_I \mathcal{U}_I(x, y), \tag{26}$$

with $d_I = \int_{[0,1]\times[0,1]} f(x, y)\mathcal{U}_I(x, y)dxdy$.

Note that we can use two distinct bases, one for each direction. In the evolutionary spectrum case, we use a 1-d basis for time direction and another 1-d basis for frequency direction.

4 Wavelet estimators of the evolutionary spectrum

Now suppose that we have an observed sequence of values $\{X_{1,T}, \cdots, X_{T,T}\}$ and based on these values we want to estimate the evolutionary spectrum. In this section we consider wavelet estimators constructed using the basis $\mathcal{B}_1$

defined in (22). Introduce a local version of the classical periodogram over a segment of length N of the tapered data $X_{t,T}$, $1 \leq t \leq T$ as:

$$I_N(u,\omega) = \frac{1}{2\pi H_N} \mid \sum_{s=0}^{N-1} h(\frac{s}{N}) X_{[uT]-\frac{N}{2}+s+1,T} \exp(-i\omega s) \mid^2, \qquad (27)$$

for $0 < u < 1$, $-\pi \leq \omega \leq \pi$, where $h : [0,1] \to [0,1]$ is a data window, $H_N = \sum_{j=0}^{N-1} h^2(\frac{j}{N}) \approx N \int_0^1 h^2(x)dx$ is the normalizing factor.

Assume $N = 2^J$, the finest level chosen to be $J = \log_2 N$ and the coarset level $l = 0$. Consider the projection of the spectrum $f(u,\omega)$ onto the 2^{2J}-dimensional subspace $\mathcal{V}_J \subset L^2(U \times \Pi)$ (on the finest scale J), denoted by $f_J(u,\omega)$. So its wavelet decomposition in terms of basis functions of $\mathcal{B}_1$ is given by

$$f_J(u,\omega) = c_{0,0} + \sum_{j=0}^{J-1} \sum_{\mathbf{K}=0}^{2^j-1} \sum_{m=h,v,d} d_{j,\mathbf{K}}^m \Psi_{j,\mathbf{K}}^m(u,\omega), \qquad (28)$$

sampled on an equally spaced grid (u_i, ω_n), $0 \leq i, n \leq N-1$, with the coefficients

$$c_{0,0} = \int_0^1 \int_{-\pi}^{\pi} f(u,\omega) du d\omega \qquad (29)$$

and

$$d_{j,\mathbf{K}}^m = \int_0^1 \int_{-\pi}^{\pi} f(u,\omega) \Psi_{j,\mathbf{K}}^m(u,\omega) du d\omega. \qquad (30)$$

Now we use the periodogram $I_N(u,\omega)$ defined in (27) to obtain the empirical coefficients $\hat{c}_{0,0}$ and $\hat{d}_{j,\mathbf{K}}^m$, where $I_N(u,\omega)$ is calculated on overlapping segments of $X_{t,T}$ of length $N = 2^J$. Let S be the shift from segment to segment, $1 \leq S \leq N$. Then, the $I_N(u,\omega)$ is calculated at the M timepoints

$$u_i = \frac{t_i}{T}, \ t_i = S \cdot i + \frac{N}{2}, \ 0 \leq i \leq M-1,$$

with $T = S(M-1) + N$ and the frequencies $\omega_n = \frac{2\pi n}{N} - \pi$, $0 \leq n \leq N-1$. So the empirical coefficients are:

$$\hat{c}_{0,0} = \frac{1}{M} \sum_{i=0}^{M-1} \int_{-\pi}^{\pi} I_N(u_i,\omega) d\omega \qquad (31)$$

and

$$\hat{d}^m_{j,\mathbf{K}} = \frac{1}{M}\sum_{i=0}^{M-1}\int_{-\pi}^{\pi} I_N(u_i,\omega)\Psi^m_{j,\mathbf{K}}(u_i,\omega)d\omega. \tag{32}$$

Note that in practice we have to choose M to be equal N to be able to use a traditional quadratic 2-d wavelet scheme.

We need some regularity assumptions on both the spectrum $f(u,\omega)$ (or $A(u,\omega)$) and on the wavelet basis functions used to obtain asymptotic results of the empirical coefficients $\hat{d}^m_{j,\mathbf{K}}$.

Assumptions:

(S1) Let $A(u,\omega)$ and $\Psi^m_{j,\mathbf{K}}(u,\omega)$ be differentiable in u and ω with uniformly bounded first partial derivatives;

(S2) The parameters N, S and T fulfill the relations

$$T^{1/4} << N << T^{1/2}/\ln T \text{ and } S = N \text{ or } \frac{S}{N} \to 0, \text{ as } T \to \infty.$$

(S3) The data-taper $h(x)$ is continuous on $[0,1]$ and twice differentiable at $x \notin p$, where p is a finite set and $\sup_{x\notin p} \mid h''(x) \mid < \infty$.

Lemma 1 (von Sachs and Schneider, 1996). Let Assumptions (S1)-(S3) be fulfilled. Then, as $T \to \infty$, uniformly over $j, \mathbf{K}$, with $2^j = o(N)$,

(a)

$$\mathrm{E}(\hat{d}^m_{j,\mathbf{K}} - d^m_{j,\mathbf{K}}) = O(2^{-j}N^{-1}) = o(T^{-1/2}), \ \forall m = h, v, d. \tag{33}$$

(b)

$$\mathrm{Var}(\hat{d}^m_{j,\mathbf{K}}) = \frac{A^m_{j,\mathbf{K}}}{T} + O(\frac{2^j N}{T^2}) + O(2^{-j}T^{-1}), \ \forall m = v, h, d; \tag{34}$$

where

$$A^m_{j,\mathbf{K}} = 2C_h \int_{U\times\Pi} \{f(u,\omega)\}^2 \Psi^m_{j,\mathbf{K}}(u,\omega)[\Psi^m_{j,\mathbf{K}}(u,\omega) + \Psi^m_{j,\mathbf{K}}(u,-\omega)]dud\omega$$

with $C_h = \frac{\int_0^1 h^4(x)dx}{(\int_0^1 h^2(x)dx)^2}$ for $S = N$ and $C_h = 1$ if $S/N \to 0$.

(c)

$$T^{L/2}\mathrm{Cum}_L\{\hat{d}^m_{j,\mathbf{K}}\} = o(1), \ \ \forall L \geq 3; \tag{35}$$

(d) $\sqrt{T}(\hat{d}^m_{j,\mathbf{K}} - d^m_{j,\mathbf{K}})$ has asymptotically a normal distribution, with mean 0 and covariance $A^m_{j,\mathbf{K}}$, $m = h, v, d$.

Note that the properties of this estimate depend on the choice of the segment length N, its optimal choice depends on the relation between the unknown smoothness of $f(u,\omega)$. To avoid a preliminaly choice of a fixed N, one possibility is using the basis $\mathcal{B}_2$. Neumann and von Sachs (1997) introduced a periodogram-like statistic $I_{t,T}$, $1 \leq t \leq T$,

$$I_{t,T}(\omega) = \frac{1}{2\pi} \sum_{|s| \leq min\{t-1, T-t\}} X_{[t-s/2],T} X_{[t+s/2],T} \exp(-i\omega s), \tag{36}$$

which can be considered as a preliminary estimate of $f(u,\omega)$.

The wavelet coefficients d_I of expansion of $f(u,\omega)$ in terms of basis functions of $\mathcal{B}_2$ are defined as:

$$d_I = \int_{U\times\Pi} f(u,\omega)\mathcal{U}_I(u,\omega)dud\omega = \int_{U\times\Pi} f(u,\omega)\psi_{j_1,k_1}(u)\tilde{\psi}_{j_2,k_2}(\omega)dud\omega \tag{37}$$

where $U \times \Pi = [0,1] \times [-\pi, \pi]$.

Hence, we have

$$f(u,\omega) = \sum_I d_I \psi_{j_1,k_1}(u)\tilde{\psi}_{j_2,k_2}(\omega). \tag{38}$$

Now we define the empirical wavelet coefficients as follows:

$$\hat{d}_I = \sum_{t=1}^{T} \int_{\frac{t-1}{T}}^{\frac{t}{T}} \psi_{j_1,k_1}(u)du \int_{-\pi}^{\pi} \tilde{\psi}_{j_2,k_2}(\omega) I_{t,T}(\omega)d\omega, \tag{39}$$

and an estimate of $f(u,\omega)$ can be set as

$$\hat{f}(u,\omega) = \sum_{I\in I_T} \hat{d}_I \psi_{j_1,k_1}(u)\tilde{\psi}_{j_2,k_2}(\omega), \tag{40}$$

where $I_T = \{I : 2^{j_1+j_2} \leq T^{1-\delta}\}$, for some $\delta > 0$.

Asymptotic results of the empirical coefficients $\hat{d}_I$ can be found in Neumann and von Sachs (1997).

5 Kernel estimators of the evolutionary spectrum

Dahlhaus(1996) proposes the following kernel estimator for the evolutionary spectrum. Let

$$\hat{f}(u,\omega) = \frac{1}{b_f}\int K_f(\frac{\omega-\mu}{b_f})I_N(u,\mu)d\mu, \tag{41}$$

where $I_N(u,\omega)$ is defined in (27), $K_f : \mathcal{R} \to [0,\infty]$ is a kernel with $K_f(x) = 0$ for $x \notin [-1/2, 1/2]$, $K_f(x) = K_f(-x)$ and $\int K_f(x)dx = 1$ and b_f is a frequency domain bandwidth.

Now define

$$K_t(x) = \{\int_0^1 h(x)^2 dx\}^{-1} h(x+1/2)^2, \quad x \in [-1/2, 1/2]$$

which has the role of a kernel in the time direction and $b_t = N/T$ the bandwidth in time direction. Then we have:

Lemma 2 (Dahlhaus, 1996). Suppose $X_{t,T}$ is a locally stationary process as defined in (4) with $\mu = 0$ and transfer function A whose derivatives $\frac{\partial^2}{\partial u^2}A$, $\frac{\partial^2}{\partial \omega^2}A$, $\frac{\partial^2}{\partial u \partial \omega}A$ are continuous. Then

(a)

$$\begin{aligned}&\mathrm{E}(\hat{f}(u,\omega)) = \\ &\quad f(u,\omega) + \frac{1}{2}b_t^2\int_{-1/2}^{1/2} x^2 K_t(x)dx \frac{\partial^2}{\partial u^2} f(u,\omega) + \\ &\quad \frac{1}{2}b_f^2\int_{-1/2}^{1/2} x^2 K_f(x)dx \frac{\partial^2}{\partial \omega^2} f(u,\omega) + o(b_t^2 + \frac{\log(b_t T)}{b_t T} + b_f^2);\end{aligned} \tag{42}$$

(b)

$$\begin{aligned}\mathrm{Var}(\hat{f}(u,\omega)) &= (b_t b_f T)^{-1} f(u,\omega)^2 \int_{-1/2}^{1/2} K_t(x)^2 dx \int_{-1/2}^{1/2} K_f(x)^2 dx \cdot \\ &\quad \cdot (2\pi + 2\pi\{\omega \equiv 0 \pmod{\pi}\}).\end{aligned} \tag{43}$$

6 Time varying linear systems

Now we consider a linear system

$$Y_{t,T} = \sum_u a_u(\frac{t}{T}) X_{t-u,T} + \epsilon_t, \tag{44}$$

where:

(i) $Y_{t,T}$, $X_{t,T}$ are locally stationary processes with zero mean;

(ii)

$$\sup_v \sum_u | a_u(v) | < \infty; \tag{45}$$

(iii) ϵ_t is a stationary series, mean zero, orthogonal to $X_{t,T}$.

On a wavelet basis ψ, we can replace $a_u(\cdot)$ by

$$a_u(\frac{t}{T}) = \sum_j \sum_k \beta_{j,k}^{(u)} \psi_{j,k}(\frac{t}{T}), \tag{46}$$

where

$$\beta_{j,k}^{(u)} = \int_0^1 a_u(z) \psi_{j,k}(z) dz. \tag{47}$$

In the following we intend to find an estimate $\hat{\beta}_{j,k}^{(u)}$ of the $\beta_{j,k}^{(u)}$'s and to derive some asymptotic properties of $\hat{\beta}_{j,k}^{(u)}$. Consequently, asymptotic properties of estimates of $a_u(\frac{t}{T})$ will follow.

First of all, for the process $X_{t,T}$, we define

$$C_{xx}(u,k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_{xx}(u,\omega) e^{i\omega k} d\omega, \tag{48}$$

the *local covariance* of lag k at time u, where $f_{xx}(u,\omega)$ is the evolutionary spectrum of $X_{t,T}$. Then we have (Dahlhaus, 1996):

$$\begin{aligned} C_{xx}^{(T)}(\frac{t}{T},k) &= Cov\{X_{[t-k/2],T}, X_{[t+k/2],T}\} \\ &= \int_{-\pi}^{\pi} A(\frac{t-k/2}{T},\omega) \overline{A(\frac{t+k/2}{T},\omega)} e^{i\omega k} d\omega \\ &= C_{xx}(\frac{t}{T},k) + C'(\frac{t}{T},k) O(\frac{k}{T}) \end{aligned} \tag{49}$$

for smooth A, with both $\sup_{t/T}\sum_k \mid C_{xx}(\frac{t}{T},k) \mid < \infty$ and $\sup_{t/T}\sum_k \mid C'(\frac{t}{T},k) \mid < \infty$.

Analogously for the processes $X_{t,T}$ and $Y_{t,T}$, we define

$$C_{xy}(u,k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_{xy}(u,\omega)e^{i\omega k}d\omega,$$

the *local cross-covariance* of lag k at time u, where $f_{xy}(u,\omega)$ is the *cross-evolutionary spectrum* of $X_{t,T}$ and $Y_{t,T}$, defined by

$$f_{xy}(u,\omega) = \lim_{T\to\infty}\frac{1}{2\pi}\sum_{s=-\infty}^{\infty} \mathrm{Cov}\{X_{[uT-s/2],T}, Y_{[uT+s/2],T}\}\exp(-i\omega s). \tag{50}$$

Now, we have, for every $m \in \mathcal{Z}$,

$$\begin{aligned}
\mathrm{Cov}\{Y_{t,T}, X_{t-m,T}\} &= \mathrm{E}\{Y_{t,T}X_{t-m,T}\} \\
&= \mathrm{E}\{[\sum_u a_u(\frac{t}{T})X_{t-u,T} + \epsilon_t]X_{t-m,T}\} \\
&= \mathrm{E}\{\sum_u a_u(\frac{t}{T})X_{t-u,T}X_{t-m,T}\} + \mathrm{E}\{\epsilon_t X_{t-m,T}\} \\
&= \sum_u a_u(\frac{t}{T})E\{X_{t-u,T}X_{t-m,T}\} \\
&= \sum_u a_u(\frac{t}{T})\mathrm{Cov}\{X_{t-u,T}, X_{t-m,T}\} \\
&= \sum_u a_u(\frac{t}{T})\{C_{xx}(\frac{t}{T},(m-u)) + O(T^{-1})\} \\
&= \sum_u a_u(\frac{t}{T})\frac{1}{2\pi}\int_{-\pi}^{\pi} f_{xx}(\frac{t}{T},\omega)e^{i\omega(m-u)}d\omega + O(T^{-1}) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_u a_u(\frac{t}{T})e^{-i\omega u}f_{xx}(\frac{t}{T},\omega)e^{i\omega m}d\omega + O(T^{-1}) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} B_a(\frac{t}{T},\omega)f_{xx}(\frac{t}{T},\omega)e^{i\omega m}d\omega + O(T^{-1})
\end{aligned}$$

where

$$B_a(\frac{t}{T},\omega) = \sum_u a_u(\frac{t}{T})e^{-i\omega u}. \tag{51}$$

Then,

$$f_{xy}(\frac{t}{T},\omega) = B_a(\frac{t}{T},\omega) f_{xx}(\frac{t}{T},\omega) + O(T^{-1}).$$

Hence $B_a(\frac{t}{T},\omega)$ can be estimated by

$$\hat{B}_a(\frac{t}{T},\omega) = \hat{f}_{xy}(\frac{t}{T},\omega)\{\hat{f}_{xx}(\frac{t}{T},\omega)\}^{-1}, \tag{52}$$

where $\hat{f}_{xx}(\frac{t}{T},\omega)$ is assumed to be nonsingular and $\hat{f}_{xx}(\frac{t}{T},\omega)$, $\hat{f}_{xy}(\frac{t}{T},\omega)$ can be replaced by estimators described in section 4 or 5.

As we have (51), then

$$a_u(\frac{t}{T}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} B_a(\frac{t}{T},\omega)e^{i\omega u}d\omega, \tag{53}$$

for $u = 0, \pm 1, \cdots$. Thus, $a_u(\frac{t}{T})$ can be estimated by

$$\hat{a}_u(\frac{t}{T}) = \frac{1}{P_T+1}\sum_{p=-P_T/2}^{P_T/2} \hat{B}_a(\frac{t}{T},\frac{2\pi p}{P_T})\exp(iu\frac{2\pi p}{P_T}), \tag{54}$$

where P_T is a sequence of integers tending to ∞ as $T \to \infty$ and $\hat{a}_u(\frac{t}{T})$ is considered as a preliminary estimate of $a_u(\frac{t}{T})$.

Now, from (47), $\hat{\beta}_{j,k}^{(u)}$ can be written as

$$\begin{aligned}
\hat{\beta}_{j,k}^{(u)} &= \frac{1}{T}\sum_{t=0}^{T-1} \hat{a}_u(\frac{t}{T})\psi_{j,k}(\frac{t}{T}) \\
&= \frac{1}{T}\sum_{t=0}^{T-1}\frac{1}{P_T+1}\sum_{p=-P_T/2}^{P_T/2} \hat{B}_a(\frac{t}{T},\frac{2\pi p}{P_T})\exp(iu\frac{2\pi p}{P_T})\psi_{j,k}(\frac{t}{T}) \\
&= \frac{1}{T}\sum_{t=0}^{T-1}\frac{1}{P_T+1}\sum_{p=-P_T/2}^{P_T/2} \hat{f}_{xy}(\frac{t}{T},\frac{2\pi p}{P_T})\{\hat{f}_{xx}(\frac{t}{T},\frac{2\pi p}{P_T})\}^{-1}\exp(iu\frac{2\pi p}{P_T})\psi_{j,k}(\frac{t}{T}). \quad (55)
\end{aligned}$$

Finally, a non-linear threshold estimator of $a_u(\frac{t}{T})$ is given as

$$\tilde{a}_u(\frac{s}{T}) = \sum_{j=0}^{J_T}\sum_k \tilde{\beta}_{j,k}^{(u)}\psi_{j,k}(\frac{s}{T}), \tag{56}$$

for J_T the largest j such that $\beta_{j,k}^{(u)} \neq 0$, where, for hard thresholding,

$$\tilde{\beta}_{j,k}^{(u)} = \delta^{(h)}(\hat{\beta}_{j,k}^{(u)}, \lambda_{j,k}) = \hat{\beta}_{j,k}^{(u)} I(\mid \hat{\beta}_{j,k}^{(u)} \mid \geq \lambda_{j,k}) \tag{57}$$

and for soft thresholding

$$\tilde{\beta}_{j,k}^{(u)} = \delta^{(s)}(\hat{\beta}_{j,k}^{(u)}, \lambda_{j,k}) = sgn(\hat{\beta}_{j,k}^{(u)})(\mid \hat{\beta}_{j,k}^{(u)} \mid - \lambda_{j,k})_+ \tag{58}$$

with threshold parameters $\lambda_{j,k}$. Since ψ has compact support, the number of k for which $\psi_{j,k}(\cdot) \neq 0$ is bounded, so only a finite number of terms are involved in (56).

There are a variety of forms of shrinkage estimates. In this paper, we consider hard thresholding.

Now we derive properties for $\hat{\beta}_{j,k}^{(u)}$ and $\tilde{a}_u(\frac{s}{T})$, using wavelet or kernel estimators, described in section 4 and 5, respectively.

6.1 Asymptotic properties of wavelet estimators

In this section, consider the basis $\mathcal{B}_1$ defined in (22). From (52),

$$\hat{B}_a(\frac{t}{T}, \omega) = \hat{f}_{xy}(\frac{t}{T}, \omega)\{\hat{f}_{xx}(\frac{t}{T}, \omega)\}^{-1}, \quad \hat{f}_{xx}(\frac{t}{T}, \omega) \text{ nonsingular},$$

where

$$\hat{f}_{xx}(\frac{t}{T}, \omega) = \hat{c}_{00}^{(xx)} + \sum_{j=0}^{J-1} \sum_{\mathbf{K}=0}^{2^j-1} \sum_{m=h,v,d} \hat{d}_{j,\mathbf{K}}^{m,(xx)} \Psi_{j,\mathbf{K}}^m(\frac{t}{T}, \omega),$$

$$\hat{c}_{00}^{(xx)} = \frac{1}{M} \sum_{i=0}^{M-1} \int_{-\pi}^{\pi} I_N^{(xx)}(\frac{t_i}{T}, \omega) d\omega,$$

$$\hat{d}_{j,\mathbf{K}}^{m,(xx)} = \frac{1}{M} \sum_{i=0}^{M-1} \int_{-\pi}^{\pi} I_N^{(xx)}(\frac{t_i}{T}, \omega) \Psi_{j,\mathbf{K}}^m(\frac{t_i}{T}, \omega) d\omega$$

and

$$I_N^{(xx)}(\frac{t}{T}, \omega) = \frac{1}{2\pi H_N} \mid d^x(\frac{t}{T}, \omega) \mid^2$$

with $d^x(\frac{t}{T},\omega) = \sum_{s=0}^{N-1} h(s/N) X_{t-\frac{N}{2}+s+1,T} e^{-i\omega s}$. Similarly,

$$\hat{f}_{xy}(\frac{t}{T},\omega) = \hat{c}_{00}^{(xy)} + \sum_{j=0}^{J-1} \sum_{\mathbf{K}=0}^{2^j-1} \sum_{m=h,v,d} \hat{d}_{j,\mathbf{K}}^{m,(xy)} \Psi_{j,\mathbf{K}}^m(\frac{t}{T},\omega),$$

$$\hat{c}_{00}^{(xy)} = \frac{1}{M} \sum_{i=0}^{M-1} \int_{-\pi}^{\pi} I_N^{(xy)}(\frac{t_i}{T},\omega) d\omega,$$

$$\hat{d}_{j,\mathbf{K}}^{m,(xy)} = \frac{1}{M} \sum_{i=0}^{M-1} \int_{-\pi}^{\pi} I_N^{(xy)}(\frac{t_i}{T},\omega) \Psi_{j,\mathbf{K}}^m(\frac{t_i}{T},\omega) d\omega$$

and

$$I_N^{(xy)}(\frac{t}{T},\omega) = \frac{1}{2\pi H_N} d^x(\frac{t}{T},\omega) d^y(\frac{t}{T},\omega)$$

with $d^y(\frac{t}{T},\omega) = \sum_{s=0}^{N-1} h(s/N) Y_{t-\frac{N}{2}+s+1,T} e^{i\omega s}$.

Using Lemma 1, we obtain the following result.

Theorem 1: Assuming assumptions (S1) through (S3), we have

(a)

$$\mathrm{E}\{\hat{\beta}_{j,k}^{(u)}\} = \beta_{j,k}^{(u)} + O(T^{-1}) + o(T^{-1}(\ln T)^{-2}), \text{ uniformly over } j,k; \tag{59}$$

(b)

$$\mathrm{Cov}\{\hat{\beta}_{j,k}^{(u)}, \hat{\beta}_{j',k'}^{(u')}\} =$$
$$= \frac{1}{T^2} \sum_{t,t'=0}^{T-1} \frac{1}{(P_T+1)^2} \sum_{p,p'=-P_T/2}^{P_T/2} \frac{\sum_{j,\mathbf{K},m} [C_{j,\mathbf{K}}^{m,(xy)} + C_{j,\mathbf{K}}^{m,(xx)} B_a(\frac{t}{T}, \frac{2\pi p}{P_T}) B_a(\frac{t'}{T}, \frac{2\pi p'}{P_T})] \Psi_{j,\mathbf{K}}^m}{f_{xx}(\frac{t}{T}, \frac{2\pi p}{P_T}) f_{xx}(\frac{t'}{T}, \frac{2\pi p'}{P_T})}$$
$$\exp[i(u\frac{2\pi p}{P_T} + u'\frac{2\pi p'}{P_T})] \psi_{j,k}(\frac{t}{T}) \psi_{j',k'}(\frac{t'}{T}) + o((T\ln T)^{-2}) + O(NT^{-3}), \tag{60}$$

where

$$C_{j,\mathbf{K}}^{m,(xx)} = \frac{A_{j,\mathbf{K}}^{m,(xx)}}{T}, \quad C_{j,\mathbf{K}}^{m,(xy)} = \frac{A_{j,\mathbf{K}}^{m,(xy)}}{T},$$

$$A_{j,\mathbf{K}}^{m,(xx)} = 2C_h \int_{U\times\Pi} \{f_{xx}(u,\omega)\}^2 [\Psi_{j,\mathbf{K}}^m(u,\omega) + \Psi_{j,\mathbf{K}}^m(u,-\omega)] du d\omega,$$

$$A_{j,\mathbf{K}}^{m,(xy)} = 2C_h \int_{U\times\Pi} \{f_{xy}(u,\omega)\}^2 [\Psi_{j,\mathbf{K}}^m(u,\omega) + \Psi_{j,\mathbf{K}}^m(u,-\omega)] du d\omega$$

with $C_h = \frac{\int_0^1 h^4(x)dx}{(\int_0^1 h^2(x)dx)^2}$ for $S = N$ and $C_h = 1$ if $S/N \to 0$,

$$\Psi^m_{j,\mathbf{k}} = \Psi^m_{j,\mathbf{K}}(\frac{t}{T},\omega)\Psi^m_{j,\mathbf{K}}(\frac{t'}{T},\omega'),$$

and $\sum_{j,\mathbf{K},m} C^{m,(xx)}_{j,\mathbf{K}}\Psi^m_{j,\mathbf{K}} = o((\ln T)^{-2})$, $\sum_{j,\mathbf{K},m} C^{m,(xy)}_{j,\mathbf{K}}\Psi^m_{j,\mathbf{K}} = o((\ln T)^{-2})$;

(c) $\hat{\beta}^{(u)}_{j,k}$ has asymptotically a normal distribution, with mean $\beta^{(u)}_{j,k}$ and covariance structure given by (60).

(d)

$$\mathrm{E}\{\tilde{a}_u(\frac{s}{T})\} = a_u(\frac{s}{T}) + O(T^{-1}2^{J_T/2}) + o(T^{-1}(\ln T)^{-2}2^{J_T/2}) + O(NT^{-3}2^{J_T/2}), \quad (61)$$

uniformly over s.

(e)

$$\begin{aligned}\mathrm{Var}\{\tilde{a}_u(\frac{s}{T})\} &= \sum_{j,j',k,k'} \mathrm{Cov}(\hat{\beta}^{(u)}_{j,k},\hat{\beta}^{(u)}_{j',k'})\psi_{j,k}(\frac{s}{T})\psi_{j',k'}(\frac{s}{T}) \\ &\quad + o((T\ln T)^{-2}2^{J_T}) + O(NT^{-3}2^{J_T}). \quad (62)\end{aligned}$$

Remark: We can also use an appropriate wavelet basis $\mathcal{B}_2$ defined in (24) to estimate $\hat{\beta}^{(u)}_{j,k}$ and $\tilde{a}_u(\frac{s}{T})$. The asymptotic properties of these estimators can be found in Chiann (1997). In practice, for our simulation example considered in section 7, these estimators did not lead to good results because the periodogram used in this case can take negative values and is very erratic.

The proof of Theorem 1 is given in the Appendix.

6.2 Asymptotic properties of kernel estimators

In this section, consider kernel estimators of $f_{xx}(\cdot,\cdot)$ and $f_{xy}(\cdot,\cdot)$ described in section 5. We have here

$$\hat{B}_a(\frac{t}{T},\omega) = \hat{f}_{xy}(\frac{t}{T},\omega)\{\hat{f}_{xx}(\frac{t}{T},\omega)\}^{-1}, \quad \hat{f}_{xx}(\frac{t}{T},\omega) \text{ nonsingular},$$

where

$$\hat{f}_{xx}(u,\omega) = \frac{1}{b_f}\int_0^1 K_f(\frac{\omega-\mu}{b_f}) I_N^{(xx)}(u,\mu)d\mu,$$

$$\hat{f}_{xy}(u,\omega) = \frac{1}{b_f^{'}}\int_0^1 K_f^{'}(\frac{\omega-\mu}{b_f^{'}}) I_N^{(xy)}(u,\mu)d\mu,$$

$$I_N^{(xx)}(\frac{t}{T},\omega) = \frac{1}{2\pi H_N} \mid d^x(\frac{t}{T},\omega) \mid^2,$$

and

$$I_N^{(xy)}(\frac{t}{T},\omega) = \frac{1}{2\pi H_N} d^x(\frac{t}{T},\omega) d^y(\frac{t}{T},\omega),$$

with $u = \frac{t}{T}$ and $d^x(\cdot,\cdot)$ and $d^y(\cdot,\cdot)$ as before.

Here we assume that $O(b_f) = O(b_f^{'})$ and $O(b_t) = O(b_t^{'})$.

Using Lemma 2, we obtain the following result.

Theorem 2: Assuming the conditions of Lemma 2 satisfied, we have:

(a)

$$\mathrm{E}\{\hat{\beta}_{j,k}^{(u)}\} = \beta_{j,k}^{(u)} + O(\frac{1}{T}) + O(\frac{1}{b_t b_f T^2}), \text{ uniformly over } j,k; \tag{63}$$

(b)

$$\begin{aligned}
&\mathrm{Var}\{\hat{\beta}_{j,k}^{(u)}\} = \\
&\frac{1}{T^2}\frac{1}{(P_T+1)^2}\sum_{t,p}\frac{1}{T[f_{xx}(\frac{t}{T},\frac{2\pi p}{P_T})]^2}[\frac{1}{b_t^{'}b_f^{'}} f_{xy}(\frac{t}{T},\frac{2\pi p}{P_T})^2 \int_{-1/2}^{1/2} K_t^{'}(x)^2dx \int_{-1/2}^{1/2} K_f^{'}(x)^2 dx \\
&+\frac{1}{b_t b_f} B_a(\frac{t}{T},\frac{2\pi p}{P_T})^2 f_{xx}(\frac{t}{T},\frac{2\pi p}{P_T})^2 \int_{-1/2}^{1/2} K_t(x)^2dx \int_{-1/2}^{1/2} K_f(x)^2dx] \\
&\cdot(2\pi + 2\pi(\frac{2\pi p}{P_T} \equiv 0 \pmod{\pi})) \exp(i2u\frac{2\pi p}{P_T})\psi_{j,k}(\frac{t}{T})^2 + O(\frac{1}{b_t b_f T^3})
\end{aligned} \tag{64}$$

(c) $\hat{\beta}_{j,k}^{(u)}$ has asymptotically a normal distribution with mean $\beta_{j,k}^{(u)}$ and variance structure given by (64).

(d)

$$\mathrm{E}\{\tilde{a}_u(\frac{s}{T})\} = a_u(\frac{s}{T}) + O((T^{-1} + (b_t b_f T^3)^{-1/2})2^{J_T/2}), \tag{65}$$

uniformly over s.

(e)

$$\mathrm{Var}\{\tilde{a}_u(\frac{s}{T})\} = \sum_{j,j',k,k'} \mathrm{Cov}\{\hat{\beta}^{(u)}_{j,k}, \hat{\beta}^{(u)}_{j',k'}\}\psi_{j,k}(\frac{s}{T})\psi_{j',k'}(\frac{s}{T}) + O((b_t b_f T^{-3})^{-1} 2^{J_T}). \tag{66}$$

The proof of Theorem 2 is given in the Appendix.

7 A simulation example

We now present a simulation example for the estimate $\tilde{a}_u(\cdot)$. Here we consider a series $X_{t,T}$ generated as a time varying AR(2), an example that can be found in Dahlhaus (1997):

$$X_{t,T} + \theta_1(\frac{t}{T})X_{t-1,T} + \theta_2(\frac{t}{T})X_{t-2,T} = \sigma(\frac{t}{T})\epsilon_t,$$

with $\theta_1(u) = -1.8\cos(1.5 - \cos 4\pi u)$, $\theta_2(u) = 0.81$, $\sigma(u) = 1$ and the ϵ_t are independent random normal variables with mean zero and variance 1. Figure 1(a) shows this series. In figure 1(b) we present a series $Y_{t,T}$ as:

$$Y_{t,T} = a_1(\frac{t}{T})X_{t-1,T} + a_0(\frac{t}{T})X_{t,T} + a_{-1}(\frac{t}{T})X_{t+1,T} + \epsilon_t,$$

with

$$a_1(\frac{t}{T}) = -1 - 2\sin(\frac{2\pi t}{T} - \pi),$$

$$a_0(\frac{t}{T}) = \frac{2\sin(\frac{4\pi t}{T})}{\frac{4\pi t}{T}},$$

and

$$a_{-1}(\frac{t}{T}) = -3 - 2\cos(\frac{2\pi t}{T} - \pi).$$

The filters $a_1(u)$, $a_0(u)$ and $a_{-1}(u)$ are presented in figure 2.

For the simulation we generated $T = 2048$ data values for $X_{t,T}$ and consequently for $Y_{t,T}$. In order to use a quadratic two-dimensional MRA, we computed the short-time periodogram defined at (27) over $M = 128$ segments of length $N = 128$, with shift $S = 15$, using Tukey Hanning data taper

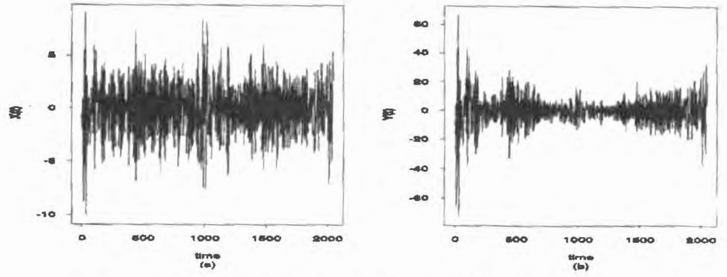

Figure 1: (a) Series $X_{t,T}$ (b) Series $Y_{t,T}$, $t = 1, \cdots, 2048$

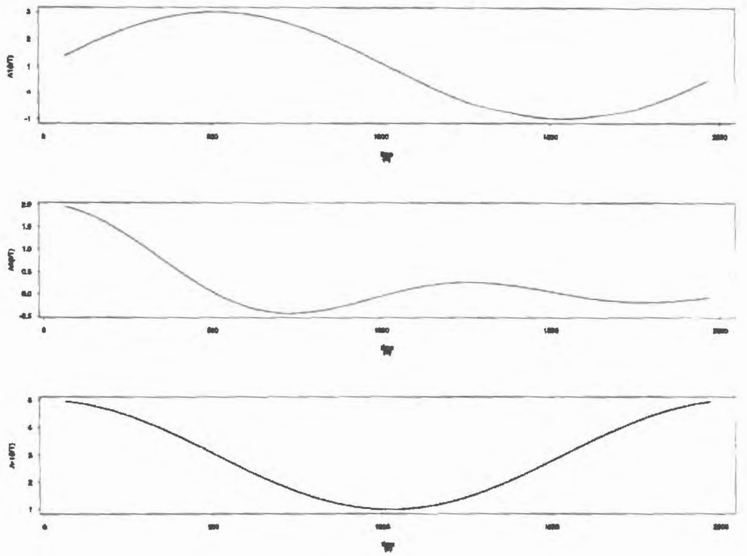

Figure 2: (a) filter $a_1(\frac{t}{T})$ (b) filter $a_0(\frac{t}{T})$ (c) filter $a_{-1}(\frac{t}{T})$

$h(x) = \frac{1}{2}[1 - \cos(2\pi x)]$. For the wavelet basis used, at first, we choose the Daublet orthogonal periodized wavelet (d8) to obtain $\hat{f}_{xx}(\frac{t}{T}, \omega)$ and $\hat{f}_{xy}(\frac{t}{T}, \omega)$, and then obtained $\hat{a}_u(\frac{s}{T})$, for $u = -1, 0, 1$ and $s = 1, 17, \cdots, 2048$. The results are showns in figure 3(a),(b) and (c). Next, in order to improve the estimates obtained above, we use biorthogonal B-spline wavelets to obtain $\tilde{a}_u(\frac{t}{T})$. The results are shown in figure 4(a),(b),(c). Note that the noises in the figure 3 are suppressed by non-linear thresholding without losing local structure of $a_{-1}(\cdot)$ and $a_1(\cdot)$.

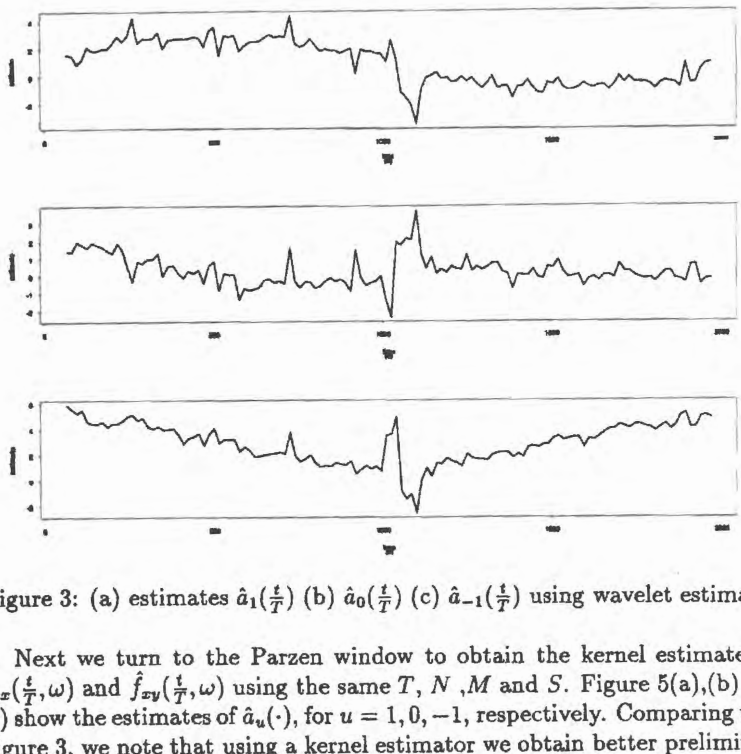

Figure 3: (a) estimates $\hat{a}_1(\frac{t}{T})$ (b) $\hat{a}_0(\frac{t}{T})$ (c) $\hat{a}_{-1}(\frac{t}{T})$ using wavelet estimator

Next we turn to the Parzen window to obtain the kernel estimates of $\hat{f}_{xx}(\frac{t}{T}, \omega)$ and $\hat{f}_{xy}(\frac{t}{T}, \omega)$ using the same T, N ,M and S. Figure 5(a),(b) and (c) show the estimates of $\hat{a}_u(\cdot)$, for $u = 1, 0, -1$, respectively. Comparing with Figure 3, we note that using a kernel estimator we obtain better preliminary estimates for the filter coefficients. Finally, Figure 6 present the improved

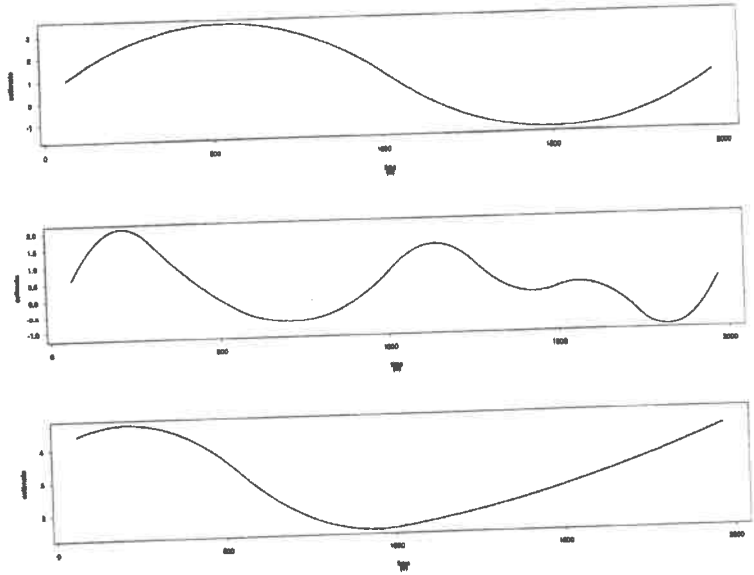

Figure 4: (a) estimates $\tilde{a}_1(\frac{t}{T})$ (b) $\tilde{a}_0(\frac{t}{T})$ (c) $\tilde{a}_{-1}(\frac{t}{T})$

estimates $\tilde{a}_u(\cdot)$ using biorthogonal B-spline wavelets. As above, we see that the non-linear threshold estimators $\tilde{a}_u(\frac{t}{T})$ are better than $\hat{a}_u(\frac{t}{T})$, except for $a_0(\cdot)$.

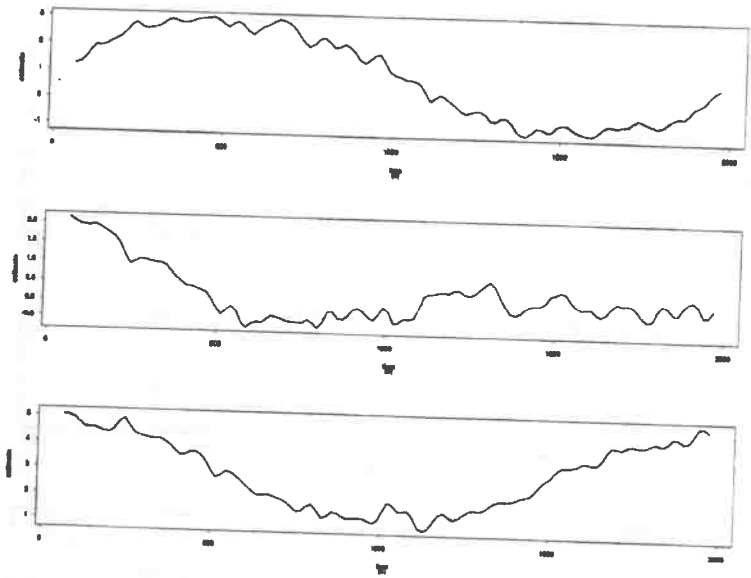

Figure 5: (a) estimates $\hat{a}_1(\frac{t}{T})$ (b) $\hat{a}_0(\frac{t}{T})$ (c) $\hat{a}_{-1}(\frac{t}{T})$ using kernel estimator

8 Further comments

In this article we considered the important problem of estimating the filter coefficients of a time-varying linear system, whose input and output are locally stationary processes in the sense of Dahlhaus. Two types of estimators were entertained: those based on kernel estimates of the evolutionary spectra and cross-spectra involved and those based on wavelet estimates of the latter. Non-linear thresholding procedures were used to obtain the final

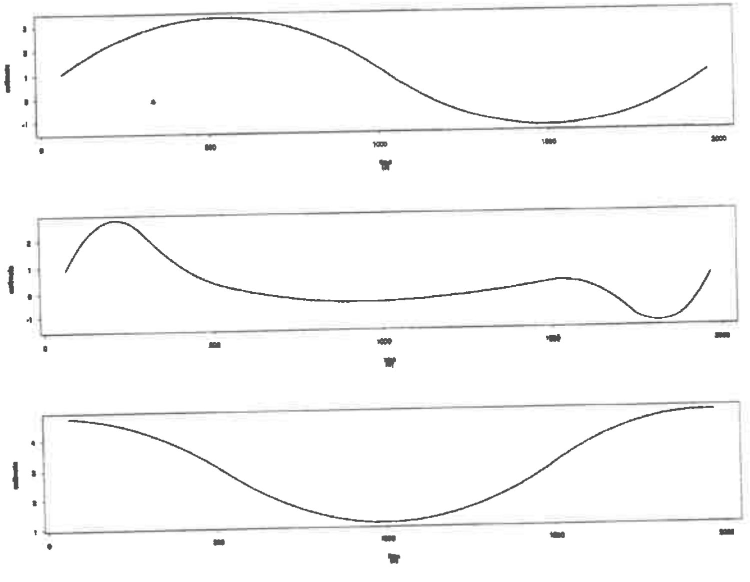

Figure 6: (a) estimates $\tilde{a}_1(\frac{t}{T})$ (b) $\tilde{a}_0(\frac{t}{T})$ (c) $\tilde{a}_{-1}(\frac{t}{T})$

estimators. In both procedures, a wavelet expansion of the filter coefficients is proposed initially. Asymptotic properties of the proposed estimators were derived under various assumptions. Some simulations were performed to assess the validity of the methodology. It was found that the performance of an estimator depends on the wavelet bases used, on the kernels used and on the combination made of these choices. Also, it seems that kernel-based estimates performed better than wavelet-based estimates, but further work in this area remains to be done. For the simulated example, both estimators did not present good results uniformly for all filter coefficients. Thus, one of the coefficients was not estimated well. A few small values obtained for the spectrum estimate can lead to bad estimates for the filters, since the preliminary estimate is a ratio. The use of basis $\mathcal{B}_2$ for the wavelet estimators gave rise to very wigly estimates, since the pre-periodogram (36), besides being itself a non-consistent estimate, can assume negative values. Perhaps the use of a tapered periodogram in this case could improve the estimation, but the asymptotic theory should be established in this situation.

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Appendix

Proof of Theorem 1:

(a) We have

$$E(\hat{f}_{xx}(\frac{t}{T},\omega) = E(\hat{c}_{00}^{(xx)}) + \sum_{j,\mathbf{K},m} E(\hat{d}_{j,\mathbf{K}}^{m,(xx)})\Psi_{j,\mathbf{K}}^{m}(\frac{t}{T},\omega).$$

By Dahlhaus(1997, Lemma A8),

$$E(\hat{c}_{00}^{(xx)}) = c_{00}^{(xx)} + o(T^{-1/2}).$$

Now, using (33),

$$\begin{aligned}\sum_{j,\mathbf{K},m} E(\hat{d}_{j,\mathbf{K}}^{m,(xx)})\Psi_{j,\mathbf{K}}^{m}(\frac{t}{T},\omega) &= \sum_{j,\mathbf{K},m}\{d_{j,\mathbf{K}}^{m,(xx)} + O(2^{-j}N^{-1})\}\Psi_{j,\mathbf{K}}^{m}(\frac{t}{T},\omega)\\ &= \sum_{j,\mathbf{K},m} d_{j,\mathbf{K}}^{m,(xx)}\Psi_{j,\mathbf{K}}^{m}(\frac{t}{T},\omega) + R_1,\end{aligned}$$

where

$$\begin{aligned}\mid R_1 \mid &\leq \sum_{j,\mathbf{K},m} O(2^{-j}N^{-1}) \mid \Psi_{j,\mathbf{K}}^{m}(\frac{t}{T},\omega) \mid\\ &\leq \sum_{j,\mathbf{K},m} O(2^{-j}N^{-1})2^{j}A.\end{aligned}$$

Thus, $R_1 = O(N^{-1}) = o(T^{-1/4})$ with $\mid \Psi^m \mid \leq A$.

Hence,

$$\mathrm{E}\{\hat{f}_{xx}(\frac{t}{T},\omega)\} = f_{xx}(\frac{t}{T},\omega) + o(T^{-1/4}). \tag{67}$$

Remark: At several places it was used the fact that, since $\psi(x)$ has compact support, for a given x, the number of k for which $\psi_{j,k}(x) \neq 0$ is bounded, uniformly in j by $2^{j/2} \mid support\psi \mid$.

Analogously, we have

$$\mathrm{E}\{\hat{f}_{xy}(\frac{t}{T},\omega)\} = f_{xy}(\frac{t}{T},\omega) + o(T^{-1/4}). \tag{68}$$

By Fuller (1976, theorem 5.4.3),

$$
\begin{aligned}
\mathrm{E}\{\hat{B}_a(\frac{t}{T},\omega)\} &= \mathrm{E}\{\hat{f}_{xy}(\frac{t}{T},\omega)[\hat{f}_{xx}(\frac{t}{T},\omega)]^{-1}\} \\
&= \frac{\mathrm{E}\{\hat{f}_{xy}(\frac{t}{T},\omega)\}}{\mathrm{E}\{\hat{f}_{xx}(\frac{t}{T},\omega)\}} + O(a_n^2) \\
&= \frac{f_{xy}(\frac{t}{T},\omega)+o(T^{-1/4})}{f_{xx}(\frac{t}{T},\omega)+o(T^{-1/4})} + o((\ln T)^{-2}) + O(T^{-1}) \\
&= \frac{B_a(\frac{t}{T},\omega)f_{xx}(\frac{t}{T},\omega)+O(T^{-1})+o(T^{-1/4})}{f_{xx}(\frac{t}{T},\omega)+o(T^{-1/4})} + o((\ln T)^{-2}) + O(T^{-1})
\end{aligned}
$$

where

$$
a_n^2 = \mathrm{E}\left| \frac{\hat{f}_{xy}(\frac{t}{T},\omega) - E[\hat{f}_{xy}(\frac{t}{T},\omega)]}{\hat{f}_{xx}(\frac{t}{T},\omega) - E[\hat{f}_{xx}(\frac{t}{T},\omega)]} \right|^2
$$

and $O(a_n^2) = o((\ln T)^{-2}) + O(T^{-1})$.

So from (55), we have

$$
\begin{aligned}
\mathrm{E}\{\hat{\beta}_{j,k}^{(u)}\} &= \frac{1}{T}\frac{1}{P_T+1}\sum_{t=0}^{T-1}\sum_{p=-P_T/2}^{P_T/2} \mathrm{E}\{\hat{B}_a(\frac{t}{T},\frac{2\pi p}{P_T})\}\exp(iu\frac{2\pi p}{P_T})\psi_{j,k}(\frac{t}{T}) \\
&= \frac{1}{T}\frac{1}{P_T+1}\sum_{t=0}^{T-1}\sum_{p=-P_T/2}^{P_T/2}\{\frac{B_a(\frac{t}{T},\frac{2\pi p}{P_T})f_{xx}(\frac{t}{T},\frac{2\pi p}{P_T})+O(T^{-1})+o(T^{-1/4})}{f_{xx}(\frac{t}{T},\frac{2\pi p}{P_T})+o(T^{-1/4})} + \\
&\quad o((\ln T)^{-2}) + O(T^{-1})\}\cdot\exp(iu\frac{2\pi p}{P_T})\psi_{j,k}(\frac{t}{T}) \\
&= \frac{1}{T}\frac{1}{P_T+1}\sum_{t=0}^{T-1}\sum_{p=-P_T/2}^{P_T/2}\frac{B_a(\frac{t}{T},\frac{2\pi p}{P_T})f_{xx}(\frac{t}{T},\frac{2\pi p}{P_T})+O(T^{-1})+o(T^{-1/4})}{f_{xx}(\frac{t}{T},\frac{2\pi p}{P_T})+o(T^{-1/4})}\exp(iu\frac{2\pi p}{P_T})\psi_{j,k}(\frac{t}{T}) \\
&\quad +\frac{1}{T}\frac{1}{P_T+1}\sum_{t=0}^{T-1}\sum_{p=-P_T/2}^{P_T/2}(o((\ln T)^{-2})+O(T^{-1}))\exp(iu\frac{2\pi p}{P_T})\psi_{j,k}(\frac{t}{T}) \\
&= S_1 + S_2.
\end{aligned}
$$

Now,

$$
| S_2 | = | \frac{1}{T}\frac{1}{P_T+1}\sum_{t=0}^{T-1}\sum_{p=-P_T/2}^{P_T/2}(o((\ln T)^{-2})+O(T^{-1}))\exp(iu\frac{2\pi p}{P_T})\psi_{j,k}(\frac{t}{T}) |
$$

$$\leq \ \mid \frac{1}{T}\sum_{t=0}^{T-1}\psi_{j,k}(\frac{t}{T}) \mid\mid \frac{1}{P_T+1}\sum_{p=-P_T/2}^{P_T/2}\exp(iu\frac{2\pi p}{P_T}) \mid (o((\ln T)^{-2})+O(T^{-1}))$$

Thus, $S_2 = o(T^{-1}(\ln T)^{-2}) + O(T^{-2})$.

As $T \to \infty$,

$$\begin{aligned}
S_1 &= \frac{1}{T}\sum_{t=0}^{T-1}\frac{1}{P_T+1}\sum_{p=-P_T/2}^{P_T/2} B_a(\frac{t}{T},\frac{2\pi p}{P_T})\exp(iu\frac{2\pi p}{P_T})\psi_{j,k}(\frac{t}{T}) \\
&= \frac{1}{T}\sum_{t=0}^{T-1}\psi_{j,k}(\frac{t}{T})\frac{1}{P_T+1}\sum_{p=-P_T/2}^{P_T/2}\{\sum_{u'} a_{u'}(\frac{t}{T})\exp(-iu'\frac{2\pi p}{P_T})\}\exp(iu\frac{2\pi p}{P_T}) \\
&= \frac{1}{T}\sum_{t=0}^{T-1}\psi_{j,k}(\frac{t}{T})\frac{1}{P_T+1}\sum_{p=-P_T/2}^{P_T/2} a_u(\frac{t}{T})\exp(-iu\frac{2\pi p}{P_T})\exp(iu\frac{2\pi p}{P_T}) \\
&\quad +\frac{1}{T}\sum_{t=0}^{T-1}\psi_{j,k}(\frac{t}{T})\frac{1}{P_T+1}\sum_{p=-P_T/2}^{P_T/2}\sum_{u'\neq u} a_{u'}(\frac{t}{T})\exp(-i\frac{2\pi p}{P_T}(u'-u)) \\
&= \frac{1}{T}\sum_{t=0}^{T-1}\psi_{j,k}(\frac{t}{T})a_u(\frac{t}{T}) + S_3
\end{aligned}$$

By a Lemma of Polya and Szego(1925), we obtain

$$\mid \frac{1}{T}\sum_{t=0}^{T-1}\psi_{j,k}(\frac{t}{T})a_u(\frac{t}{T}) - \int a_u(\frac{t}{T})\psi_{j,k}(\frac{t}{T})dt \mid \leq \frac{V}{T},$$

where V is the variation of $a_u(\frac{t}{T})\psi_{j,k}(\frac{t}{T})$. Thus,

$$\begin{aligned}
\frac{1}{T}\sum_{t=0}^{T-1}\psi_{j,k}(\frac{t}{T})a_u(\frac{t}{T}) &= \int a_u(\frac{t}{T})\psi_{j,k}(\frac{t}{T})dt + O(T^{-1}) \\
&= \beta_{j,k}^{(u)} + O(T^{-1})
\end{aligned}$$

and

$$\begin{aligned}
\mid S_3 \mid &= \mid \frac{1}{T}\sum_{t=0}^{T-1}\psi_{j,k}(\frac{t}{T})\frac{1}{P_T+1}\sum_{p=-P_T/2}^{P_T/2}\sum_{u'\neq u} a_{u'}(\frac{t}{T})\exp(-i\frac{2\pi p}{P_T}(u'-u)) \mid \\
&\leq \mid \frac{1}{T}\sum_{t=0}^{T-1}\psi_{j,k}(\frac{t}{T}) \mid \frac{1}{P_T+1}\sum_{p=-P_T/2}^{P_T/2}\sum_{u'\neq u} \mid a_{u'}(\frac{t}{T}) \mid\mid \exp(-i\frac{2\pi p}{P_T}(u'-u)) \mid
\end{aligned}$$

Thus, $S_3 = O(T^{-1})$ and $S_1 = \beta_{j,k}^{(u)} + O(T^{-1})$.

Hence, we obtain finally

$$\mathrm{E}\{\hat{\beta}_{j,k}^{(u)}\} = \beta_{j,k}^{(u)} + O(T^{-1}) + o(T^{-1}(\ln T)^{-2}).$$

(b) Now, we want to obtain $\mathrm{Cov}\{\hat{\beta}_{j,k}^{(u)}, \hat{\beta}_{j',k'}^{(u')}\}$.

First, using (34),

$$\begin{aligned}
&\mathrm{Cov}\{\hat{f}_{xx}(\frac{t}{T},\omega), \hat{f}_{xx}(\frac{t'}{T},\omega')\} \\
&= \mathrm{Cov}\{\hat{c}_{00}^{(xx)} + \sum_{j,\mathbf{K},m} \hat{d}_{j,\mathbf{K}}^{m,(xx)} \Psi_{j,\mathbf{K}}^{m}(\frac{t}{T},\omega), \hat{c}_{00}^{(xx)} + \sum_{j',\mathbf{K}',m'} \hat{d}_{j',\mathbf{K}'}^{m',(xx)} \Psi_{j',\mathbf{K}'}^{m'}(\frac{t'}{T},\omega')\} \\
&= \sum_{j,j'} \sum_{\mathbf{K},\mathbf{K}'} \sum_{m,m'} \mathrm{Cov}\{\hat{d}_{j,\mathbf{K}}^{m,(xx)}, \hat{d}_{j',\mathbf{K}'}^{m',(xx)}\} \Psi_{j,\mathbf{K}}^{m}(\frac{t}{T},\omega) \Psi_{j',\mathbf{K}'}^{m'}(\frac{t'}{T},\omega') + R_2 + R_3 + R_4 \\
&= \sum_{j=j'} \sum_{\mathbf{K}=\mathbf{K}'} \sum_{m=m'} \mathrm{Var}\{\hat{d}_{j,\mathbf{K}}^{m,(xx)}\} \Psi_{j,\mathbf{K}}^{m}(\frac{t}{T},\omega) \Psi_{j,\mathbf{K}}^{m}(\frac{t'}{T},\omega') \\
&+ \sum_{j\neq j'} \sum_{\mathbf{K}\neq\mathbf{K}'} \sum_{m\neq m'} \mathrm{Cov}\{\hat{d}_{j,\mathbf{K}}^{m,(xx)}, \hat{d}_{j',\mathbf{K}'}^{m',(xx)}\} \Psi_{j,\mathbf{K}}^{m}(\frac{t}{T},\omega) \Psi_{j',\mathbf{K}'}^{m'}(\frac{t'}{T},\omega') \\
&+ R_2 + R_3 + R_4 \\
&= \sum_{j,\mathbf{K},m} C_{j,\mathbf{K}}^{m,(xx)} \Psi_{j,\mathbf{K}}^{m} + o((\ln T)^{-2}) + O(NT^{-1}),
\end{aligned}$$

with

$$R_2 = \mathrm{Cov}\{\hat{c}_{00}^{(xx)}, \hat{c}_{00}^{(xx)}\} = O(T^{-1}),$$

$$R_3 = \mathrm{Cov}\{\hat{c}_{00}^{(xx)}, \sum_{j,\mathbf{K},m} \hat{d}_{j,\mathbf{K}}^{m,(xx)} \Psi_{j,\mathbf{K}}^{m}(\frac{t}{T},\omega)\} = O(NT^{-1}),$$

and

$$R_4 = \mathrm{Cov}\{\hat{c}_{00}^{(xx)}, \sum_{j',\mathbf{K}',m'} \hat{d}_{j',\mathbf{K}'}^{m',(xx)} \Psi_{j',\mathbf{K}'}^{m'}(\frac{t'}{T},\omega')\} = O(NT^{-1}),$$

where

$$C_{j,\mathbf{K}}^{m,(xx)} = \frac{A_{j,\mathbf{K}}^{m,(xx)}}{T} \quad \text{and } \Psi_{j,\mathbf{K}}^{m} = \Psi_{j,\mathbf{K}}^{m}(\frac{t}{T},\omega)\Psi_{j,\mathbf{K}}^{m}(\frac{t'}{T},\omega').$$

Hence,

$$\mathrm{Cov}\{\hat{f}_{xx}(\frac{t}{T},\omega),\hat{f}_{xx}(\frac{t'}{T},\omega')\} = \sum_{j,\mathbf{K},m} C_{j,\mathbf{K}}^{m,(xx)}\Psi_{j,\mathbf{K}}^{m} + o((\ln T)^{-2}) + O(NT^{-1}). \tag{69}$$

Analogously

$$\mathrm{Cov}\{\hat{f}_{xy}(\frac{t}{T},\omega),\hat{f}_{xy}(\frac{t'}{T},\omega')\} = \sum_{j,\mathbf{K},m} C_{j,\mathbf{K}}^{m,(xy)}\Psi_{j,\mathbf{K}}^{m} + o((\ln T)^{-2}) + O(NT^{-1}). \tag{70}$$

where

$$C_{j,\mathbf{K}}^{m,(xy)} = \frac{A_{j,\mathbf{K}}^{m,(xy)}}{T}.$$

Using a Taylor expansion, we obtain

$$\begin{aligned}
&\mathrm{Cov}\{\hat{B}_a(\frac{t}{T},\frac{2\pi p}{P_T}),\hat{B}_a(\frac{t'}{T},\frac{2\pi p'}{P_T})\}\\
&= Cov\{\frac{\hat{f}_{xy}(\frac{t}{T},\frac{2\pi p}{P_T})}{\hat{f}_{xx}(\frac{t}{T},\frac{2\pi p}{P_T})},\frac{\hat{f}_{xy}(\frac{t'}{T},\frac{2\pi p'}{P_T})}{\hat{f}_{xx}(\frac{t'}{T},\frac{2\pi p'}{P_T})}\}\\
&= \frac{1}{f_{xx}(\frac{t}{T},\frac{2\pi p}{P_T})f_{xx}(\frac{t'}{T},\frac{2\pi p'}{P_T})}\{Cov[\hat{f}_{xy}(\frac{t}{T},\frac{2\pi p}{P_T}),\hat{f}_{xy}(\frac{t'}{T},\frac{2\pi p'}{P_T})]\\
&- B_a(\frac{t}{T},\frac{2\pi p}{P_T})Cov[\hat{f}_{xy}(\frac{t'}{T},\frac{2\pi p'}{P_T}),\hat{f}_{xx}(\frac{t}{T},\frac{2\pi p}{P_T})]\\
&- B_a(\frac{t'}{T},\frac{2\pi p'}{P_T})Cov[\hat{f}_{xy}(\frac{t}{T},\frac{2\pi p}{P_T}),\hat{f}_{xx}(\frac{t'}{T},\frac{2\pi p'}{P_T})]\\
&+ B_a(\frac{t}{T},\frac{2\pi p}{P_T})B_a(\frac{t'}{T},\frac{2\pi p'}{P_T})Cov[\hat{f}_{xx}(\frac{t}{T},\frac{2\pi p}{P_T}),\hat{f}_{xx}(\frac{t'}{T},\frac{2\pi p'}{P_T})]\} + o((\ln T)^{-2}) + O(T^{-1})\\
&= \frac{1}{f_{xx}(\frac{t}{T},\frac{2\pi p}{P_T})f_{xx}(\frac{t'}{T},\frac{2\pi p'}{P_T})}\{\sum_{j,\mathbf{K},m} C_{j,\mathbf{K}}^{m,(xy)}\Psi_{j,\mathbf{K}}^{m} + \sum_{j,\mathbf{K},m} C_{j,\mathbf{K}}^{m,(xx)}\Psi_{j,\mathbf{K}}^{m}B_a(\frac{t}{T},\frac{2\pi p}{P_T})B_a(\frac{t'}{T},\frac{2\pi p'}{P_T})\}\\
&+ o((\ln T)^{-2}) + O(\frac{N}{T}).
\end{aligned}$$

Finally,

$$\text{Cov}\{\hat{\beta}^{(u)}_{j,k},\hat{\beta}^{(u')}_{j',k'}\} = \frac{1}{T^2}\sum_{t,t'}\frac{1}{(P_T+1)^2}\sum_{p,p'}\exp[i(u\frac{2\pi p}{P_T}+u'\frac{2\pi p'}{P_T})]Cov[\hat{B}_a(\frac{t}{T},\frac{2\pi p}{P_T}),\hat{B}_a(\frac{t'}{T},\frac{2\pi p'}{P_T})]\psi_{j,k}(\frac{t}{T})\psi_{j',k'}(\frac{t'}{T}$$

and we obtain (60).

(c) By Dahlhaus(1997, lemma 2(d)), Brillinger (1975, theorem A2, theorem 3.1.1 and proof of theorem 8.10.1), we obtain the result of (c).

(d) We have

$$\tilde{a}_u(\frac{s}{T}) = \sum_{j=0}^{J_T}\sum_k \tilde{\beta}^{(u)}_{j,k}\psi_{j,k}(\frac{s}{T}) \text{ and } a_u(\frac{s}{T}) = \sum_{j,k}\beta^{(u)}_{j,k}\psi_{j,k}(\frac{s}{T}),$$

so,

$$\begin{aligned}\tilde{a}_u(\frac{s}{T}) - a_u(\frac{s}{T}) &= \sum_{j,k}(\tilde{\beta}^{(u)}_{j,k}\psi_{j,k}(\frac{s}{T}) - \beta^{(u)}_{j,k}\psi_{j,k}(\frac{s}{T}))\\ &= \sum_{j,k}\{\hat{\beta}^{(u)}_{j,k}\psi_{j,k}(\frac{s}{T}) - \beta^{(u)}_{j,k}\psi_{j,k}(\frac{s}{T})\} - \sum_{j,k} I(\mid \hat{\beta}^{(u)}_{j,k} \mid < \lambda_{j,k})\hat{\beta}^{(u)}_{j,k}\psi_{j,k}(\frac{s}{T})\\ &= S_1 - S_2\end{aligned}$$

$$\begin{aligned}\text{E}\mid S_1\mid &\leq \sum_{j,k}\mid \text{E}\{\hat{\beta}^{(u)}_{j,k} - \beta^{(u)}_{j,k}\}\mid\mid \psi_{j,k}(\frac{s}{T})\mid\\ &\leq \sum_{j,k}\{O(T^{-1}) + o(T^{-1}(\ln T)^{-2})\}2^{j/2}\mid A\mid\\ &= O(T^{-1}2^{J_T/2}) + o(T^{-1}(\ln T)^{-2}2^{J_T/2}).\end{aligned}$$

with $\mid \psi \mid \leq A$. By Schwarz's inequality,

$$\begin{aligned}\text{E}\mid S_2\mid &\leq \{P(\mid \hat{\beta}^{(u)}_{j,k} \mid < \lambda_{j,k})\}^{1/2}\{\text{E}[(\hat{\beta}^{(u)}_{j,k})^2]\}^{1/2}2^{j/2}A\\ &\leq \{o((T\ln T)^{-2}) + O(NT^{-3})\}2^{J_T/2}A,\end{aligned}$$

so we obtain $S_2 = o((T\ln T)^{-2}2^{J_T}) + O(NT^{-3}2^{J_T})$.

(e) As we have

$$\tilde{a}_u(\frac{s}{T}) - a_u(\frac{s}{T}) = \sum_{j,k}\{\hat{\beta}^{(u)}_{j,k}\psi_{j,k}(\frac{s}{T}) - \beta^{(u)}_{j,k}\psi_{j,k}(\frac{s}{T})\} - \sum_{j,k} I(\mid \hat{\beta}^{(u)}_{j,k} \mid < \lambda_{j,k})\hat{\beta}^{(u)}_{j,k}\psi_{j,k}(\frac{s}{T}),$$

so,

$$\begin{aligned}
\{\tilde{a}_u(\frac{s}{T}) - a_u(\frac{s}{T})\}^2 &= \{\sum_{j,k}\{\hat{\beta}^{(u)}_{j,k}\psi_{j,k}(\frac{s}{T}) - \beta^{(u)}_{j,k}\psi_{j,k}(\frac{s}{T})\}\}^2 \\
&\quad -2\{\sum_{j,k}\{\hat{\beta}^{(u)}_{j,k}\psi_{j,k}(\frac{s}{T}) - \beta^{(u)}_{j,k}\psi_{j,k}(\frac{s}{T})\}\}\{\sum_{j,k} I(|\,\hat{\beta}^{(u)}_{j,k}\,| < \lambda_{j,k})\hat{\beta}^{(u)}_{j,k}\psi_{j,k}(\frac{s}{T})\} \\
&\quad +\{\sum_{j,k} I(|\,\hat{\beta}^{(u)}_{j,k}\,| < \lambda_{j,k})\hat{\beta}^{(u)}_{j,k}\psi_{j,k}(\frac{s}{T})\}^2 \\
&= S_1 + S_2 + S_3
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\{S_1\} &= \mathrm{E}\{\sum_{j,j',k,k'} \psi_{j,k}(\frac{s}{T})\psi_{j',k'}(\frac{s}{T})\{\hat{\beta}^{(u)}_{j,k} - \beta^{(u)}_{j,k}\}\{\hat{\beta}^{(u)}_{j',k'} - \beta^{(u)}_{j',k'}\}\} \\
&= \sum_{j,j',k,k'} \psi_{j,k}(\frac{s}{T})\psi_{j',k'}(\frac{s}{T})\mathrm{Cov}(\hat{\beta}^{(u)}_{j,k}, \hat{\beta}^{(u)}_{j',k'})
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\,|\,S_3\,| &= \mathrm{E}\,|\sum_{j,j',k,k'} I(|\,\hat{\beta}^{(u)}_{j,k}\,| < \lambda_{j,k})I(|\,\hat{\beta}^{(u)}_{j',k'}\,| < \lambda_{j',k'})\hat{\beta}^{(u)}_{j,k}\hat{\beta}^{(u)}_{j',k'}\psi_{j,k}(\frac{s}{T})\psi_{j',k'}(\frac{s}{T})\,| \\
&\leq \sum_{j,j',k,k'} \psi_{j,k}(\frac{s}{T})\psi_{j',k'}(\frac{s}{T})\{P(|\,\hat{\beta}^{(u)}_{j,k}\,| < \lambda_{j,k})\}^{1/4}\{P(|\,\hat{\beta}^{(u)}_{j',k'}\,| < \lambda_{j',k'})\}^{1/4}\{E(\hat{\beta}^{(u)}_{j,k}\hat{\beta}^{(u)}_{j'\,k'})^2\}^{1/2} \\
&\leq \sum_{j,j',k,k'} \psi_{j,k}(\frac{s}{T})\psi_{j',k'}(\frac{s}{T})\{\mathrm{E}(\hat{\beta}^{(u)}_{j,k})^2\mathrm{E}(\hat{\beta}^{(u)}_{j'\,k})^2 + 2\{\mathrm{E}(\beta^{(u)}_{j,k}\hat{\beta}^{(u)}_{j',k'})\}^2\},
\end{aligned}$$

hence,

$$S_3 = o((T\ln T)^{-2}2^{J_T}) + O(NT^{-3}2^{J_T}).$$

Analogously, we have

$$S_2 = o((T\ln T)^{-2}2^{J_T}) + O(NT^{-3}2^{J_T}).$$

Proof of Theorem 2:

(a) Using (42),

$$\mathrm{E}(\hat{f}_{xx}(u,\omega)) = f_{xx}(u,\omega) + V_T^{(xx)}$$

with

$$
\begin{aligned}
V_T^{(xx)} &= \frac{1}{2}b_t^2\int_{-1/2}^{1/2}x^2K_t(x)dx\frac{\partial^2}{\partial u^2}f_{xx}(u,\omega)+\frac{1}{2}b_f^2\int_{-1/2}^{1/2}x^2K_f(x)dx\frac{\partial^2}{\partial\omega^2}f_{xx}(u,\omega)\\
&+ o(b_t^2+\frac{\log(b_tT)}{b_tT}+b_f^2).
\end{aligned}
$$

Analogously, we have

$$
\mathrm{E}(\hat{f}_{xy}(u,\omega)) = f_{xy}(u,\omega)+V_T^{(xy)}
$$

with

$$
\begin{aligned}
V_T^{(xy)} &= \frac{1}{2}b_t'^2\int_{-1/2}^{1/2}x^2K_t'(x)dx\frac{\partial^2}{\partial u^2}f_{xy}(u,\omega)+\frac{1}{2}b_f'^2\int_{-1/2}^{1/2}x^2K_f'(x)dx\frac{\partial^2}{\partial\omega^2}f_{xy}(u,\omega)\\
&+ o(b_t'^2+\frac{\log(b_t'T)}{b_t'T}+b_f'^2).
\end{aligned}
$$

By Fuller (1976, theorem 5.4.3),

$$
\begin{aligned}
\mathrm{E}\{\hat{B}_a(\frac{t}{T},\omega)\} &= \mathrm{E}\{\hat{f}_{xy}(\frac{t}{T},\omega)[\hat{f}_{xx}(\frac{t}{T},\omega)]^{-1}\}\\
&= \frac{\mathrm{E}\{\hat{f}_{xy}(\frac{t}{T},\omega)\}}{\mathrm{E}\{\hat{f}_{xx}(\frac{t}{T},\omega)\}}+O(a_n^2)\\
&= \frac{f_{xy}(\frac{t}{T},\omega)+V_T^{(xy)}}{f_{xx}(\frac{t}{T},\omega)+V_T^{(xy)}}+O(\frac{1}{b_tb_fT})\\
&= \frac{B_a(\frac{t}{T},\omega)f_{xx}(\frac{t}{T},\omega)+O(T^{-1})+V_T^{(xy)}}{f_{xx}(\frac{t}{T},\omega)+V_T^{(xx)}}+O(\frac{1}{b_tb_fT})
\end{aligned}
$$

where

$$
a_n^2 = \mathrm{E}\left|\frac{\hat{f}_{xy}(\frac{t}{T},\omega)-E[\hat{f}_{xy}(\frac{t}{T},\omega)]}{\hat{f}_{xx}(\frac{t}{T},\omega)-E[\hat{f}_{xx}(\frac{t}{T},\omega)]}\right|^2
$$

and $O(a_n^2)=O(\frac{1}{b_tb_fT})$.

So from (55), we have

$$
\mathrm{E}\{\hat{\beta}_{j,k}^{(u)}\} = \frac{1}{T}\frac{1}{P_T+1}\sum_{t=0}^{T-1}\sum_{p=-P_T/2}^{P_T/2}\mathrm{E}\{\hat{B}_a(\frac{t}{T},\frac{2\pi p}{P_T})\}\exp(iu\frac{2\pi p}{P_T})\psi_{j,k}(\frac{t}{T})
$$

$$
\begin{aligned}
&= \frac{1}{T}\frac{1}{P_T+1}\sum_{t=0}^{T-1}\sum_{p=-P_T/2}^{P_T/2}\{\frac{B_a(\frac{t}{T},\frac{2\pi p}{P_T})f_{xx}(\frac{t}{T},\frac{2\pi p}{P_T})+O(T^{-1})+V_T^{(xy)}}{f_{xx}(\frac{t}{T},\frac{2\pi p}{P_T})+V_T^{(xx)}}+ \\
&\quad O(\frac{1}{b_tb_fT})\}\cdot\exp(iu\frac{2\pi p}{P_T})\psi_{j,k}(\frac{t}{T}) \\
&= \frac{1}{T}\frac{1}{P_T+1}\sum_{t=0}^{T-1}\sum_{p=-P_T/2}^{P_T/2}\frac{B_a(\frac{t}{T},\frac{2\pi p}{P_T})f_{xx}(\frac{t}{T},\frac{2\pi p}{P_T})+O(T^{-1})+V_T^{(xy)}}{f_{xx}(\frac{t}{T},\frac{2\pi p}{P_T})+V_T^{(xx)}}\exp(iu\frac{2\pi p}{P_T})\psi_{j,k}(\frac{t}{T}) \\
&\quad +\frac{1}{T}\frac{1}{P_T+1}\sum_{t=0}^{T-1}\sum_{p=-P_T/2}^{P_T/2}(O(\frac{1}{b_tb_fT}))\exp(iu\frac{2\pi p}{P_T})\psi_{j,k}(\frac{t}{T}) \\
&= S_1+S_2.
\end{aligned}
$$

Now,

$$
\begin{aligned}
|S_2| &= |\frac{1}{T}\frac{1}{P_T+1}\sum_{t=0}^{T-1}\sum_{p=-P_T/2}^{P_T/2}(O(\frac{1}{b_tb_fT}))\exp(iu\frac{2\pi p}{P_T})\psi_{j,k}(\frac{t}{T})| \\
&\le |\frac{1}{T}\sum_{t=0}^{T-1}\psi_{j,k}(\frac{t}{T})||\frac{1}{P_T+1}\sum_{p=-P_T/2}^{P_T/2}\exp(iu\frac{2\pi p}{P_T})|O(\frac{1}{b_tb_fT})
\end{aligned}
$$

Thus, $S_2 = O(\frac{1}{b_tb_fT^2})$.

As $T\to\infty$,

$$
\begin{aligned}
S_1 &= \frac{1}{T}\sum_{t=0}^{T-1}\frac{1}{P_T+1}\sum_{p=-P_T/2}^{P_T/2}B_a(\frac{t}{T},\frac{2\pi p}{P_T})\exp(iu\frac{2\pi p}{P_T})\psi_{j,k}(\frac{t}{T}) \\
&= \beta_{j,k}^{(u)}+O(T^{-1})
\end{aligned}
$$

Hence, we obtain

$$
\mathrm{E}\{\hat{\beta}_{j,k}^{(u)}\} = \beta_{j,k}^{(u)}+O(T^{-1})+O(\frac{1}{b_tb_fT^2}).
$$

(b) Using (43), we have

$$
\begin{aligned}
\mathrm{Var}(\hat{f}_{xx}(\frac{t}{T},\omega)) &= (b_tb_fT)^{-1}f_{xx}(\frac{t}{T},\omega)^2\int_{-1/2}^{1/2}K_t(x)^2dx\int_{-1/2}^{1/2}K_f(x)^2dx\cdot \\
&\quad\cdot(2\pi+2\pi\{\omega\equiv 0 \pmod{\pi}\}).
\end{aligned}
$$

and

$$\text{Var}(\hat{f}_{xy}(\frac{t}{T},\omega)) = (b_t^{'}b_f^{'}T)^{-1} f_{xy}(\frac{t}{T},\omega)^2 \int_{-1/2}^{1/2} K_t^{'}(x)^2 dx \int_{-1/2}^{1/2} K_f^{'}(x)^2 dx \cdot$$
$$\cdot(2\pi + 2\pi\{\omega \equiv 0 \pmod{\pi}\}).$$

Thenfore,

$$Var(\hat{B}_a(\frac{t}{T},\frac{2\pi p}{P_T}) = \text{Var}\{\frac{\hat{f}_{xy}(\frac{t}{T},\frac{2\pi p}{P_T})}{\hat{f}_{xx}(\frac{t}{T},\frac{2\pi p}{P_T})}\}$$
$$= \frac{1}{[f_{xx}(\frac{t}{T},\frac{2\pi p}{P_T})]^2}\{Var[\hat{f}_{xy}(\frac{t}{T},\frac{2\pi p}{P_T})] - 2B_a(\frac{t}{T},\frac{2\pi p}{P_T})Cov[\hat{f}_{xy}(\frac{t}{T},\frac{2\pi p}{P_T}),\hat{f}_{xx}(\frac{t}{T},\frac{2\pi p}{P_T})]$$
$$+Ba^2(\frac{t}{T},\frac{2\pi p}{P_T})Var(\hat{f}_{xx}(\frac{t}{T},\frac{2\pi p}{P_T}))\} + O(a_n^2)$$
$$= \frac{1}{[f_{xx}(\frac{t}{T},\frac{2\pi p}{P_T})]^2}[\frac{1}{b_t b_f T} f_{xy}(\frac{t}{T},\frac{2\pi p}{P_T})^2 \int_{-1/2}^{1/2} K_t(x)^2 dx \int_{-1/2}^{1/2} K_f(x)^2 dx$$
$$\cdot(2\pi + 2\pi\{\frac{2\pi p}{P_T} \equiv 0 \pmod{\pi}\} - O(\frac{1}{b_t b_f T}) + Ba^2(\frac{t}{T},\frac{2\pi p}{P_T})\frac{1}{b_t b_f T} f_{xx}(\frac{t}{T},\frac{2\pi p}{P_T})^2$$
$$\cdot\int_{-1/2}^{1/2} K_t(x)^2 dx \int_{-1/2}^{1/2} K_f(x)^2 dx \cdot (2\pi + 2\pi\{\frac{2\pi p}{P_T} \equiv 0 \pmod{\pi}\} + O(\frac{1}{b_f b_t T})$$

Now,

$$\text{Var}(\hat{\beta}_{j,k}^{(u)}) = \frac{1}{T^2}\frac{1}{(P_T+1)^2}\sum_{t,t'}\sum_{p,p'}\text{Cov}\{\hat{B}_a(\frac{t}{T},\frac{2\pi p}{P_T}\hat{B}_a(\frac{t'}{T},\frac{2\pi p'}{P_T})\}\exp(iu(\frac{2\pi p}{P_T}+\frac{2\pi p'}{P_T}))\psi_{j,k}(\frac{t}{T})\psi_{j,k}(\frac{t'}{T})$$
$$= \frac{1}{T^2}\frac{1}{(P_T+1)^2}\{\sum_{t=t'}\sum_{p=p'} Var(\hat{B}_a(\frac{t}{T},\frac{2\pi p}{P_T}))\exp(i2u\frac{2\pi p}{P_T})\psi_{j,k}(\frac{t}{T})^2$$
$$+\sum_{t\neq t'}\sum_{p\neq p'} Cov(\hat{B}_a(\frac{t}{T},\frac{2\pi p}{P_T})\hat{B}_a(\frac{t'}{T},\frac{2\pi p'}{P_T}))\exp(iu(\frac{2\pi p}{P_T}+\frac{2\pi p'}{P_T}))\psi_{j,k}(\frac{t}{T})\psi_{j,k}(\frac{t'}{T})]$$
$$= \frac{1}{T^2}\frac{1}{(P_T+1)^2}[\sum_{t,p}\frac{1}{[f_{xx}(\frac{t}{T},\frac{2\pi p}{P_T})]^2 T}[\frac{1}{b_t^{'}b_f^{'}} f_{xy}(\frac{t}{T},\frac{2\pi p}{P_T})^2 \int_{-1/2}^{1/2} K_t^{'}(x)^2 dx \int_{-1/2}^{1/2} K_f^{'}(x)^2 dx$$
$$+\frac{1}{b_t b_f} Ba^2(\frac{t}{T},\frac{2\pi p}{P_T}) f_{xx}(\frac{t}{T},\frac{2\pi p}{P_T})^2 \int_{-1/2}^{1/2} K_t(x)^2 dx \int_{-1/2}^{1/2} K_f(x)^2 dx]$$
$$\cdot(2\pi + 2\pi\{\omega \equiv 0 \pmod{\pi}\})\exp(i2u\frac{2\pi p}{P_T})\psi_{j,k}(\frac{t}{T})^2 + \sum_{t,p} O(\frac{1}{b_f b_t T})\exp(i2u\frac{2\pi p}{P_T})\psi_{j,k}(\frac{t}{T})^2$$
$$+\sum_{t\neq t'}\sum_{p\neq p'} O(\frac{1}{b_t b_f T})\exp(iu(\frac{2\pi p}{P_T}+\frac{2\pi p'}{P_T}))\psi_{j,k}(\frac{t}{T})\psi_{j,k}(\frac{t'}{T})],$$

and the sum of the last two terms is of order $(\frac{1}{b_t b_f T^3})$. Hence we obtain (64).

(c) The proof of (c) is omitted. It is similar to the proof of Lemma A.10 in Dahlhaus (1997) and the proof of Theorem 1(c).

(d) The proof (d) is similar to the proof (d) of Theorem 1.

(e) The proof (e) is similar to the proof (e) of Theorem 1.

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