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# *PARTIAL EXCHANGEABILITY AND EXTENDIBILITY*

*by*

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# Partial exchangeability and extendibility

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**Abstract**

In this paper we present conditions to finite and infinite extendibility for some of process partially exchangeable associated with several sequences of random variables $X_{i1}, X_{i2}, \ldots$, $i = 1, 2, \ldots, m$ taking values for countable sets. We derive a representation theorem in some cases and we show using this result that the problem of extendibility can be reduced to the multidimensional moment problems. The connections with ordinary exchangeability are established.

**Key words:** de Finetti's Theorem, exchangeability, partial exchangeability, extendibility, binomial distributions, geometric distributions, uniform distributions.

## 1 Introduction

It is known that the de Finetti representation of 0-1 random variables and its extensions do not hold in general for finite sequences (see for example, Diaconis, 1977 and Spizzichino, 1982). The problem is that not every finite symmetric sequence is part of an infinitely long exchangeable sequence with the same type of symmetry. When this is possible we say that the sequence is extendible or representable. In this work we explore the problem of extendibility for partial exchangeability in the sense of de Finetti (1938, 1959). We consider exchangeable sequences $X_{i1}, X_{i2}, \ldots$ for random quantities observed in each $i \in \{1, 2, \ldots, m\}$ contexts, with exchangeability within of each group. In general, for this class of sequences there is a representation theorem. For example, if $X_{i1}, X_{i2}, \ldots$, $i = 1, 2, \ldots, m$ are infinitely exchangeable sequences of 0-1 random quantities with joint probability measure $P$, there exists a probability measure $\mu$ on $[0,1]^m$ such that

$$\begin{aligned} & P(X_{ij} = x_{ij},\ j = 1, 2, \ldots, n_i;\ i = 1, 2, \ldots, m) \\ = & \int_{[0,1]^m} \prod_{i=1}^{m} \theta_i^{t_i(n_i)} (1-\theta_i)^{n_i - t_i(n_i)} d\mu(\theta_1, \theta_2, \ldots, \theta_m) \end{aligned} \tag{1}$$

where $t_i(n_i) = x_{i1} + x_{i2} + \cdots + x_{in_i}$. Extension to this theorem in more general cases can be obtained using the notion of predictive sufficient statistics (Bernardo, J.M. and Smith, A.F.M, 1994).

As in the case of ordinary exchangeability, representation analogous to (1) does not hold in general for finite partially exchangeable sequences. To discuss this problem, let us call a partially exchangeable sequence $X_{i1}, X_{i2}, \ldots, X_{in_i}$, $i = 1, 2, \ldots, m$, $(N_1, N_2, \ldots, N_m)$-extendible if it is part of the longer sequences $X_{i1}, X_{i2}, \ldots, X_{iN_i}$, $i = 1, 2, \ldots, m$ with $N_i > n_i$, $i = 1, 2, \ldots, m$ which are partially exchangeable. Infinite extendibility corresponds to $(N_1, N_2, \ldots, N_m)$-extendibility for all $N_i > n_i$, $i = 1, 2, \ldots, m$. The purpose of this work is to give conditions of extendibility for class of partial exchangeability. We consider types of partially exchangeable sequences $X_{i1}, X_{i2}, \ldots, X_{iN_i}$, $i = 1, 2, \ldots, m$ taking values in $\mathcal{X}$ (countable) with the property that

$$\begin{aligned} & P(X_{ij} = x_{ij};\ j = 1, 2, \ldots, N_i,\ i = 1, 2, \ldots, m \mid T_i(N_i) = t_i,\ i = 1, \ldots, m) \\ = & \prod_{i=1}^{m} P(X_{ij} = x_{ij};\ j = 1, 2, \ldots, N_i \mid T_i(N_i) = t_i) \end{aligned}$$

where $T_i(N_i)$ is a function of $X_{i1}, X_{i2}, \ldots, X_{iN_i}$ and $P(\cdot \mid T_i(N_i) = t_i)$ is uniform.

In section 2 we show that when $\mathcal{X} = \{0, 1\}$ or $\mathcal{X} = \mathbb{N}$ and $T_i(N_i) = \sum_{j=1}^{N_i} X_{ij}$ the conditions for extendibility are proved using the Multidimensional Hausdorff Moment Theorem (Shohat, J.A. and Tamarkin, J.D., 1943).

The case where $\mathcal{X} = \mathbb{N}$ and $T_i(N_i) = \max\{X_{i1}, \ldots, X_{iN_i}\}$ is presented in section 3. We derive a representation theorem and a simple necessary and sufficient condition is obtained based on the distribution of the vector $(T_1(N_1), T_2(N_2), \ldots, T_m(N_m))$.

Finally in section 4 finite forms and finite extendibility are discussed using the results for ordinary exchangeability.

# 2 Infinite extendibility and Moment Hausdorff Problem

Let $X_{i1}, X_{i2}, \ldots, X_{iN_i}$, $i = 1, 2, \ldots, m$ be finite sequences of random variables with joint probability measure $P^N$, where $N = (N_1, N_2, \ldots, N_m)$. We say that the sequence is partially exchangeable if for each $\pi_i$ permutation of $\{1, 2, \ldots, N_i\}$, $i = 1, 2, \ldots, m$,

$$(X_1, X_2, \ldots, X_m) \stackrel{\mathbb{D}}{=} (X_{\pi_1}, X_{\pi_2}, \ldots, X_{\pi_m}) \tag{2}$$

where $X_i = (X_{i1}, X_{i2}, \ldots, X_{iN_i})$, $X_{\pi_i} = (X_{i\pi_i(1)}, X_{i\pi_i(2)}, \ldots, X_{i\pi_i(N_i)})$ and $\mathbb{D}$ denotes equality in distribution. For infinite sequences the definition above is true for every $N_i \in \mathbb{N}$, $i = 1, 2, \ldots, m$. This type of invariance is sometimes called unrestricted exchangeabilty.

For 0 - 1 random variables it is not difficult to see that finite partial exchangeability implies that

$$\begin{aligned} & P^N(X_{ij} = x_{ij},\ j = 1, 2, \ldots, N_i,\ i = 1, 2, \ldots, m \mid T_1(N_1) = t_1, \ldots, T_m(N_m) = t_m) \\ = & \prod_{i=1}^{m} \binom{N_i}{t_i}^{-1} I_{\{t_i\}} \left( \sum_{j=1}^{N_i} X_{ij} \right) \end{aligned} \tag{3}$$

where $T_i(N_i) = \sum_{j=1}^{N_i} X_{ij}$. That means that condicionally on the totals, the sequences $X_{i1}, X_{i2}, \ldots, X_{iN_i}$, $i = 1, 2, \ldots, m$ are condicionally independents, each one uniformly distributed on the set $\{(y_1, \ldots, y_{N_i}) : \sum_{j=1}^{N_i} y_j = t_i,\ i = 1, 2, \ldots, m\}$.

Note that identity (3) implies that

$$P^N(X_{ij} = x_{ij},\ j = 1, 2, \ldots, N_i;\ i = 1, 2, \ldots, m)$$

$$= \prod_{i=1}^{m} \binom{N_i}{t_i} P(T_1(N_1) = t_1, \ldots, T_m(N_m) = t_m) \tag{4}$$

where $t_i = \sum_{j=1}^{N_i} x_{ij}$, $i = 1, 2, \ldots, m$.

Obviously this is true for $N$-dimensional distribution of infinitelly partial exchangeable sequences of 0 - 1 random quantities. Hence, the condition for infinite extendibility is given in terms of the distribution of the totals $T_1(N_1), \ldots, T_m(N_m)$. Consequently, if $X_{i1}, \ldots, X_{iN_i}$, $i = 1, 2, \ldots, m$ is a partially exchangeable sequence of 0 - 1 random variables with joint probability measure $P^N$, $N = (N_1, N_2, \ldots, N_m)$, then the sequence is infinitely extendible if, and only if, there exists a probability measure $\mu$ on $[0,1]^m$ such that

$$P^N(T_1(N_1) = t_1, \ldots, T_m(N_m) = t_m) = \int_{[0,1]^m} \prod_{i=1}^{m} \binom{N_i}{t_i} \theta_i^{t_i}(1-\theta_i)^{N_i - t_i} \mathrm{d}\mu(\theta_1, \ldots, \theta_m) \tag{5}$$

for all $(t_1, t_2, \ldots, t_m) \in X_{i=1}^{m}\{0, 1, \ldots, N_i\}$, where $X$ denotes cartesian product. Note that if $P^N(T_1(N_1) = t_1, \ldots, T_m(N_m) = t_m) = 0$ for some $(t_1, t_2, \ldots, t_m)$ then the sequence cannot be extended.

A condition for infinite extendibility can be obtained from condition (5) using the solution to the Multidimensional Hausdorff Moment Problem.

Let $\mu_{i_1 i_2 \cdots i_m}$ $(i_1, i_2, \ldots, i_m = 0, 1, 2, \ldots)$ be a real sequence. We attribute to the expression

$$\Delta_1^{n_1} \Delta_2^{n_2} \cdots \Delta_m^{n_m} \mu_{i_1 i_2 \cdots i_m}$$

the usual meaning where the first operator $\Delta_1^{n_1}$ applies to the first subscript $i_1$, the second operator $\Delta_2^{n_2}$ applies independently to the second subscript $i_2$, and so on, where

$$\Delta^k \mu_n = \sum_{m=0}^{k} (-1)^m \binom{k}{m} \mu_{n+m} \cdot$$

Let $\nabla_i = \Delta_i^{-1}$, $i = 1, 2, \ldots, m$.

**Proposition 2.1** *Let $X_{i1}, X_{i2}, \ldots, X_{iN_i}$, $i = 1, 2, \ldots, m$ be a partially exchangeable sequence of 0 - 1 random quantities with joint probability measure $P^N$. If the sequence is* *extendible then*

$$\Delta_1^{i_1}\Delta_2^{i_2}\cdots\Delta_m^{i_m}\prod_{i=1}^{m}\binom{N_i}{t_i}^{-1}P^N(T_1(N_1)=t_1,\ldots,T_m(N_m)=t_m)\geq 0 \tag{6}$$

*for* $i_k = 0,1,\ldots,t_k;\ k=1,2,\ldots,m.$

**Proof:** The hypothesis of extendibility implies that there is $\mu$, a probability measure on $[0,1]^m$, such that

$$\begin{aligned}
&\prod_{i=1}^{m}\binom{N_i}{t_i}^{-1}P^N(T_1(N_1)=t_1,\ldots,T_m(N_m)=t_m)\\
=\ &\int_{[0,1]^m}\prod_{i=1}^{m}\theta_i^{t_i}(1-\theta_i)^{N_i-t_i}\mathrm{d}\mu(\theta_1,\theta_2,\ldots,\theta_m)\\
=\ &\int_{[0,1]^m}\prod_{i=1}^{m}\theta_i^{t_i}\sum_{r_i=0}^{N_i-t_i}\binom{N_i-t_i}{r_i}(-1)^{r_i}\theta_i^{r_i}\mathrm{d}\mu(\theta_1,\ldots,\theta_m)\\
=\ &\Delta_1^{N_1-t_1}\Delta_2^{N_2-t_2}\cdots\Delta_m^{N_m-t_m}\int_{[0,1]^m}\prod_{i=1}^{m}\theta_i^{t_i}\mathrm{d}\mu(\theta_1,\ldots,\theta_m)\,.
\end{aligned}$$

Taking inverse operation both sides we have

$$\begin{aligned}
&\nabla_1^{N_1-t_1}\cdots\nabla_m^{N_m-t_m}\prod_{i=1}^{m}\binom{N_i}{t_i}^{-1}P^N(T_1(N_1)=t_1,\ldots,T_m(N_m)=t_m)\\
=\ &\int_{[0,1]^m}\prod_{i=1}^{m}\theta_i^{t_i}\mathrm{d}\mu(\theta_1,\ldots,\theta_m)\,.
\end{aligned}$$

Hence, extendibility implies that the constants on the left side of above expression corresponds to moments of some distribution on $[0,1]^m$. Consequently, they are solutions to the Multidimensional Hausdorff Moment Problem.

In the particular case of $m=2$ and $N_1=N_2=2$, the condition is Proposition 2.1 can be summarized by the following inequality

$$Ap\geq 0$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 1/2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1/2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & -1/2 & 0 & -1/2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} P_{00} \\ P_{10} \\ P_{20} \\ P_{01} \\ P_{11} \\ P_{21} \\ P_{02} \\ P_{12} \\ P_{22} \end{bmatrix}$$

with $P_{ij} = P(T_1(2) = i, T_2(2) = j)$, $i, j = 0, 1, 2$.

**Example 2.1** Let $X_{11}, X_{12}, X_{21}, X_{22}$ be partially exchangeable 0 - 1 variables with joint probability measure $P$ defined by

$$P(T_1(2) = t_1, T_2(2) = t_2) = \frac{4}{49}(\frac{1}{2})^{t_1 - t_2};\ t_1, t_2 = 0, 1, 2.$$

By simple checking we see that the first inequality in (6) cannot be verified.

**Remark 2.1** Note from expression (5) that extendibility implies extendibility of each sequence $X_{i1}, \ldots, X_{iN_i}$, $i = 1, 2, \ldots, m$, but the converse is not true. Also, if $N_i = 1$, $i = 1, 2, \ldots, m$ then the sequence is always extendible, but this case is unrealistic.

The problem of infinite extendibility for finite partially exchangeable sequence of 0-1 random quantities has been studied by Plato (1991) who gives a necessary condition. The condition presented in our work is an alternative to the Plato's condition. For $N_1 = N_2 = 2$, Plato's condition can be translated as

$$P(X_{11} = 1, X_{12} = 1)P(X_{21} = 1, X_{22} = 1) \geq [P(X_{11} = 1, X_{21} = 1)]^2. \quad (7)$$

In Example 2.1 this condition is satisfied, consequently nothing can be said from (7) with respect to extendibility. On the other hand, if the distribution of $X_{11}, X_{12}, X_{21}, X_{22}$ is given by

$$P(T_1(2) = t_1, T_2(2) = t_2) = \frac{16}{49}(\frac{1}{2})^{t_1 + t_2};\ t_1, t_2 = 0, 1, 2,$$

then the inequalities in (6) are true and, the Plato condition does not hold. However if we impose tha additional condition that

$$
\begin{aligned}
& P(X_{11}=1, X_{21}=1, X_{21}=1, X_{22}=0) = P(X_{11}=1, X_{12}=0, X_{21}=1, X_{22}=1) \\
= \; & P(X_{11}=1, X_{12}=0, X_{21}=1, X_{22}=0) = 0
\end{aligned}
$$

then condition (6) implies (7).

For the class of measures satisfying the condition above, Plato's condition for non-extendibility is stronger than the condition in Proposition 2.1.

We consider the following partially exchangeable sequences $X_{i1}, X_{i2}, \ldots, i = 1, 2, \ldots, m$ taking values in $\mathbb{N}$ with the property that for all $N_i \in \mathbb{N}$, $i = 1, 2, \ldots, m$

$$
\begin{aligned}
& P(X_{ij} = x_{ij},\ j = 1,2,\ldots,N_i,\ i = 1,2,\ldots,m \mid T_i(N_i) = t_i,\ i = 1,2,\ldots,m) \\
= \; & \prod_{i=1}^{m} P(X_{ij} = x_{ij},\ j = 1,2,\ldots,N_i \mid T_i(N_i) = t_i) \qquad (8)
\end{aligned}
$$

where $T_i(N_i) = X_{i1} + X_{i2} + \cdots + X_{iN_i}$ and $P(\cdot \mid T_i(N_i) = t_i)$ is uniform on the set $\{(y_1, y_2, \ldots, y_{N_i}) : \sum_{j=1}^{N_i} y_j = t_i,\ i = 1, \ldots, m\}$.

As before, extendibility of a finite sequence in this case will depend on the joint distribution of the totals.

It is known (Diaconis and Freeedman, 1987) that for each $i \in I$ fixed, $X_{i1}$, $X_{i2}$, $\ldots$, $X_{iN_i}$, $\ldots$ are conditionally independent and identically distributed geometric random variables with sucess probability $\theta_i$. This fact jointly with condition (8) and the law of large numbers implies that there is a probability measure $\mu$ over $[0,1]^m$ such that for all $N_i \in \mathbb{N}$, $i = 1, 2, \ldots, m$

$$
\begin{aligned}
& P(X_{ij} = x_{ij},\ j = 1,2,\ldots,N_i;\ i = 1,2,\ldots,m) \\
= \; & \int_{[0,1]^m} \prod_{i=1}^{m} (1-\theta_i)^{N_i} \theta_i^{t_i(N_i)} \, \mathrm{d}\mu(\theta_1, \ldots, \theta_m) \qquad (9)
\end{aligned}
$$

where $t_i(N_i) = x_{i1} + \cdots + x_{iN_i}$, $i = 1, 2, \ldots, m$ and $x_{ij} \in \mathbb{N}$. Thus, for all $N_i$, $i = 1, 2, \ldots, m$

$$\frac{P(T_i(N_i) = t_i;\ i = 1, 2, \ldots, m)}{\prod_{i=1}^{m} \binom{N_i + T_i - 1}{t_i}} = \int_{[0,1]^m} \prod_{i=1}^{m} (1 - \theta_i)^{N_i} \theta_i^{t_i(N_i)} \mathrm{d}\mu(\theta_1, \ldots, \theta_m), \tag{10}$$

$t_i \in \mathbb{N}$, $i = 1, 2, \ldots, m$. Now

$$\begin{aligned}
&\int_{[0,1]^m} \prod_{i=1}^{m} (1 - \theta_i)^{N_i} \theta_i^{t_i(N_i)} \mathrm{d}\mu(\theta_1\ \ldots, \theta_m) \\
&= \int_{[0,1]^m} \prod_{i=1}^{m} \theta_i^{t_i(N_i)} \sum_{s_i=0}^{N_i} \binom{N_i}{s_i} (-1)^{s_i} \theta_i^{s_i} \mathrm{d}\mu(\theta_1, \ldots, \theta_m) \\
&= \int_{[0,1]^m} \prod_{i=1}^{m} \sum_{s_i=0}^{N_i} (-1)^{s_i} \binom{N_i}{s_i} \theta_i^{s_i + t_i(N_i)} \mathrm{d}\mu(\theta_1, \ldots, \theta_m) \\
&= \Delta_1^{N_1} \Delta_2^{N_2} \cdots \Delta_m^{N_m} \int_{[0,1]^m} \prod_{i=1}^{m} \theta_i^{t_i(N_i)} \mathrm{d}\mu(\theta_1, \ldots, \theta_m)\,.
\end{aligned}$$

Taking inverse operation both sides we have that

$$\begin{aligned}
&\nabla_1^{N_1} \nabla_2^{N_2} \cdots \nabla_m^{N_m} \prod_{i=1}^{m} \binom{N_i + t_i - 1}{t_i}^{-1} P(T_i(N_i) = t_i,\ i = 1, 2, \ldots, m) \\
&= \int_{[0,1]^m} \prod_{i=1}^{m} \theta_i^{t_i(N_i)} \mathrm{d}\mu(\theta_1, \ldots, \theta_m)
\end{aligned} \tag{11}$$

$t_i(N_i) \in \mathbb{N}$; $i = 1, 2, \ldots, m$.

In this manner extendibility to this class of sequences partially exchangeable is actually Multidimension Hausdorff Moment Problem.

**Proposition 2.2** *Let $X_{i1}, \ldots, X_{iN_i}$, $i = 1, 2, \ldots, m$ be a finite sequence partially exchangeable taking values in $\mathbb{N}$ satisfying condition (8). The sequence is extendible to the class of infinite sequences satisfying the representation in (9) if and only if*

$$\Delta_1^{k_1} \Delta_2^{k_2} \cdots \Delta_m^{k_m} \prod_{i=1}^{m} \binom{N_i + t_i - 1}{t_i}^{-1} P(T_i(N_i) = t_i;\ i = 1, \ldots, m) \geq 0$$

*for each $k_i \in \mathbb{N}$, $t_i \in \mathbb{N}$; $i = 1, 2, \ldots, m$.*

**Proof:** It follows from identity (9) and the Multidimensional Hausdorff Moment Problem.

**Remark 2.2** It is also clear that in this case extendibility implies extendibility of each one of the sequences $X_{i1}, X_{i2}, \ldots, X_{iN_i}$. On the other hand, if $N_i = 1$, the sequence is not necessarily extendible.

In fact, applying this condition for $N_i = 1$ and $m = 2$ we have that

$$\Delta_1^{k_1}\Delta_2^{k_2} P(X_{11} = x_{11}, X_{12} = x_{12}) \geq 0$$

for $k_i \in \mathbb{N}$, $i = 1, 2$.

For $k_1 = k_2 = 1$ this implies that

$$\begin{aligned} & P(X_{11} = x_{11}, X_{12} = x_{12}) + P(X_{11} = x_{11} + 1, X_{12} = x_{12} + 1) \\ \geq\ & P(X_{11} = x_{11} + 1, X_{12} = x_{12}) + P(X_{11} = x_{11}, X_{12} = x_{12} + 1) \end{aligned}$$

for $x_{ij} \in \mathbb{N}$. This inequality is not satisfied for arbitrary random variables taking values in $\mathbb{N}$. The conclusion is obvious because we consider only the subclass of partially exchangeable sequences taking values in $\mathbb{N}$.

# 3 Infinite extendibility to Mixture of Product of Uniform Distribution

In this section we consider the case in that $T_i(N_i) = \max_{1 \leq j \leq N_i}\{X_{ij}\}$, $i = 1, 2, \ldots, m$. In this case we have a reparametrization theorem.

**Theorem 3.1** *Let $X_{i1}, X_{i2}, \ldots$, $i = 1, 2, \ldots, m$, be an infinite partially exchangeable sequence taking values in $\mathbb{N}$ satisfying condition (8). Then exists a probability measure $\mu$ on $\mathbb{N}^m$ such that, for each $n_i \in \mathbb{N}$, $i = 1, 2, \ldots, m$,*

$$\begin{aligned} & P(X_{ij} = x_{ij}\ ;\ \ j = 1, 2, \ldots, n_i,\ i = 1, 2, \ldots, m) \\ = & \int_{\mathbb{N}^m} \frac{1}{(\theta_1 + 1)^{n_1}} \cdots \frac{1}{(\theta_m + 1)^{n_m}} I_{X_{i=1}^m\{0,1,\ldots,\theta_i\}}\Big(\max_{1 \leq j \leq n_i}\{x_{ij}\};\ i = 1, 2, \ldots, m\Big)\, d\mu(\theta_1, \ldots, \theta_m). \end{aligned} \tag{12}$$

**Proof.** For each $n_i \le N_i$, $i = 1, 2, \ldots, m$, we have that

$$P\left(X_{ij} = x_{ij} \; ; \; \begin{matrix} j = 1,2,\ldots,n_i \\ i = 1,2,\ldots,m \end{matrix} \Big| \max_{1 \le j \le N_i}\{X_{ij}\} = m_i \; , \quad i = 1,2,\ldots,m\right)$$

$$= \prod_{i=1}^{m} P\left(X_{ij} = x_{ij} \; , \; j = 1,2,\ldots,n_i \Big| \max_{1 \le j \le N_i}\{X_{ij}\} = m_i\right) .$$

Now, for each $i \in \{1, 2, \ldots, m\}$

$$P\left(X_{ij} = x_{ij} \; ; \; j = 1,2,\ldots,n_i \Big| \max_{1 \le j \le N_i}\{X_{ij}\} = m_i\right)$$

$$= \begin{cases} \dfrac{1}{(m_i+1)^{n_i}} \left\{ \dfrac{1 - \left(\frac{m_i}{m_i+1}\right)^{N_i - n_i}}{1 - \left(\frac{m_i}{m_i+1}\right)^{N_i}} \right\} & \text{if } \max_{1 \le j \le n_i}\{x_{ij}\} < m_i \\ \dfrac{1}{(m_i+1)^{n_i}} \left\{ \dfrac{1}{1 - \left(\frac{m_i}{m_i+1}\right)^{N_i}} \right\} & \text{if } \max_{1 \le j \le n_i}\{x_{ij}\} = m_i . \end{cases}$$

Hence,

$$P\left(X_{ij} = x_{ij} \; ; \; \begin{matrix} j = 1,2,\ldots,n_i \\ i = 1,2,\ldots,m \end{matrix}\right)$$

$$= \sum_{m_m > \max_{1 \le j \le n_m}\{x_{ij}\}} \cdots \sum_{m_1 > \max_{1 \le j \le n_1}\{x_{ij}\}} P\left(X_{ij} = x_{ij} \; ; \; \begin{matrix} j = 1,2,\ldots,n_i \\ i = 1,2,\ldots,m \end{matrix} \Big| \begin{matrix} T_1(N_1) = m_1, \ldots, \\ T_m(N_m) = m_m \end{matrix}\right)$$
$$\times P(T_1(N_1) = m_1, \ldots, T_m(N_m) = m_m)$$

$$+ \sum_{m_m = \max_{1 \le j \le n_m}\{x_{ij}\}} \cdots \sum_{m_1 = \max_{1 \le j \le n_1}\{x_{ij}\}} P\left(X_{ij} = x_{ij} \; ; \; \begin{matrix} j = 1,2,\ldots,n_i \\ i = 1,2,\ldots,m \end{matrix} \Big| \begin{matrix} T_1(N_1) = m_1, \ldots, \\ T_m(N_m) = m_m \end{matrix}\right)$$
$$\times P(T_1(N_1) = m_1, \ldots, T_m(N_m) = m_m)$$

$$= \sum_{m_m > \max_{1 \le j \le n_m}\{x_{ij}\}} \cdots \sum_{m_1 > \max_{1 \le j \le n_1}\{x_{ij}\}} \prod_{i=1}^{m} \frac{1}{(m_i+1)^{n_i}} \left\{ \frac{1 - \left(\frac{m_i}{m_i+1}\right)^{N_i - n_i}}{1 - \left(\frac{m_i}{m_i+1}\right)^{N_i}} \right\}$$
$$\times P(T_1(N_1) = m_1, \ldots, T_m(N_m) = m_m)$$

$$+ \sum_{m_m = \max_{1 \le j \le n_m}\{x_{ij}\}} \cdots \sum_{m_1 = \max_{1 \le j \le n_1}\{x_{ij}\}} \prod_{i=1}^{m} \frac{1}{(m_i+1)^{n_i}} \left\{ \frac{1}{1 - \left(\frac{m_i}{m_i+1}\right)^{N_i}} \right\}$$

$$\times P(T_1(N_1) = m_1, \ldots, T_m(N_m) = m_m)$$

$$= \int_{N^m} \left( \prod_{i=1}^{m} \frac{1}{(\theta_i+1)^{n_i}} \left\{ \frac{1-\left(\frac{\theta_i}{\theta_i+1}\right)^{N_i-n_i}}{1-\left(\frac{\theta_i}{\theta_i+1}\right)^{N_i}} \right\} I_{(\theta_1>\max_{1\le j\le n_1}\{x_{1j}\}\cdots\theta_m>\max_{1\le j\le n_m}\{x_{ij}\})} \right.$$

$$\left. + \prod_{i=1}^{m} \frac{1}{(\theta_i+1)^{n_i}} \left\{ \frac{1}{1-\left(\frac{\theta_i}{\theta_i+1}\right)^{N_i}} \right\} I_{(\theta_1=\max_{1\le j\le n_1}\{x_{i_j}\}\cdots\theta_m=\max_{1\le j\le n_m}\{x_{ij}\})} \right) d\mu_{\underset{\sim}{N}}(\theta_1, \theta_2, \ldots, \theta_m) \quad (13$$

where $\mu_{\underset{\sim}{N}} = \mu_{N_1} \times \mu_{N_2} \times \cdots \times \mu_{N_m}$ for $\underset{\sim}{N} = (N, N, \ldots, N)$ is the $P$-law of $(T_1(N_1), T_2(N_2), \ldots, T_m(N_m))$.

Now, for each $i \in \{1, 2, \ldots, m\}$ fixed, $\{\mu_{N_i} = \mu_{i1} \times \mu_{i2} \times \cdots \times \mu_{iN_i}\}_{N_i \in \mathbb{N}}$, the $P$-law of $T_i(N_i)$ is tight (Iglesias-Zuazola, Pereira and Tanaka, 1993, p.17). Consequently (Billingsley, 1968, p.41) $\{\mu_{\underset{\sim}{N}}\}_{\underset{\sim}{N} \in \mathbb{N}^m}$ is tight. Hence each sequence in family $\{\mu_{\underset{\sim}{N}}\}_{\underset{\sim}{N} \in \mathbb{N}^m}$ has a subsequence converging weakly to a probability measure $\mu$ on $\mathbb{N}^m$. Taking limit in (13) through subsequences we have the result.

**Corollary 3.1** *Let $X_{i1}, X_{i2}, \ldots, X_{iN}$, $i = 1, 2, \ldots, m$ be a finite partially exchangeable sequence taking values in $\mathbb{N}$. The sequence is infinitely extendible if, and only if, there exists a probability measure $\mu$ on $\mathbb{N}^m$ such that*

$$P(T_1(N_1) = m_1, \ldots, T_m(N_m) = m_m)$$
$$= \int_{(\theta_1,\ldots,\theta_m)\ge(m_1,\ldots,m_m)} \frac{(m_1+1)^{N_1} - m_1^{N_1}}{(\theta_1+1)^{N_1}} \cdots \frac{(m_m+1)^{N_m} - m_m^{N_m}}{(\theta_m+1)^{N_m}} d\mu(\theta_1, \ldots, \theta_m) \; . \quad (14)$$

**Proof.** By the hypothesis of the extendibility and Theorem 2.1 there exists a probability measure on $\mathbb{N}^m$ such that

$$P\left(X_{ij} = x_{ij} \,, \begin{array}{c} j = 1, 2, \ldots, N_i \\ i = 1, 2, \ldots, m \end{array}\right)$$
$$= \int_{(\theta_1,\ldots,\theta_m)\ge(m_1,\ldots,m_m)} \frac{1}{(\theta_1+1)^{N_1}} \cdots \frac{1}{(\theta_m+1)^{N_m}} d\mu(\theta_1, \ldots, \theta_m)$$

where $m_i = \max\{X_{i1}, \ldots, X_{iN_i}\}$, $i = 1, 2, \ldots, m$ e $(x_{i1}, \ldots, x_{iN_i}) \in \mathbb{N}^{N_i}$.

Using the fact that

$$P(T_1(N_1) = m_1, \ldots, T_m(N_m) = m_m) = \left[(m_1+1)^{N_1} - m_1^{N_1}\right] \cdots \left[(m_m+1)^{N_m} - m_m^{N_m}\right] \times P\left(X_{ij} = x_{ij}\ ,\ \begin{matrix} j = 1, 2, \ldots, N_i \\ i = 1, 2, \ldots, m \end{matrix}\right) \tag{15}$$

the necessity is proved.

Reciprocally, suppose that exists a probability measure $\mu$ on $\mathbb{N}^m$ such that (14) is satisfied. Now,

$$\begin{aligned} & P\left(X_{ij} = x_{ij}\ ,\ \begin{matrix} j = 1, 2, \ldots, N_i \\ i = 1, 2, \ldots, m \end{matrix}\right) \\ = \ & P\left(X_{ij} = x_{ij}\ ,\ \begin{matrix} j = 1, 2, \ldots, N_i \\ i = 1, 2, \ldots, m \end{matrix} \middle| T_1(N_1) = m_1, \ldots, T_m(N_m) = m_m\right) \\ & \times P(T_1(N_1) = m_1, \ldots, T_m(N_m) = m_m) \\ = \ & \int_{(\theta_1, \ldots, \theta_m) \geq (m_1, \ldots, m_m)} P\left(X_{ij} = x_{ij}\ ,\ \begin{matrix} j = 1, 2, \ldots, N_i \\ i = 1, 2, \ldots, m \end{matrix} \middle| T_1(N_1) = m_1, \ldots, T_m(N_m) = m_m\right) \\ & \times \frac{(m_1+1)^{N_1} - m_1^{N_1}}{(\theta_1+1)^{N_1}} \cdots \frac{(m_m+1)^{N_m} - m_m^{N_m}}{(\theta_m+1)^{N_m}} \mathrm{d}\mu(\theta_1, \ldots, \theta_m)\ . \end{aligned}$$

Using (15) we have that

$$\begin{aligned} & P\left(X_{ij} = x_{ij}\ ,\ \begin{matrix} j = 1, 2, \ldots, N_i \\ i = 1, 2, \ldots, m \end{matrix}\right) \\ & = \int_{(\theta_1, \ldots, \theta_m) \geq (m_1, \ldots, m_m)} \frac{1}{(\theta_1+1)^{N_1}} \cdots \frac{1}{(\theta_m+1)^{N_m}} \mathrm{d}\mu(\theta_1, \ldots, \theta_m)\ . \end{aligned}$$

Consider now $Z_{i1}, Z_{i2}, \ldots$, $i = 1, 2, \ldots, m$ be an infinite partially exchangeable sequence taking values in $\mathbb{N}$ such that, for each $N_i \in \mathbb{N}$, $i = 1, 2, \ldots, m$,

$$\begin{aligned} & P\left(Z_{ij} = z_{ij}\ ,\ \begin{matrix} j = 1, 2, \ldots, N_i \\ i = 1, 2, \ldots, m \end{matrix}\right) \\ & = \int_{(\theta_1, \ldots, \theta_m) \geq (m_1, \ldots, m_m)} \frac{1}{(\theta_1+1)^{N_1}} \cdots \frac{1}{(\theta_m+1)^{N_m}} \mathrm{d}\mu(\theta_1, \ldots, \theta_m)\ . \end{aligned}$$

Then,

$$(X_1, X_2, \ldots, X_m) \overset{\mathbb{D}}{=} (Z_1, Z_2, \ldots, Z_m)$$

where $X_i = (X_{i1}, \ldots, X_{iN_i})$ and $Z_i = (Z_{i1}, \ldots, Z_{iN_i})$, $i = 1, 2, \ldots, m$. Hence the sequence is infinitely extendible.

**Corollary 3.2** *Let $X_{i1}, X_{i2}, \ldots, X_{iN_i}$, $i = 1, 2$, be a finite partially exchangeable sequence taking values in $\mathbb{N}$. The sequence is infinitely extendible if and only if, the function $f : \mathbb{N}^2 \to \mathbb{R}$ defined by*

$$f(m_1, m_2) = (m_1+1)^{N_1}(m_2+1)^{N_2} \sum_{r_1=0}^{1} \sum_{r_2=0}^{1} (-1)^{r_1+r_2} \binom{1}{r_1}\binom{1}{r_2} \frac{P(T_1(N_1) = m_1 + r_1, T_2(N_2) = m_2 + r_2)}{[(m_1+r_1+1)^{N_1} - (m_1+r_1)^{N_1}][(m_2+r_2+1)^{N_2} - (m_2+r_2)^{N_2}]} \quad (16)$$

*is a probability measure on $\mathbb{N}^2$.*

**Proof.** If the finite sequence is extendible, we have by Corollary 2.1 that exists a probability measure $\mu$ on $\mathbb{N}^m$ such that

$$\begin{aligned}
A &=: P(T_1(N_1) = m_1, T_2(N_2) = m_2) \\
&= \int_{(\theta_1,\theta_2) \geq (m_1,m_2)} \frac{(m_1+1)^{N_1} - m_1^{N_1}}{(\theta_1+1)^{N_1}} \frac{(m_2+1)^{N_2} - m_2^{N_2}}{(\theta_2+1)^{N_2}} d\mu(\theta_1, \theta_2) \\
&= \sum_{\theta_1=m_1}^{\infty} \sum_{\theta_2=m_2}^{\infty} \frac{(m_1+1)^{N_1} - m_1^{N_1}}{(\theta_1+1)^{N_1}} \frac{(m_2+1)^{N_2} - m_2^{N_2}}{(\theta_2+1)^{N_2}} \mu(\{\theta_1, \theta_2\}) \\
&= \sum_{\theta_1=m_1}^{\infty} \frac{(m_1+1)^{N_1} - m_1^{N_1}}{(\theta_1+1)^{N_1}} \frac{(m_2+1)^{N_2} - m_2^{N_2}}{(m_2+1)^{N_2}} \mu(\{\theta_1, m_2\}) \\
&\quad + \sum_{\theta_1=m_1}^{\infty} \sum_{\theta_2=m_2+1}^{\infty} \frac{(m_1+1)^{N_1} - m_1^{N_1}}{(\theta_1+1)^{N_1}} \frac{(m_2+1)^{N_2} - m_2^{N_2}}{(\theta_2+1)^{N_2}} \mu(\{\theta_1, \theta_2\}) \\
&= \frac{(m_1+1)^{N_1} - m_1^{N_1}}{(m_1+1)^{N_1}} \frac{(m_2+1)^{N_2} - m_2^{N_2}}{(m_2+1)^{N_2}} \mu(\{m_1, m_2\})
\end{aligned}$$

$$+\sum_{\theta_1=m_1+1}^{\infty}\frac{(m_1+1)^{N_1}-m_1^{N_1}}{(\theta_1+1)^{N_1}}\frac{(m_2+1)^{N_2}-m_2^{N_2}}{(m_2+1)^{N_2}}\mu(\{\theta_1,m_2\})$$
$$+\sum_{\theta_2=m_2+1}^{\infty}\frac{(m_1+1)^{N_1}-m_1^{N_1}}{(m_1+1)^{N_1}}\frac{(m_2+1)^{N_2}-m_2^{N_2}}{(\theta_2+1)^{N_2}}\mu(\{m_1,\theta_2\})$$
$$+\sum_{\theta_1=m_1+1}^{\infty}\sum_{\theta_2=m_2+1}^{\infty}\frac{(m_1+1)^{N_1}-m_1^{N_1}}{(\theta_1+1)^{N_1}}\frac{(m_2+1)^{N_2}-m_2^{N_2}}{(\theta_2+1)^{N_2}}\mu(\{\theta_1,\theta_2\}) \quad (17)$$

In analogous manner

$$\begin{aligned} B &=: P(T_1(N_1)=m_1+1, T_2(N_2)=m_2+1) \\ &= \sum_{\theta_1=m_1+1}^{\infty}\sum_{\theta_2=m_2}^{\infty}\frac{(m_1+2)^{N_1}-(m_1+1)^{N_1}}{(\theta_1+1)^{N_1}}\frac{(m_2+2)^{N_2}-(m_2+1)^{N_2}}{(\theta_2+1)^{N_2}}\mu(\{\theta_1,\theta_2\}) \end{aligned} \quad (18)$$

$$\begin{aligned} & P(T_1(N_1)=m_1, T_2(N_2)=m_2+1) \\ &= \sum_{\theta_1=m_1}^{\infty}\sum_{\theta_2=m_2+1}^{\infty}\frac{(m_1+1)^{N_1}-m_1^{N_1}}{(\theta_1+1)^{N_1}}\frac{(m_2+2)^{N_2}-(m_2+1)^{N_2}}{(\theta_2+1)^{N_2}}\mu(\{\theta_1,\theta_2\}) \end{aligned} \quad (19)$$

and

$$\begin{aligned} & P(T_1(N_1)=m_1+1, T_2(N_2)=m_2) \\ &= \sum_{\theta_1=m_1+1}^{\infty}\sum_{\theta_2=m_2}^{\infty}\frac{(m_1+2)^{N_1}-(m_1+1)^{N_1}}{(\theta_1+1)^{N_1}}\frac{(m_2+1)^{N_2}-m_2^{N_2}}{(\theta_2+1)^{N_2}}\mu(\{\theta_1,\theta_2\}) . \end{aligned} \quad (20)$$

Subtracting $B$ from $A$ we have

$$\begin{aligned} & \frac{(m_1+1)^{N_1}-m_1^{N_1}}{(m_1+1)^{N_1}}\frac{(m_2+1)^{N_2}-m_2^{N_2}}{(m_2+1)^{N_2}}\mu(\{m_1,m_2\}) \\ &= A-\frac{(m_1+1)^{N_1}-m_1^{N_1}}{(m_1+2)^{N_1}-(m_1+1)^{N_1}}\frac{(m_2+1)^{N_2}-m_2^{N_2}}{(m_2+2)^{N_2}-(m_2+1)^{N_2}}B \\ & -\frac{(m_2+1)^{N_2}-m_2^{N_2}}{(m_2+1)^{N_2}}C-\frac{(m_1+1)^{N_1}-m_1^{N_1}}{(m_1+1)^{N_1}}D . \end{aligned} \quad (21)$$

where,

$$\begin{aligned} C &= \sum_{\theta_1=m_1+1}^{\infty}\frac{(m_1+1)^{N_1}-m_1^{N_1}}{(\theta_1+1)^{N_1}}\mu(\{\theta_1,m_2\}) \quad \text{and} \\ D &= \sum_{\theta_2=m_2+1}^{\infty}\frac{(m_2+1)^{N_2}-m_2^{N_2}}{(\theta_2+1)^{N_2}}\mu(\{m_1,\theta_2\}) . \end{aligned}$$

Now, from (19) and (20), respectively, we have that

$$\frac{(m_1+1)^{N_1}-m_1^{N_1}}{(m_1+1)^{N_1}}D = \frac{(m_2+1)^{N_2}-m_2^{N_2}}{(m_2+2)^{N_2}-(m_2+1)^{N_2}}P(T_1(N_1)=m_1, T_2(N_2)=m_2+1)$$
$$-\frac{(m_1+1)^{N_1}-m_1^{N_1}}{(m_1+2)^{N_1}-(m_1+1)^{N_1}}\frac{(m_2+1)^{N_2}-m_2^{N_2}}{(m_2+2)^{N_2}-(m_2+1)^{N_2}}B\,, \tag{22}$$

and

$$\frac{(m_2+1)^{N_2}-m_2^{N_2}}{(m_2+1)^{N_2}}C = \frac{(m_1+1)^{N_1}-m_1^{N_1}}{(m_1+2)^{N_1}-(m_1+1)^{N_1}}P(T_1(N_1)=m_1+1, T_2(N_2)=m_2+1)$$
$$-\frac{(m_1+1)^{N_1}-m_1^{N_1}}{(m_1+2)^{N_1}-(m_1+1)^{N_1}}\frac{(m_2+1)^{N_2}-m_2^{N_2}}{(m_2+2)^{N_2}-(m_2+1)^{N_2}}B\,. \tag{23}$$

Summing (22) and (23) and replacing the result in (21) we have

$$\begin{aligned}\mu(\{m_1,m_2\}) &= (m_1+1)^{N_1}(m_2+1)^{N_2}\sum_{r_1=0}^{1}\sum_{r_2=0}^{1}(-1)^{r_1+r_2}\binom{1}{r_1}\binom{1}{r_2} \\ &\quad\times\frac{P(T_1(N_1)=m_1+r_1, T_2(N_2)=m_2+r_2)}{[(m_1+r_1+1)^{N_1}-(m_1+r_1)^{N_1}][(m_2+r_2+1)^{N_2}-(m_2+r_2)^{N_2}]}\,.\end{aligned}$$

Hence $f$ defines a probability measure on $\mathbb{N}^2$.

Reciprocally, suppose that $f$ defines a probability measure $\mu_f$ on $\mathbb{N}^2$, i.e. $\mu_f(\{m_1, m_2\}) = f(m_1,m_2)$, $(m_1,m_2)\in\mathbb{N}^2$. Let $Z_{i1}, Z_{i2}, \ldots$, $i=1,2$, be an infinite sequence partially exchangeable with joint probability measure $P$ given by

$$P\left(Z_{ij}=z_{ij}\ ,\ \begin{matrix} j=1,2,\ldots,N_i \\ i=1,2\end{matrix}\right) = \int\limits_{(\theta_1,\theta_2)\geq(m_1,m_2)} \frac{1}{(\theta_1+1)^{N_1}}\frac{1}{(\theta_2+1)^{N_2}}\mathrm{d}\mu_f(\{\theta_1,\theta_2\})\,,$$

$N_i\in\mathbb{N}$, $i=1,2$; $m_i=\max\{z_{i1},\ldots,z_{iN_i}\}$ and $(z_{i1},\ldots,z_{iN_i})\in\mathbb{N}^{N_i}$ for $i=1,2$.

Now,

$$\begin{aligned}&P(\max\{Z_{11},\ldots,Z_{1N_1}\}=m_1\ ,\ \max\{Z_{21},\ldots,Z_{2N_2}\}=m_2) \\ &= \sum_{\theta_1=m_1}^{\infty}\sum_{\theta_2=m_2}^{\infty}\frac{(m_1+1)^{N_1}-m_1^{N_1}}{(\theta_1+1)^{N_1}}\frac{(m_2+1)^{N_2}-m_2^{N_2}}{(\theta_2+1)^{N_2}}f(\theta_1,\theta_2)\,. \end{aligned}\tag{24}$$

Making $\mu=\mu_f$ on $\mathbb{N}^2$ and using Corollary 2.1 we have that the finite sequence is extendible.

Although this case let more simple than the previous, the result is more stronger, in the sense that the Corollary 2.2 exhibt the form of the measure in the mixture, in the case of the finite sequence be extendible. The following example illustrate this fact.

**Example 3.1** Let $X_{i1}, X_{i2}, \ldots, X_{iN_i}$; $i = 1,2$, be a finite sequence of random variables taking values in $\mathbb{N}$ and let $T_i(N_i) = \max\{X_{i1}, X_{i2}, \ldots, X_{iN_i}\}$, $i = 1,2$. If

$$\begin{aligned} & X_{11}, X_{12}, \ldots, X_{1N_1}, X_{21}, X_{22}, \ldots, X_{2N_2} \mid (T_1(N_1) = m_1, T_2(N_2) = m_2) \\ \sim\ & \bigcup \{(x_{i1}, x_{i2}, \ldots, x_{iN_i}) \in \mathbb{N}^{N_i} : \max\{x_{i1}, \ldots, x_{iN_i}\} = m_i;\ i = 1,2\} \end{aligned}$$

and if $P(T_1(N_1) = M_1 + r_1, T_2(N_2) = M_2 + r_2) = 0$ when $r_1 = r_2 = 0$ and $r_1 = r_2 = 1$ for some $M_1 \in \mathbb{N}$ and some $M_2 \in \mathbb{N}$, $P(T_1(N_1) = m_1 + r_1,\ T_2(N_2) = m_2 + r_2) > 0$ for other values of $r_1$ and $r_2$ and for each $m_1 \neq M_1$ and each $m_2 \neq M_2$, then the finite sequence $X_{i1}, X_{i2}, \ldots, X_{iN_i}$, $i = 1,2$, is not extendible for a infinite sequence partially exchangeable of random variables. In fact,

$$\begin{aligned} f(M_1, M_2) \ =\ & (M_1+1)^{N_1}(M_2+1)^{N_2} \left\{ -\frac{P(T_1(N_1) = M_1, T_2(M_2) = M_2 + 1)}{[(M_1+1)^{N_1} - M_1^{N_1}][(M_2+2)^{N_2} - (M_2+1)^{N_2}]} \right. \\ & \left. - \frac{P(T_1(N_1) = M_1 + 1, T_2(N_2) = M_2)}{[(M_1+2)^{N_1} - (M_1+1)^{N_1}][(M_2+1)^{N_2} - M_2^{N_2}]} \right\} < 0\,, \end{aligned}$$

hence $f$ no define a probability measure. Therefore by Corollary 2.2 we have that the finite sequence $X_{i1}, X_{i2}, \ldots, X_{iN_i}$; $i = 1,2$, is not extendible.

# 4 Finite extendibility

Let $X_{i1}, X_{i2}, \ldots, X_{iN_i}$, $i = 1, 2, \ldots, m$ be a finite partially exchangeable sequence taking values in $\mathcal{X}$ (countable) and $T_i(N_1)$ a function of $X_{i1}, X_{i2}, \ldots, X_{iN_i}$, $i = 1, 2, \ldots, m$. As in the previous section consider that the conditional distribution of $X_{i1}, X_{i2}, \ldots, X_{iN_i}$; $i = 1, 2, \ldots, m$, given $(T_1(N_1), \ldots, T_m(N_m))$ is uniform. Consider also $\mathcal{X}^{N_i} = \mathcal{X} \times \mathcal{X} \times \cdots \mathcal{X}$ a $N_i$-fold product from a set $\mathcal{X}$ and $\mathcal{I}_i$ the image of $\mathcal{X}^{N_i}$ by $T_i(N_i)$, $i = 1, 2, \ldots, m$.

Then, for each $n_i \le N_i$, $i = 1, 2, \ldots, m$,

$$\begin{aligned} & P(X_{ij} = x_{ij}\ ,\ \ j = 1, 2, \ldots, n_i,\ i = 1, 2, \ldots, m) \\ = & \sum_{t_1 \in \mathcal{I}_1} \cdots \sum_{t_m \in \mathcal{I}_m} \prod_{i=1}^{m} P\left( X_{ij} = x_{ij}\ ,\ \begin{array}{c} j = 1, 2, \ldots, n_i \\ i = 1, 2, \ldots, m \end{array} \middle| \, T_i(N_i) = t_i,\ i = 1, 2, \ldots, m \right) \\ & \times P(T_1(N_1) = t_1, \ldots, T_m(N_m) = t_m) \end{aligned}$$

for $x_{ij} \in \mathcal{X}$, $j = 1, 2, \ldots, n_i$ and $i = 1, 2, \ldots, m$.

This is an exact finite representation. Using this representation and the argument given in the previous section we can conclude that the finite sequence $X_{i1}, X_{i2}, \ldots, X_{in_i}$, $i = 1, 2, \ldots, m$ of partially exchangeable random variables taking values in $\mathcal{X}$ is $(N_1, N_2, \ldots, N_m)$-extendible, with $N_i \ge n_i$, for $i = 1, 2, \ldots, m$, if and only if there exists a probability measure $W$ on $\times_{i=1}^{m} \mathcal{I}$ such that

$$\begin{aligned} & P(T_i(n_i) = t_i;\ i = 1, 2, \ldots, m) \qquad (25) \\ = & \sum_{j_1 \in \mathcal{I}_1} \cdots \sum_{j_m \in \mathcal{I}_m} \prod_{i=1}^{m} P\big( X_{ij} = x_{ij}, j = 1, 2, \ldots, n_1 \big| T_i(N_i) = t_i, i = 1, 2, \ldots, m \big) W(j_1, \ldots, j_m) \end{aligned}$$

where $\times$ denotes cartesian product.

In the case that $\mathcal{X} = \{0, 1\}$ and $T_i(N_i) = \sum_{j=1}^{N_i} X_{ij}$, for $i = 1, 2, \ldots, m$, we have

$$P\left( X_{ij} = x_{ij};\ j = 1, 2, \ldots, n_i \middle| \, T_i(N_i) = t_i,\ i = 1, 2, \ldots, m \right) = \frac{\binom{j_i}{t_i}\binom{N_i - j_i}{n_i - t_i}}{\binom{N_i}{n_i}},$$

for $i = 1, 2, \ldots, m$; when $\mathcal{X} = \mathbb{N}$ and $T_i(N_i) = \sum_{j=1}^{N_i} X_{ij}$, for $i = 1, 2, \ldots, m$,

$$P\left( X_{ij} = x_{ij};\ j = 1, 2, \ldots, n_i \middle| \, T_i(N_i) = t_i \right) = \binom{N_i - n_i + j_i - t_i}{j_i - t_i} \binom{N_i + j_i - 1}{j_i}^{-1},$$

$i = 1, 2, \ldots, m$, and finally, when $\mathcal{X} = \mathbb{N}$ and $T_i(N_i) = \max\{X_{i1}, \ldots, X_{iN_i}\}$; for $i = 1, 2, \ldots, m$, we have

$$P\left( X_{ij} = x_{ij};\ j = 1, 2, \ldots, n_i \middle| \, T_i(N_i) = m_i,\ i = 1, 2, \ldots, m \right)$$

$$= \begin{cases} \dfrac{1}{(m_i+1)^{n_i}} \left\{ \dfrac{1-\left(\frac{m_i}{m_i+1}\right)^{N_i-n_i}}{1-\left(\frac{m_i}{m_i+1}\right)^{N_i}} \right\} & \text{if } \max_{1\le j\le N_i}\{x_{ij}\} < m_i \\ \dfrac{1}{(m_i+1)^{n_i}} \left\{ \dfrac{1}{1-\left(\frac{m_i}{m_i+1}\right)^{N_i}} \right\} & \text{if } \max_{1\le j\le N_i}\{x_{ij}\} = m_i . \end{cases}$$

Using the total probability theorem, marginalization and extendibility we have the following proposition:

**Proposition 4.1** *Let $X_{i1}, X_{i2}, \ldots, X_{in_i}$, $i = 1, 2, \ldots, m$, be a finite partially exchangeable sequence taking values in $\mathcal{X}$ (countable). If the sequence is $(N_1, N_2, \ldots, N_m)$-extendible ($N_i > n_i$, $i = 1, 2, \ldots, m$) then for each $i = 1, 2, \ldots, m$ the sequence $X_{i1}, X_{i2}, \ldots, X_{in_i}$ is $N_i$-extendible.*

The converse to the last proposition is not true in general, as the next example shows.

**Example 4.1** Let $X_{11}, X_{12}, X_{21}, X_{22}$ be a partially exchangeable random variables taking values in $\{0,1\}$ with joint probability measure determined by

$$P(T_1(2) = 1, T_2(2) = 1) = \frac{1}{6}$$

and

$$P(T_1(2) = 1) = P(T_2(2) = 1) = \frac{1}{2} .$$

Then, $X_{11}, X_{12}, X_{21}, X_{22}$ is not (3,3)-extendible, but for each $i = 1, 2$ we have that $X_{i1}, X_{i2}$ is 3-extendible.

In fact, it follows from (25) that if $X_{11}, X_{12}, X_{21}, X_{22}$ is (3,3)-extendible, then,

$$1 - \frac{3}{2}P(T_1(2) = 1) - \frac{3}{2}P(T_2(2) = 1) + \frac{9}{4}P(T_1(2) = 1, T_2(2) = 1) \ge 0 .$$

As in this example such inequality does not hold, the sequence is not extendible.

Now, by Diaconis and Freedman (1980), we have that, for each $i = 1, 2$, $X_{i1}, X_{i2}$ is 3-extendible if, and only if, there exist weights $W(j)$, $j = 0, 1, 2, 3$, satisfying $W(j) \geq 0$, $j = 0, 1, 2, 3$; $\sum_{j=0}^{3} W(j) = 1$ and

$$\begin{bmatrix} 1 & 1/3 & 0 & 0 \\ 0 & 2/3 & 2/3 & 0 \\ 0 & 0 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} W(0) \\ W(1) \\ W(2) \\ W(3) \end{bmatrix} = \begin{bmatrix} P(T_i(2) = 0) \\ P(T_i(2) = 1) \\ P(T_i(2) = 2) \end{bmatrix}, \quad i = 1, 2.$$

By simple computation we have that when $P(T_i(2) = 1) \leq 2/3$ for $i = 1, 2$, then for each $i = 1, 2$, $X_{i1}, X_{i2}$ is 3-extendible.

Finite forms of the representation exhibited in the previous section can be obtained by using the finite forms for ordinary exchangeable as in Diaconis and Freedman (1980, 1986), Iglesias-Zuazola (1993) and the fact that if $Q_i$ and $P_i$ are probability measures such that

$$\|Q_i - P_i\| \leq C_i \,, \quad i = 1, 2, \ldots, m, \tag{26}$$

then

$$\| \times_{i=1}^{m} Q_i - \times_{i=1}^{m} P_i \| \leq \sum_{i=1}^{m} C_i \,. \tag{27}$$

$\| \cdot \|$ denotes the total variation distance, $\times$ denotes the product measure and $C_i$, $i = 1, 2, \ldots, m$ are constants.

We denote by $Q_{t_i}$, $i = 1, 2, \ldots, m$ the uniform probability distribution on the set

$$\{(x_{i1}, x_{i2}, \ldots, x_{iN_i}) \in \mathcal{X}^{N_i} : T_i(N_i) = t_i;\ i = 1, 2, \ldots, m\},\ t_i \in \mathcal{I}_i;$$

$Q_{t_i}^{n_i}$ the $n_i$-dimensional marginal probability distribution of $Q_{t_i}$, $i = 1, 2, \ldots, m$ and $P_{\theta_i}^{n_i}$ the probability distribution of the representation under extendibility. Then, using Diaconis and Freedman (1980, 1986) and (26) for $Q_{t_i}^{n_i}$ and $P_{t_i}^{n_i}$, $i = 1, 2, \ldots, m$, we have that:

When $\mathcal{X} = \{0, 1\}$, $T_i(N_i) = \sum_{j=1}^{N_i} X_{ij}$; $i = 1, 2, \ldots, m$, and $P_{\theta_i}^{n_i}$ denote the law of $n_i$ independent random variables with the common binomial $(1, \theta_i)$ distribution for $i =$

$1, 2, \ldots, m$, then, for $\bar{t}_i = \frac{T_i(N_i)}{N_i}$, $i = 1, 2, \ldots, m$, the right side of (27) is given by

$$2 \sum_{i=1}^{m} \frac{n_i}{N_i} \, ;$$

When $\mathcal{X} = \mathbb{N}$, $T_i(N_i) = \sum_{j=1}^{N_i} X_{ij}$, $i = 1, 2, \ldots, m$, $P_{\theta_i}^{n_i}$ denote the law of $n_i$ independent random variables with the common geometric $(\theta_i)$ distribution for $i = 1, 2, \ldots, m$, then, for $\bar{t}_i = \frac{t_i}{N_i + t_i}$, $i = 1, 2, \ldots, m$, the right side of (27) is given by

$$2 \sum_{i=1}^{m} \left( \frac{N_i^2}{(N_i - n_i - 1)(N_i - n_i - 2)} - 1 \right)$$

for $1 \leq n_i \leq N_i - 3$, $i = 1, 2, \ldots, m$,

and finally, when $\mathcal{X} = \mathbb{N}$, $T_i(N_i) = \max\{X_{i1}, \ldots, X_{iN_i}\}$, $i = 1, 2, \ldots, m$, and the Theorem 1 (Iglesias-Zuazola, Pereira and Tanaka, 1993, p.6) we have that the right side of (27) is given by

$$2 \sum_{i=1}^{m} \frac{n_i}{N_i} \, .$$

Finite versions follow using these facts and the convexity of the total variation distance.


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