LINEAR AUTONOMOUS NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS IN THE PHASE SPACE OF REGULATED FUNCTIONS

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1. Introduction

This paper intends to present a result contained in [LF2], namely, the extension of the spectral results of [Henry 1] for the flow of the Linear Autonomous NFDEs to the context of regulated right-continuous functions.

If $[a,b]$ is an interval of the real line and X is a Banach space, we write $G([a,b],X)$ for the space of the functions $\psi : [a,b] \to X$ for which there exist the limits $\psi(t^+)$ for every $t \in [a,b[$ and $\psi(t^-)$ for every $t \in]a,b]$. Such functions are called regulated functions.

In [LF2] we extend some results obtained by J. Hale ([Hale]) and D. Henry ([Henry 1]) for the so-called Neutral Functional Differential Equations (NFDEs), which have the form $\frac{d}{dt}(x(t) - f(t,x_t)) = g(t,x_t)$, from the context of continuous functions to the context of regulated functions. The motivation for this extension is the fact that the fundamental matrix, which appears in the variation-of-constants formula of the linear non-homogeneous NFDE ([Hale], [Henry 1]), is regulated and not continuous in t. So, the space of regulated functions appears as a natural context to include the fundamental matrix or the resolvent, in the case we consider a generic Banach space X. In this general context, Hönig ([H1], [H2]) studied the Volterra-Stieltjes linear Integral Equations. We applied this results, since the initial value problem of a linear NFDE leads to such an integral equation ([LF1]).

2. The main result

Let $\mathbf{E}^n$ denote the Euclidean space of real or complex n-vectors and let r be a fixed positive number. $\mathcal{G}^+ = G^+([-r,0],\mathbf{E}^n)$ is the space of the regulated right-continuous functions $\varphi : [-r,0] \to \mathbf{E}^n$, which is complete with the norm $\|\varphi\| = \sup_{-r\leq\theta\leq 0}\|\varphi(\theta)\|$.

We call $C = C([-r,0],\mathbf{E}^n)$ the closed subspace of $\mathcal{G}^+$ of continuous functions. If x is a regulated right-continuous map of $[a-r,b]$ into $\mathbf{E}^n$, then $x_t \in \mathcal{G}^+$ is given, for each $a \le t \le b$, by $x_t(\theta) = x(t+\theta)$, $-r \le \theta \le 0$.

Let D, L be fixed continuous linear functionals from $\mathcal{G}^+$ into $\mathbf{E}^n$, with integrals representations given by $D\varphi = \varphi(0) - \int_{-r}^{0} d\mu(\theta)\varphi(\theta)$ and $L\varphi = \int_{-r}^{0} d\eta(\theta)\varphi(\theta)$ for $\varphi \in \mathcal{G}^+$; where μ, η are matrix-valued functions (from $[-r,0]$ into $\mathcal{L}(\mathbf{E}^n)$) of bounded variation which vanish at $\theta = 0$ and are left-continuous. For these representations, we utilize the Interior Integral which extends the Riemann-Stieltjes Integral (see [H1]). We assume here that μ has no singular part, i.e., $\int_{-r}^{0} d\mu(\theta)\varphi(\theta) = \sum_{k=1}^{\infty} A_k\varphi(-r_k) + \int_{-r}^{0} A(\theta)\varphi(\theta)d\theta$ $\forall\, \varphi \in \mathcal{G}^+$, where $0 < r_k \le r$ and $A_k \in \mathcal{L}(\mathbf{E}^n)$ for $k \in \mathrm{N}$ and $A \in L_1([-r,0],\mathcal{L}(\mathbf{E}^n))$.

In this situation, the initial value problem is well posed for the NFDE:

$$\text{(N)} \qquad \frac{d}{dt}Dx_t = Lx_t, \quad t \ge 0\,,$$

that is, for $\varphi \in \mathcal{G}^+$ we have the unique regulated right-continuous solution $x = x(0,\varphi)$ of (N) for $t \ge 0$ with $x_0 = \varphi$. We have, then, well defined the flow of (N), $\{T(t)\}_{t\ge 0}$, semigroup of bounded linear operators on $\mathcal{G}^+$ given by $T(t)\varphi = x_t(0,\varphi)$ for $\varphi \in \mathcal{G}^+$ and $t \ge 0$.

Let D^0 be the jump part of D, that is, $D^0\varphi = \varphi(0) - \sum_{k=1}^{\infty} A_k\varphi(-r_k)$ for $\varphi \in \mathcal{G}^+$. We denote by $\mathcal{G}^+_{D^0}$ the kernel of D^0. The initial value problem is also well posed for the difference equation $(D)_0 :\ D^0 x_t = 0,\ t \ge 0$. This defines the flow of $(D)_0$, $\{T^0(t)\}_{t\ge 0}$, semigroup of bounded linear operators on $\mathcal{G}^+_{D^0}$.

We known that C is invariant under $T(t)$ $(t \ge 0)$ and $C_{D^0} \overset{\text{def.}}{=} \mathcal{G}^+_{D^0} \cap C$ is invariant under $T^0(t)$ $(t \ge 0)$, that is, the solution of (N) or $(D)_0$ is continuous whenever the initial data is a continuous function.

Daniel Henry ([Henry 1], [Henry 2]) gives a complete description of the spectrum of the operators $T^0(t)_{|C_{D^0}}$ and $T(t)_{|C}$ for $t \ge 0$, using the infinitesimal generator $\mathbf{A}^0$ of $\{T^0(t)\,|_{C_{D^0}}\}_{t\ge 0}$ and $\mathbf{A}$ of $\{T(t)_{|C}\}_{t\ge 0}$. The restriction of each flow, as above, is a strongly continuous semigroup of linear operators which admits a closed infinitesimal generator with dense domain in C_{D^0} and C, respectively. For the spectrum of these generators, we have:

$$\sigma(\mathbf{A}^0) = P\sigma(\mathbf{A}^0) = \{\lambda \in \mathbf{C} \mid \det H(\lambda) = 0\}$$

$$\sigma(\mathbf{A}) = P\sigma(\mathbf{A}) = \{\lambda \in \mathbf{C} \mid \det\Delta(\lambda) = 0\}$$

where $\det H(\lambda) = 0$ and $\det\Delta(\lambda) = 0$ are the respective characteristic equations of $(D)_0$ and (N), i.e.:

$$H(\lambda) \;=\; I - \sum_{k=1}^{\infty} A_k e^{-\lambda r_k} \;=\; D^0(e^{\lambda \cdot} I)$$

and $\Delta(\lambda) = \lambda H(\lambda) - \lambda \int_{-r}^{0} A(\theta)e^{\lambda\theta}d\theta - \int_{-r}^{0} d\eta(\theta)e^{\lambda\theta} = \lambda D(e^{\lambda\cdot}I) - L(e^{\lambda\cdot}I)$.

Henry shows that:

$$\sigma(T^0(t)\,|_{C_{D^0}})\backslash\{0\} \;=\; \overline{e^{t\sigma(\mathbf{A}^0)}}\backslash\{0\} \quad \text{a.e.} \quad \text{in } t \geq 0$$

and $T(t)_{|C} - T^0(t) \circ \Psi_{|C} : C \to C$ is a compact operator for each $t \geq 0$, (where the map Ψ above is a continuous projection from $\mathcal{G}^+$ onto $\mathcal{G}^+_{D^0}$ such that $\Psi(C) \subset C_{D^0}$, defined in the next section), and with these facts he concludes that:

$$\sigma(T(t)_{|C})\backslash\{0\} \;=\; \overline{e^{t\sigma(\mathbf{A})}}\backslash\{0\} \quad \text{a.e.} \quad \text{in } t \geq 0.$$

The flows of $(D)_0$ and (N) are neither strongly continuous nor something like "strongly regulated". In fact, they have infinite oscillation around each $t \geq 0$ when we fix a non-continuous initial function, and then, we cannot extend the infinitesimal generators to dense domains in $\mathcal{G}^+_{D^0}$ and $\mathcal{G}^+$ respectively. Nevertheless, we still can show that the results obtained by Henry are extensibles for $\mathcal{G}^+_{D^0}$ and $\mathcal{G}^+$ respectively. This is done in the next sections.

3. The difference equation

Let $\mathcal{E}^+ \subset \mathcal{G}^+$ the subspace of step-functions, that is:

$$\mathcal{E}^+ = \{\varphi \in \mathcal{G}^+ \mid \varphi = \sum_{i=1}^{k} c_i \chi_{[\theta_i,0]} \text{ for some } k \in \mathbf{N}^*,\ c_i \in \mathbf{E}^n \text{ and } -r \leq \theta_i \leq 0,$$
$$i = 1, 2, \ldots, k\}.$$

$\mathcal{E}^+$ is dense in $\mathcal{G}^+$ (see [H1]).

Let $G^-BV_0 = G^-BV_0([-r,0],(\mathbf{E}^n)')$ the space of applications $\alpha : [-r,0] \to (\mathbf{E}^n)' = \mathcal{L}(\mathbf{E}^n, \mathbf{E})$ with bounded variation which vanish at $\theta = 0$ and are left-continuous.

We have the following immediate lemmas:

Lemma 1. For each $\varphi \in \mathcal{E}^+$, there is a sequence $\{\varphi_m\}_{m\in\mathbf{N}}$, $\varphi_m \in \mathcal{C}$, such that

$$\varphi_m(0) = \varphi(0), \quad \|\varphi_m\| = \|\varphi\| \quad \forall\, m \in \mathbf{N} \qquad \text{and}$$

$$\int_{-r}^{0} d\alpha(\theta)\varphi_m(\theta) \overset{m\to\infty}{\longrightarrow} \int_{-r}^{0} d\alpha(\theta)\varphi(\theta) \qquad \forall\, \alpha \in G^-BV_0.$$

Lemma 2. For $\varphi \in \mathcal{G}^+$, if $\int_{-r}^{0} d\alpha(\beta)\varphi(\beta) = 0 \quad \forall\, \alpha \in G^-BV_0$, then $\varphi(\theta) = 0$ for $-r \le \theta < 0$.

For a linear operator L, we denote by $\mathcal{N}(L)$ and $\mathcal{R}(L)$ the kernel and the range respectively.

Remark 1. In [Hale], cap. 12.3, it is given a continuous projection $\Psi : \mathcal{C} \to \mathcal{C}_{D^0}$ such that $\Psi = I_\mathcal{C} - \Phi D^0$, where $\Phi = (\phi_1, \dots, \phi_n)$, $\phi_i \in \mathcal{C}$, satisfies $D^0\Phi = I$, I is the identity matrix $n \times n$ and $I_\mathcal{C}$ is the identity of $\mathcal{C}$.

So, $\mathcal{C} = \mathcal{C}_{D^0} \oplus \mathcal{N}(\Psi)$ and $\dim\mathcal{N}(\Psi) = n$ because $\mathcal{N}(\Psi) = \mathcal{R}(\Phi D^0)$ has $\Phi = (\phi_1, \dots, \phi_n)$ as a basis. Putting $\varphi^0 = \Psi\varphi$, we have, for $\varphi \in \mathcal{C}$:

$$\varphi = \varphi^0 + \Phi D^0\varphi = \varphi^0 + \sum_{i=1}^{n}(D^0\varphi)_i\phi_i = \varphi^0 + \sum_{i=1}^{n}(\varphi(0)_i - \int_{-r}^{0} d\overline{\mu}_i(\theta)\varphi(\theta))\phi_i$$

where $(D^0\varphi)_i$ is the i-th component of the vector $D^0\varphi \in \mathbf{E}^n$ and $\overline{\mu}_i(\theta)$ is the i-th line of the matrix $\overline{\mu}(\theta) = -\sum_{k=1}^{\infty} A_k\chi_{]-\infty,-r_k]}(\theta)$. Thus, $\overline{\mu}_i \in G^-BV_0$, $i = 1,2,\dots,n$.

For $\varphi \in \mathcal{G}^+$, it is also true that $D^0(\varphi - \Phi D^0\varphi) = 0$, then we have Ψ extended to $\mathcal{G}^+$ and the same decomposition as above holds for $\varphi \in \mathcal{G}^+$.

From this remark and lemma 1, it follows easily the:

Lemma 3. For $\varphi \in \mathcal{E}^+$, let $\varphi_m \in \mathcal{C}$, $m \in \mathbf{N}$, as in lemma 1. Then

$$\int_{-r}^{0} d\alpha(\beta)\varphi_m^0(\beta) \overset{m\to\infty}{\longrightarrow} \int_{-r}^{0} d\alpha(\beta)\varphi^0(\beta) \quad \forall\, \alpha \in G^-BV_0,$$

where $\varphi^0 = \Psi\varphi$ as in remark 1. We also have $\varphi_m^0(0) \xrightarrow{m\to\infty} \varphi^0(0)$.

Note that $\mathcal{E}_{D^0}^+ \stackrel{\text{def.}}{=} \Psi(\mathcal{E}^+)$ is dense in $\mathcal{G}_{D^0}^+$.

Lemma 4. If $T \in \mathcal{L}(\mathcal{G}_{D^0}^+, \mathcal{G}^+)$ with $(T\varphi)(\theta) = \int_{-\bar{r}}^0 d_\beta K(\theta-\beta)\varphi(\beta)$, where we have $-r \le -\bar{r} < 0$ and $K : [-r, \bar{r}] \to \mathcal{L}(\mathbf{E}^n)$ has bounded variation and is right-continuous; then, for each $\alpha \in G^- BV_0$, there is a $\tilde{\alpha} \in G^- BV_0$, such that:

$$\int_{-r}^0 d\alpha(\theta)(T\varphi)(\theta) = \int_{-r}^0 d\tilde{\alpha}(\beta)\varphi(\beta) \quad \forall\, \varphi \in \mathcal{G}_{D^0}^+.$$

Proof: We use the theorem 2.4 of [H2], which says that:

$$\int_{-r}^0 d\alpha(\theta)\int_{-\bar{r}}^0 d_\beta K(\theta-\beta)\varphi(\beta) = \int_{-\bar{r}}^0 d_\beta\left[\int_{-r}^0 d\alpha(\theta)K(\theta-\beta)\right]\varphi(\beta)$$

to construct a suitable $\tilde{\alpha} \in G^- BV_0$. ∎

Remark 2. In [LF2] we show that the variation-of-constants formula for the linear NFDEs ([Hale], [Henry 1]) remains the same in the context of regulated functions. For $\varphi \in \mathcal{G}_{D^0}^+$, the solution y of $(D)_0$, for $t \ge 0$, with $y_0 = \varphi$ is given by

$$(\rho_{D^0}) \qquad y(t) = -\sum_{k=1}^{\infty}\int_{-r_k}^0 d_\beta X(t-\beta-r_k)A_k\varphi(\beta), \quad t \ge 0,$$

where X is the fundamental matrix given by the conditions $D^0 X_t = I$ for $t \ge 0$, $X(0) = I$ and $X(t) \equiv 0$ for $t < 0$.

We have, by [Henry 1], lemma 3.5, the following result: if $a \in \mathbf{R}$ is such that $\det H(\lambda) \ne 0$ in some strip $|\mathrm{Re}\,\lambda - a| \le \delta$, $\delta > 0$, then we may decompose $X(t) = X^P(t) + X^Q(t)$ (if $a = 0$ we will have $X(t) = X^P(t) + X^Q(t) + \text{cte.}$), X^P can be extended for $t \le 0$ and we have the estimates:

$$\mathrm{Var}_{[t-r,t]}[X^Q] \le M\, e^{(a-\delta)t} \quad \text{for } t \ge 0$$

$$\mathrm{Var}_{[t-r,t]}[X^P] \le M\, e^{(a+\delta)t} \quad \text{for } t \le 0$$

for some constant M.

Theorem 1. Suppose $a \in \mathbb{R}$ is such that $\det H(\lambda) \neq 0$ in some strip $|Re\lambda - a| \leq \delta$, $\delta > 0$. Then $\mathcal{G}^+_{D^0} = P \oplus Q$ the direct sum of closed subspaces P, Q, invariant under $T^0(t)$, $t \geq 0$. The semigroup $\{T^0(t)_{|P}\}_{t\geq 0}$ may be extended uniquely as a group $(-\infty < t < \infty)$ of operators on P. There exists a constant M' such that: $\|T^0(t)_{|Q}\| \leq M'.e^{(a-\delta)t}$ for $t \geq 0$ and $\|T^0(t)_{|P}\| \leq M'.e^{(a+\delta)t}$ for $t \leq 0$.

Proof. As in remark 2, we have the split of matrix X and, for $\varphi \in \mathcal{G}^+_{D^0}$, we have the solution of $(D)_0$, $y(t)$, given by formula (ρ_{D^0}). We may write: $y(t) = y^P(t) + y^Q(t)$, where $y^{P,Q}(t) = -\sum_{k=1}^{\infty} \int_{-r_k}^0 d_\beta X^{P,Q}(t-\beta-r_k)A_k\varphi(\beta)$, $t \geq 0$, and we can take $-\infty < t < \infty$ for y^P. From remark 2, we obtain also the estimates:

$$(\bullet)\qquad \begin{cases} \|y^Q(t)\| \leq (\sum_{k=1}^{\infty}|A_k|).M.e^{(a-\delta)t}\|\varphi\| \text{ for } t \geq 0 \\ \|y^P(t)\| \leq (\sum_{k=1}^{\infty}|A_k|).M.e^{(a+\delta)t}\|\varphi\| \text{ for } t \leq 0 \end{cases}$$

Define $T^0(t)^P\varphi = y^P_t$ for $t \in \mathbb{R}$ and $T^0(t)^Q\varphi = y^Q_t$ for $t \geq r$.

By the majorations above we see that $T^0(t)^{P,Q} \in \mathcal{L}(\mathcal{G}^+_{D^0}, \mathcal{G}^+)$. Let $\pi_P \overset{\text{def}}{=} T^0(0)^P \in \mathcal{L}(\mathcal{G}^+_{D^0}, \mathcal{G}^+)$.

In [Henry 1] theorem 3.1, it is shown that $\pi_P |_{C_{D^0}} \in \mathcal{L}(C_{D^0})$ and it is idempotent.

We will show that $\pi_P \in \mathcal{L}(\mathcal{G}^+_{D^0})$ and is also idempotent.

We begin with the step-functions. Let $\varphi \in \mathcal{E}^+$ and $\varphi_m \in C$, $m \in \mathbb{N}$, as in lemma 1. By remark 1, we have: $\varphi = \varphi^0 + \sum_{i=1}^{n}(\varphi(0)_i - \int_{-r}^0 d\overline{\mu}_i(\theta)\varphi(\theta))\phi_i$. Using the formula of $\pi_P\varphi^0 = y^P_0$ and lemmas 3 and 4 we obtain $\int_{-r}^0 d\alpha(\beta)\pi_P\varphi^0_m(\beta) \xrightarrow{m\to\infty} \int_{-r}^0 d\alpha(\beta)\pi_P\varphi^0(\beta)$ $\forall\ \alpha \in G^-BV_0$. By the formula of $\pi_P\varphi^0_m(\theta)$, for each $\theta \in [-r, 0]$, and the fact that $\|\varphi_m\| = \|\varphi\|$ (and then $\|\varphi^0_m\| \leq \|\Psi\|.\|\varphi\|$ $\forall\ m \in \mathbb{N}$), we obtain $\pi_P\varphi^0_m(\theta) \xrightarrow{m\to\infty} \pi_P\varphi^0(\theta)$ and, in particular, $\pi_P\varphi^0_m(0) \xrightarrow{m\to\infty} \pi_P\varphi^0(0)$. So, $0 = D^0(\pi_P\varphi^0_m) = \pi_P\varphi^0_m(0) - \int_{-r}^0 d\overline{\mu}(\beta)(\pi_P\varphi^0_m)(\beta) \xrightarrow{m\to\infty} \pi_P\varphi^0(0) - \int_{-r}^0 d\overline{\mu}(\beta)(\pi_P\varphi^0)(\beta) = D^0(\pi_P\varphi^0)$. Then $\pi_P\varphi^0 \in \mathcal{G}^+_{D^0}$, that is, $\pi_P(\mathcal{E}^+_{D^0}) \subset \mathcal{G}^+_{D^0}$ and taking the closure of $\mathcal{E}^+_{D^0}$, we have $\pi_P \in \mathcal{L}(\mathcal{G}^+_{D^0})$. Now it makes sense to take π^2_P. To show that π_P is a projection, we note first that $\int_{-r}^0 d\alpha(\beta)\pi^2_P\varphi^0_m(\beta) \xrightarrow{m\to\infty} \int_{-r}^0 d\alpha(\beta)\pi^2_P\varphi^0(\beta)$, $\forall\, \alpha \in G^-BV_0$, a consequence of the formula of $\pi^2_P\varphi^0(\beta)$ and lemma 4. We know that $\pi^2_P\varphi^0_m = \pi_P\varphi^0_m$, since $\varphi^0_m \in C_{D^0}$. Therefore, $\int_{-r}^0 d\alpha(\beta)[\pi^2_P - \pi_P]\varphi^0(\beta) = 0$ $\forall\, \alpha \in G^-BV_0$ and from lemma 2 we have $\pi^2_P\varphi^0(\theta) = \pi_P\varphi^0(\theta)$ for $-r \leq \theta < 0$. For $\theta = 0$, observe that $D^0(\pi^2_P\varphi^0) = \pi^2_P\varphi^0(0) - \int_{-r}^0 d\overline{\mu}(\theta)\pi^2_P\varphi^0(\theta) = \pi^2_P\varphi^0(0) - \pi_P\varphi^0(0) + D^0(\pi_P\varphi^0)$, but $D^0(\pi^2_P\varphi^0) = D^0(\pi_P\varphi^0) = 0$ for $\mathcal{G}^+_{D^0}$ is invariant under π_P. This completes the proof that π_P is idempotent in $\mathcal{E}^+_{D^0}$ and so in $\mathcal{G}^+_{D^0}$.

Then we have the closed subspaces of $\mathcal{G}^+_{D^0} : P = \mathcal{N}(\pi_P)$, $Q = \mathcal{R}(\pi_P)$; $\mathcal{G}^+_{D^0} = P \oplus Q$ and π_P is a projection on P along Q.

By [Henry 1] theorem 3.1, we have $T^0(t)\pi_P\varphi^0_m(\theta) = \pi_P T^0(t)\varphi^0_m(\theta) = T^0(t)^P\varphi^0_m(\theta)$ $\forall\, t \geq 0$, $\forall\, \theta \in [-r, 0]$ and each of these expressions converges when $m \to \infty$ to the respective expression with φ^0 instead of φ^0_m (this can be shown by using the formulas of π_P, $T^0(t)$, $T^0(t)^P$ and lemma 4). Therefore, $T^0(t)\pi_P = \pi_P T^0(t) = T^0(t)^P$ in $\mathcal{E}^+_{D^0}$ and in $\mathcal{G}^+_{D^0}$, for $t \geq 0$. For $t \in \mathbf{R}$, we also obtain $0 = D^0(T^0(t)^P\varphi^0_m) \xrightarrow{m\to\infty} D^0(T^0(t)^P\varphi^0)$ and this allows us to define $T^0(t) = T^0(t)^P$ in P for $t \leq 0$ and to obtain the group of isomorphisms $\{T^0(t)_{|P}\}_{t\in\mathbf{R}}$; for, when we have the backward continuation of the solution of equations like $(D)_0$ in the whole line, this continuation is unique (see [Henry 3]).

The inequalities stated in the theorem follow immediately from inequalities in $(*)$.∎

Remark 3. The subspaces $P \cap C_{D^0}$ and $Q \cap C_{D^0}$ are characterized in [Henry 1] theorem 3.1, in terms of the generalized eigenspaces corresponding to the eigenvalues of the infinitesimal generator $\mathbf{A}^0$ which have the real parts bigger than α and smaller than α, respectively.

We can extend, now, the theorem 3.2 of [Henry 1],

Theorem 2. Let $Z \overset{\text{def}}{=} \{\mathrm{Re}\lambda \mid \det H(\lambda) = 0\}$ be nonempty; then, for $t \geq 0$, we have:

$$\overline{\{e^{\lambda t} \mid \det H(\lambda) = 0\}} \subset \sigma(T^0(t)) \subset \{\mu \mid |\mu| = e^{\xi t},\ \xi \in \overline{Z}\} \cup \{0\}.$$

If $\alpha \notin \overline{Z}$ and $\mathcal{G}^+_{D^0} = P \oplus Q$ is the decomposition given by theorem 1, then

$$\sigma(T^0(t)_{|P}) \subset \{\mu \mid |\mu| = e^{\xi t},\ \xi \in \overline{Z} \text{ and } \xi > \alpha\}$$

$$\sigma(T^0(t)_{|Q}) \subset \{\mu \mid |\mu| = e^{\xi t},\ \xi \in \overline{Z} \text{ and } \xi < \alpha\}$$

If Z is empty, then $\sigma(T^0(t)) = \{0\}$ for $t > 0$; in fact, $T^0(t) = 0$ for $t \geq r.n.$

Proof. It is the same of theorem 3.2 of [Henry 1], using now theorem 1. We recall that for $\varphi \in \mathcal{E}^+$ and $\varphi_m \in C$, $m \in \mathbf{N}$, as in lemma 1, we will have $T^0(t)\varphi^0_m(\theta) \xrightarrow{m \to \infty} T^0(t)\varphi^0(\theta)$ and then $T^0(t)_{|C_{D^0}} = 0 \Rightarrow T^0(t) = 0$. ∎

Remark 4. From theorem 2 and from [Henry 2] theorem 5.1, we have that:

$$\sigma(T^0(t))\backslash\{0\} = \overline{e^{t\sigma(A^0)}}\backslash\{0\} \quad \text{a.e. in } t \geq 0.$$

4. The Neutral FDE

Passing now to equation (N) of section 2, we first generalize the lemma 4.1 of [Henry 1].

Lemma 1. To the equations (N) and $(D)_0$ and their flows, given in section 2, we have: $T(t) - T^0(t) \circ \Psi : \mathcal{G}^+ \to \mathcal{G}^+$ is a compact operator for each $t \geq 0$, where Ψ is the projection given in remark 1 of section 3.

Proof: Analogous to the lemma 4.1 of [Henry 1]. Recall that $\mathcal{R}(I_{\mathcal{G}^+} - \Psi)$ has finite dimension. ∎

We denote by $P\sigma(L)$, $R\sigma(L)$ and $C\sigma(L)$ the point, the residual and the continuous parts of the spectrum of a linear operator L.

We generalize now the theorem 4.1 of [Henry 1].

Theorem 1. With the notation of section 2, for the flow of equation (N), we have:

i) $P\sigma(T(t))\backslash\{0\} = P\sigma(T(t)_{|C})\backslash\{0\} = \{e^{\lambda t} \mid \det\Delta(\lambda) = 0\}$

ii) $R\sigma(T(t)) \cup C\sigma(T(t)) \subset \{\mu \mid |\mu| = e^{\xi t},\ \xi \in \overline{Z}\} \cup \{0\}$

where Z is given in theorem 2 of section 3.

iii) $\sigma(T(t))\backslash\{0\} = \overline{e^{t\sigma(A)}}\backslash\{0\}$ a.e. in $t \geq 0$.

Proof: i) Suppose $t_0 > 0$, $\lambda_0 \in \mathbf{C}$ and $\varphi \in \mathcal{G}^+$ such that $T(t_0)\varphi = e^{\lambda_0 t_0}\varphi \neq 0$.

We show that there is a $c \in \mathbf{E}^n$ such that $y(t) = c.e^{\lambda t} \neq 0$ is the solution of (N) with initial data $\psi(\theta) = c.e^{\lambda\theta}$, where $\lambda = \lambda_0 + \frac{2\pi i n}{t_0}$ for some $n \in \mathbf{N}$, that is; we find a continuous (in fact, exponential) eigenvector for the eigenvalue $e^{\lambda_0 t_0}$ of $T(t_0)$.

Let $x_t = T(t)\varphi$, $t \geq 0$. The function $t \to e^{-\lambda_0 t}x(t)$ is periodic of period t_0 and then there is a $n \in \mathbf{N}$ such that the n-th Fourier coefficient is nonzero, that is,

$$c = \frac{1}{t_0}\int_0^{t_0} e^{\frac{-2\pi i n}{t_0}\cdot s} x(s)e^{-\lambda_0 s}ds = \frac{1}{t_0}\int_0^{t_0} x(s)e^{-\lambda s}ds \neq 0$$

and we have

$$\begin{aligned} ce^{\lambda(t+\theta)} &= \frac{1}{t_0}\int_0^{t_0} x(s)e^{\lambda(t+\theta-s)}ds = \\ &= \frac{1}{t_0}\int_0^{t_0} x(t+u+\theta)e^{-\lambda u}du = \frac{1}{t_0}\int_0^{t_0} [T(t)x_u](\theta)e^{-\lambda u}du = \\ &= [T(t)(\frac{1}{t_0}\int_0^{t_0} x_u(\cdot)e^{-\lambda u}du)](\theta) = [T(t)(c.e^{\lambda\cdot})](\theta). \end{aligned}$$

By the same arguments of theorem 4.1 of [Henry 1] we prove ii), using the result of Gohberg and Krein in the version of lemma 2 of [Henry 1] and using theorem 2 of section 3 and lemma 1.

For iii) we use also theorem 5.1 of [Henry 2]. ∎

In the same way of theorem 4.2 of [Henry 1] we can show that:

Theorem 2. Suppose $\alpha \notin \overline{\mathrm{Re}\,\sigma(\mathbf{A})}$, i.e., $\det\Delta(\lambda) \neq 0$ in some strip $|\mathrm{Re}\lambda - \alpha| < \delta$, $\delta > 0$. Then $\mathcal{G}^+ = P \oplus Q$, where P, Q are closed subspaces invariant under $T(t)$.

The restriction of the semigroup to P may be extended to a group $\{T(t)_{|P}\}_{t\in\mathbf{R}}$ of isomorphisms of P. Finally, there exists a constant M such that $\|T(t)_{|Q}\| \leq M\,e^{(\alpha-\delta)t}$ for $t \geq 0$ and $\|T(t)_{|P}\| \leq M\,e^{(\alpha+\delta)t}$ for $t \leq 0$ (see also [LF2], cap.II, §4, theorems 3 and 4.)

We now study the corresponding decomposition for the nonhomogeneous equation

$$(\mathrm{N})_H: \qquad \frac{d}{dt}(Dx_t - H(t)) = Lx_t$$

for a given regulated right-continuous forcing function H.

For each $t,\ t_0 \in \mathbf{R},\ t \geq t_0,$ we have a bounded linear operator $\mathcal{K}(t,t_0) \in \mathcal{L}(\mathcal{G}^+([t_0,t],\mathbf{E}^n),\mathcal{G}^+)$ such that the solution of $(\mathrm{N})_H$ for $t \geq t_0$ with $x_{t_0} = \varphi \in \mathcal{G}^+$ is given by $x_t(t_0,\varphi,H) = T(t-t_0)\varphi + \mathcal{K}(t,t_0)H$, where $\{T(t)\}_{t\geq 0}$ is the flow of (N) as in section 2.

From [LF2] we have that $\mathcal{K}(t,t_0)H = \chi_\circ H(t) - T(t-t_0)\chi_\circ H(t_0) - \int_{t_0}^t d_\sigma T(t-\sigma)\chi_\circ H(\sigma)$ where, for $p \in \mathbf{E}^n$, we have $\chi_\circ p(\theta) = 0$ for $-r \leq \theta < 0$ and $\chi_\circ p(0) = p$ and $(\int_{t_0}^t d_\sigma T(t-\sigma)\chi_\circ H(\sigma))(\theta) = \int_{t_0}^t d_\sigma X(t+\theta-\sigma)H(\sigma)$, the integral being in $\mathbf{E}^n$, and the fundamental matrix $X(t) \in \mathcal{L}(\mathbf{E}^n)$ given by $X(t)p = T(t)\chi_\circ p(0)$.

In [LF2] (see cap.II, §4, remark 7), we generalize the theorem 4.3 of [Henry 1] as a consequence of theorem 2 and the variation-of-constants formula for equation $(\mathrm{N})_H$. Then we have:

Theorem 3. Suppose $\alpha \notin \overline{\mathrm{Re}\,\sigma(\mathrm{A})}$ and $\mathcal{G}^+ = P \oplus Q$ is the decomposition provided by theorem 2. We write $\varphi = \varphi^P + \varphi^Q \in P \oplus Q$. Then, there exist matrix-functions of bounded variation X^P and X^Q, such that:

$$x_t^Q(t_0,\varphi,H) = T(t-t_0)\varphi^Q + (\chi_\circ H(t))^Q - T(t-t_0)(\chi_\circ H(t_0))^Q - \int_{t_0}^t d_\sigma T(t-\sigma)[\chi_\circ H(\sigma)]^Q$$

and

$$x_t^P(t_0,\varphi,H) = T(t-t_0)\varphi^P + (\chi_\circ H(t))^P - T(t-t_0)(\chi_\circ H(t_0))^P - \int_{t_0}^t d_\sigma T(t-\sigma)[\chi_\circ H(\sigma)]^P$$

for $t \geq t_0$, where

$$\left(\int_{t_0}^t d_\sigma T(t-\sigma)[\chi_\circ H(\sigma)]^{P,Q}\right)(\theta) = \int_{t_0}^t d_\sigma X^{P,Q}(t+\theta-\sigma)H(\sigma) \quad \text{for } \theta \in [-r,\theta].$$

Remark 2. The matrix functions X^Q and X^P are the same given in theorem 4.3 of [Henry 1] (he calls there $X^{P,Q}(t+\theta) = T(t)\chi_\circ^{P,Q}(\theta)$) and we have the estimates given in that theorem.

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