Matemática Contemporânea, Vol 33, 123-137

CONFORMALLY FLAT LORENTZIAN HYPERSURFACES AND CURVED FLATS

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Dedicated to Francesco Mercuri on his sixtieth birthdays

Abstract

We consider conformally flat Lorentzian hypersurfaces in the conformal compactification of Lorentz space $\mathbb{R}_1^{n+1}$ assuming that locally the shape operator has the same algebraic type and its eigenvalues has constant multiplicity. We prove that to every conformally flat Lorentzian hypersurface one can associate a curved flat in the Grassmannian of Lorentzian n-planes in $\mathbb{R}_2^{n+3}$, allows so a connection to the Guichard's net when $n = 3$.

1 Introduction

A pseudo-Riemannian manifold (M, g) is called *conformally flat* if, for any $x \in M$, there exists a neighborhood U of x and a function $u : U \to \mathbb{R}$ such that $(U, e^{2u}g)$ is flat ([1]).

In similar form as happens in the Riemannian case, any 2-dimensional pseudo-Riemannian manifold is conformally flat. For dimension $n \geq 4$, the necessary and sufficient condition for conformal flatness condition is given by the vanishing of the Weyl tensor, and in dimension $n = 3$, however, the criterium for conformally flatness is that the Schouten tensor is a Codazzi tensor.

In the hypersurfaces setting, recently U. Hertrich-Jeromin studied conformally flat hypersurfaces in the sphere S^{n+1} using a different point of view that used by Cartan, to classify locally that kind of hypersurfaces. More particularly, U. Hertrich-Jeromin used in [7], [8], [9], Möbius geometry and its

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Key words: Conformally flat, Lorentzian hypersurfaces, Channel hypersurfaces, conformal compactification, Spherical congruence, Curved flats.

projective model in $\mathbb{R}^{n+3}_1$, to reprove the classic Cartan's classification and also for giving a different vision to the 3-dimensional case: the existence of a 1-1 correspondence between conformally flat hypersurfaces in the conformal sphere S^4 with three distinct principal curvatures and the Guichard's nets. The started point in all those studies is to consider the conformal sphere as the conformal compactification of the Euclidean space $\mathbb{R}^{n+1}$. Already in the Lorentzian setting, in a recent work of the author with M. Magid, in [4], were considered the conformally flat Lorentzian hypersurfaces through of Möbius geometry and its projective model in $\mathbb{R}^{n+3}_2$. In fact, considering the different algebric types of the shape operators in the Lorentzian case, it was established a local characterization to the conformally flat Lorentzian hypersurfaces in the conformal compactification of $\mathbb{R}^{n+1}_1$. We observe that in this case, the ambient space corresponds to the projectivized light cone in $\mathbb{R}P^{n+2}$ induced from $\mathbb{R}^{n+3}_2$ ([4]) which we denote henceforth by $\hat{\mathbb{R}}^{n+1}_1$.

From another hand, the conformally flat hypersurfaces in the conformal sphere S^{n+1}, were proved to be also important in the context of the integrable systems. More specifically, in [7] and [9], U. Hertrich-Jeromin showed that to each of them, one can associate a curved flat constituted by the generalized Gauss map of the flat lift in the light cone L^{n+2} in the Lorentzian space $\mathbb{R}^{n+3}_1$. Moreover, if the hypersurface is 3-dimensional with three distinct real eigenvalues the curved flat becomes to be a regular map, that assures the existence of canonical principal curvature coordinates, which are justly the canonical Guichard coordinates existing on every conformally flat hypersurface (see [7]).

The main goal in this note it is to prove that one can also associate to every conformally flat Lorentzian hypersurface in $\hat{\mathbb{R}}^{n+1}_1$ a curved flat in the

Grassmannian $\frac{O_2(n+3)}{O_1(n)\times O_1(3)}$ of Lorentzian n-planes in $\mathbb{R}_2^{n+3}$. In fact, following the U. Hertrich-Jeromin's ideas in [7], we prove, assuming that locally the shape operator has the same algebraic type and its eigenvalues has constant multiplicity, the following theorem.

Theorem 1.1. *the light cone representative $f : M_1^n \to L_1^{n+2}$ of a Lorentzian hypersurface in $\hat{\mathbb{R}}_1^{n+1}$ induces a flat metric on M_1^n if and only if its generalized Gauss map is a curved flat.*

As a consequence we have the next result.

Corollary 1.1. *Given a conformally flat Lorentzian hypersurface in $\hat{\mathbb{R}}_1^{n+1}$, the generalized Gauss map of any flat light cone lift is a curved flat.*

Hence for the particular case of conformally flat hypersurfaces M_1^3 of $\hat{\mathbb{R}}_1^4$ with three distinct real eigenvalues, one gets an interpretation of the Guichard's nets through of the curved flats in similar form as the defined positive case: In the Lorentzian case the curved flat is also a regular map, which gives the canonical principal curvatures coordinates that are justly the canonical Guichard's net existing on the conformally flat Lorentzian hypersurface in $\hat{\mathbb{R}}_1^4$ with three distinct real eigenvalues ([4]).

2 Known previous results

For hypersurfaces M^n in the sphere S^{n+1}, we have in dimensions $n \geq 4$ a classic result by Cartan. The induced metric of a hypersurface M^n in the sphere S^{n+1} ($n \geq 4$) is conformally flat if and only if it is a branched channel hypersurface, that means if at least $n-1$ of the principal curvatures coincide at each point. For $n = 3$, the result of Cartan no longer holds in its full generality, since there exist examples of 3-dimensional conformally flat hypersurfaces in S^4 with exactly 3 different principal curvatures. In particular, in this case, U. Hertrich-Jeromin in ([8], [9]), used Möbius geometry for establishing a relation between that kind of hypersurfaces and the Guichard‘s nets. To be more especific, recalling that a triple orthogonal system of surfaces in a three dimensional Riemannian manifold

$$X = (x_1, x_2, x_3) : (M^3, <, >) \to \mathbb{R}^3_2 \cong \mathbb{R} \times i\mathbb{R} \times i\mathbb{R}$$

is called a *Guichard net* if its Lamé functions $l_i := ||\frac{\partial}{\partial x_i}||,\ 1 \leq i \leq 3$, satisfy the condition $\sum l_i^2 = 0$, U. Hertrich-Jeromin proves in ([8]) the following structural theorem: for a hypersurface M^3 of S^4 with three different principal curvatures to be conformally flat, it must allow the existence of a Guichard‘s net. Its converse also holds: given a Guichard net X, there exists locally a conformally flat hypersurface in S^4 whose second fundamental form has three distinct principal curvatures with conformal fundamental forms dx_i.

In another interesting direction, the conformally flat hypersurfaces are also related to the integrable system constituted by the equations of the curved flats. In fact in [7], U. Hertrich-Jeromin also used the projective model in Möbius geometry to prove that to each conformally flat hypersurface in the conformal sphere one can associate a curved flat constituted by the generalized Gauss map of a flat lift in the light cone L^{n+2} in the Lorentzian space $\mathbb{R}^{n+3}_1$. More specifically, he proved the following theorem:

Theorem 2.1. *[7] The light cone representative $f : M \to L^{n+2}$ of a hypersurface in S^{n+1} induces a flat metric on M if and only if its generalized Gauss map $\gamma_f : M^n \to \frac{O_1(n+3)}{O(n)\times O_1(3)}$ is a curved flat.*

It follows from Theorem 2.1 that given a conformally flat hypersurface, the generalized Gauss map of any flat light cone lift is a curved flat.

In the case of hypersurfaces 3-dimensional with three distinct real eigenvalues, it was proved in [7] that the induced metric from a curved flat associated to a flat light cone lift (representative of the conformally flat hypersurface) is a multiple of the conformal metric which is non-degenerated since the three eigenvalues are distinct. Hence the curved flat is a regular map, and so, one has canonical principal curvature coordinates which always come with regular curved flat. Finally U. Hertrich-Jeromin showed that the canonical principal curvature coordinates coming from the regular curved flat yields just the canonical Guichard net existing on every conformally flat hypersurface (for more details see [7]).

2.1 Lorentzian Setting

Since in this note we are interested in to start a study of the connection between the conformally flat Lorentzian hypersurfaces in $\hat{\mathbb{R}}_1^{n+1}$ and the curved flats, we review now some basic facts about the conformal geometry and the projective compactification of the Lorentz space $\mathbb{R}_1^{n+1}$, as well as some results of [4], about the Guichard coordinates system which is allowed in any 3-dimensional conformally flat Lorentzian hypersurface with three different real principal curvatures.

Let $\mathbb{R}_2^{n+3}$ be $\mathbb{R}^{n+3}$ with the metric $\langle \vec{v}, \vec{w} \rangle = -v_1 w_1 + \sum_{i=2}^{n+1} v_i w_i + v_{n+2} w_{n+3} + v_{n+3} w_{n+2}$, for $\vec{v} = (v_1, ..., v_{n+3})$, $\vec{w} = (w_1, ..., w_{n+3})$. Let $\mathbb{R}P^{n+2}$ denote the real projective space of lines passing through the origin in $\mathbb{R}^{n+3}$, π the projection from $\mathbb{R}^{n+3} - \{0\}$ to $\mathbb{R}P^{n+2}$, $L_1^{n+2} = \{v \in \mathbb{R}_2^{n+3} | \langle v, v \rangle = 0\}$ the light cone. Then the projection of $L_1^{n+2} - \{0\}$ is homeomorphic to our ambient space $\hat{\mathbb{R}}_1^{n+1}$. From [10] we know that there is a bijection between points in $\mathbb{R}P^{n+2}$ and quadrics and planes in $\mathbb{R}_1^{n+1}$. In particular, spacelike points correspond to Lorentzian spheres and timelike planes, while the timelike points correspond to hyperbolic spaces and spacelike planes. Moreover, the Lorentzian spheres in $\mathbb{R}_1^{n+1}$ correspond to points in $\mathbb{R}P_+^{n+2}$, the projectivized spacelike points, while hyperbolic spaces correspond to points in $\mathbb{R}P_-^{n+2}$, the projectivized timelike points. Finally, points in $\mathbb{R}_1^{n+1}$ or S_1^{n+1} or H_1^{n+1} are identified with points in $\hat{\mathbb{R}}_1^{n+1} = \mathbb{R}P_0^{n+2}$ the projectivized light cone in $\mathbb{R}P^{n+2}$ induced from $\mathbb{R}_2^{n+3}$.

By the natural constructions made in [10], one can also define the spherical congruence and their envelopes, as follows:

A differential n-parameter family of spheres $S : M_1^n \to \mathbb{R}P_+^{n+2}$ is called a *spherical congruence*. It corresponds to a family of Lorentzian spheres in $\mathbb{R}_1^{n+1}$.

A differential map $f : M_1^n \to \hat{\mathbb{R}}_1^{n+1}$ *envelopes* a spherical congruence S if, for all $p \in M_1^n$, $f(p) \in S(p)$ and $T_{f(p)} f(M_1^n) \subset T_{f(p)} S(p)$.

In particular, by rescaling one can assume that S takes values in S_2^{n+2}, so an equivalent condition to be an envelope for spherical congruence is

$$\langle f, S \rangle = 0 \quad and \quad \langle df, S \rangle = 0.$$

Next we define the f-adapted frame which are adequate to establish the fundamental equations for the hypersurface. $\{e_1, ..., e_{n+3}\}$ is a pseudo-orthonormal basis of $\mathbb{R}_2^{n+3}$ if $< e_i, e_j >= \pm\delta_{ij}$ for $1 \leq i, j \leq n+1$, with -1 for $i = 1$ and $+1$ otherwise, and $e_{n+2}, e_{n+3} \in \{e_1, ..., e_{n+1}\}^{\perp}$ are null vectors with $< e_{n+2}, e_{n+3} >= 1$. By an orthonormal basis $\{v_1, ...v_n\}$ in a Lorentzian n-dimensional space one means that $< v_1, v_1 >= -1$, $< v_i, v_j >= \delta_{ij}$, and $< v_1, v_j >= 0$ for $2 \leq i, j \leq n$.

So in analogy to the positive definite case, one defines a *strip* as a pair of smooth maps $(f, S) : M_1^n \to L_1^{n+2} \times S_2^{n+2}$, where f is a immersion and S is a spherical congruence enveloped by f. For an *f-adapted frame* for the strip (f, S), one means a map $F : M_1^n \to O_2(n+3)$ such that $S = Fe_{n+1}$, $f = Fe_{n+2}$ and such that, for all $p \in M_1^n$, $\text{span}\{Fe_1, ..., Fe_n\}_p = df_p(T_pM_1^n)$, for $\{e_i\}_{i=1}^{n+3}$ a pseudo-orthonormal basis of $\mathbb{R}_2^{n+3}$.

Then, assuming locally that the shape operator has the same algebraic type, this means, it can locally be diagonalizable over $\mathbb{R}$ or $\mathbb{C}$, or not diagonalizable with one eigenvalue of multiplicity two or three, the following lemma is proved in [4]:

Lemma 2.1. *([4]) If the metric of a light cone representative $f : M_1^n \to \mathbb{R}_2^{n+3}$ of a Lorentzian hypersurface in the projectivized light cone $\hat{\mathbb{R}}_1^{n+1}$ is flat, then its normal bundle is flat (as an immersion into $\mathbb{R}_2^{n+3}$).*

The natural extension of the branched channel hypersurface to the Lorentzian setting, is as following. A regular map $f : M_1^n \to \hat{\mathbb{R}}_1^{n+1}$ is called a *branched channel hypersurface* if it envelopes a spherical congruence S with $rank\ dS \leq 1$.

The definition above simplifies for dimension $n \geq 4$, saying that $f : M_1^n \to \hat{\mathbb{R}}_1^{n+1}$, $n \geq 4$ is branched channel hypersurface if the Weingarten tensor field A_S with respect to any enveloped spherical congruence S, has an eigenvalue of multiplicity $n-1$ ([4]).

So, from [4], we know that $f : M_1^n \to \hat{\mathbb{R}}_1^{n+1}$, $n \geq 4$, is conformally flat iff it is a branched channel hypersurface. Now, in the case of dimension $n = 3$, that result doesn‘t hold even that the second fundamental form was digonalizable over $\mathbb{R}$. But in this last case, if a_1, a_2, a_3 denote the three distinct eigenvalues of the shape operator, then the next theorem was proved.

Theorem 2.2. *([4]) If $f : M_1^3 \to \hat{\mathbb{R}}_1^4$ is real diagonalizable with three distinct eigenvalues, then it is conformally flat iff the conformal fundamental forms γ_i given by*

$$\begin{cases} \gamma_1 = \sqrt{(a_1 - a_2)(a_1 - a_3)}\omega_1, \\ \gamma_2 = \sqrt{(a_1 - a_2)(a_3 - a_2)}\omega_2, \\ \gamma_3 = \sqrt{(a_1 - a_3)(a_2 - a_3)}\omega_3, \end{cases} \tag{1}$$

are closed, where ω_i are the dual 1-forms to the vector fields S_i of a pseudo-orthonormal f-adapted frame $F = (S_1, S_2, S_3, S, f, \hat{f})$ for the strip (f, S).

Theorem 2.2 assures the existence of a coordinates system x_1, x_2, x_3 such that $\gamma_i = dx_i$, for which the coordinates surfaces $x_i = cte$ constitute a triple orthogonal system whose Láme functions $l_i = ||\frac{\partial}{\partial x_i}||$ satisfy $\sum_i l_i^2 = 0$. Here one observes that assuming for instance $a_1 < a_2 < a_3$, x_2 is the only imaginary coordinate function being the other real coordinates functions. So if the conformally flat Lorentzian hypersurface has second fundamental form diagonalizable over $\mathbb{R}$, our Guichard nets take values in $\mathbb{R} \times i\mathbb{R} \times \mathbb{R}$.

Hence a necessary condition of conformally flatness is the existence of a Guichard net ([4]):

Theorem 2.3. *On every conformally flat Lorentzian hypersurface in $\hat{\mathbb{R}}_1^4$ with shape operator diagonalizable over $\mathbb{R}$ and with three distinct eigenvalues, there is a Guichard net consisting of surfaces of principal curvatures.*

To finish this section, we just observe that in the 3-dimensional real diagonalizable case, the conformal metric C can be recovered from the conformal fundamental forms (1) as $-\gamma_1^2 + \gamma_2^2 + \gamma_3^2$.

3 The Curved Flats

The main goal of this section is to prove Theorem 1.1 and to establish, in the case of 3-dimensional conformally flat Lorentzian hypersurfaces with 3 different principal curvatures, an interpretation of the Guichard nets through of the curved flats. In order to do that, we follow the U. Hertrich-Jeromin‘s ideas in [7] and use the basic facts reviewed in Section 2.1 above, as well as Lemma 2.1.

A curved flat is a map into a symmetric space which is tangent at each point to a flat of the symmetric space, i.e., each tangent space is abelian (see [5]). Formally it is defined as:

Definition 3.1. *A map $\varphi : M^n \to \frac{G}{K}$ into a (pseudo) Riemmannian symmetric space is called a curved flat if for any lift $F : M^n \to G$ of φ we have $[\Phi_{\mathcal{P}} \wedge \Phi_{\mathcal{P}}] = 0$, where*

$$F^{-1}dF = \Phi = \Phi_{\mathcal{K}} + \Phi_{\mathcal{P}} : TM \to \mathcal{G} = \mathcal{K} \oplus \mathcal{P}$$

is the symmetric decomposition of the connection form of F.

Here we recall that the product $[\Phi \wedge \Psi]$ is defined by $[\Phi \wedge \Psi](x, y) := [\Phi(x), \Psi(y)] - [\Phi(y), \Psi(x)]$, and that in the matrix Lie algebras context $[\Phi \wedge \Phi](x, y) = 2\Phi \wedge \Phi(x, y)$ where the second $\wedge$ means matrix multiplication with $\wedge$ for the components.

Since a frame F is determined by its Maurer-Cartan form $\Phi = F^{-1}dF$ which satisfies the integrability condition $d\Phi + \frac{1}{2}[\Phi \wedge \Phi] = 0$, that in components is:

$$\begin{cases} d\Phi_{\mathcal{K}} + \frac{1}{2}[\Phi_{\mathcal{K}} \wedge \Phi_{\mathcal{K}}] + \frac{1}{2}[\Phi_{\mathcal{P}} \wedge \Phi_{\mathcal{P}}] = 0 \\ d\Phi_{\mathcal{P}} + [\Phi_{\mathcal{K}} \wedge \Phi_{\mathcal{P}}] = 0, \end{cases}$$

it follows that $\varphi : M^n \to \frac{G}{K}$ is a curved flat if and only if those equations decouple further to give

$$\begin{cases} d\Phi_{\mathcal{K}} + \frac{1}{2}[\Phi_{\mathcal{K}} \wedge \Phi_{\mathcal{K}}] = 0 \\ d\Phi_{\mathcal{P}} + [\Phi_{\mathcal{K}} \wedge \Phi_{\mathcal{P}}] = 0 \\ [\Phi_{\mathcal{P}} \wedge \Phi_{\mathcal{P}}] = 0. \end{cases} \tag{2}$$

Now we just recall that the curved flats come in 1-parameter families, this means, their integrability equations can be rewritten as a zero-curvature condition involving an auxiliary parameter. That in particular, assures that the geometries associated to curved flats can be handled in the context of integrable system theory ([5]).

Now we will focus to prove Theorem 1.1. We begin identifying the Gauss-Codazzi-Ricci equations for the Lorentzian immersion as the integrability condition of a 1-form connection associated to a f-adapted frame to the strip (f, S):

Let $f : M_1^n \to L_1^{n+2} \subset \mathbb{R}_2^{n+3}$ be a representative of the immersion into $\hat{\mathbb{R}}_1^{n+1}$ and $S : M_1^n \to S_2^{n+2} \subset \mathbb{R}_2^{n+3}$ be a spherical congruence enveloped by f. Now let F be a f-adapted pseudo-orthonormal framing for the strip (f, S), given by

$$F = (S_1, ..., S_n, S, f, \widehat{f}) : M_1^n \to O_2(n+3) \tag{3}$$

with $\text{span}\{S_1, ..., S_n\}_p = df_p(T_pM_1^n)$ for all $p \in M_1^n$. In particular $S_i = Fe_i$, $i = 1, ..., n$, $S = Fe_{n+1}$, $f = Fe_{n+2}$, and $\widehat{f} := Fe_{n+3}$. Moreover, $\{S_1 ..., S_n\}$ forms an orthonormal set with $\langle S_1, S_1 \rangle = -1$, where f and $\widehat{f}$ are null vectors such that $\langle f, \widehat{f} \rangle = 1$.

As usual, $dFe_B = \sum_A \omega_{AB} Fe_A$, so the connection form $\Phi = F^{-1}dF$: $TM \to o_2(n+3)$ is given by

$$\Phi = \begin{pmatrix} \omega & \eta \\ -\eta^* & \nu \end{pmatrix} = \begin{pmatrix} \omega & \eta \\ -J'\eta^t I_{1,n-1} & \nu \end{pmatrix}$$

where

$$J' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \omega = \big(\omega_{ij}\big) : TM \to o_1(n)$$

$$\eta = \begin{pmatrix} -\psi_1 & -w_1 & -\zeta_1 \\ \psi_2 & w_2 & \zeta_2 \\ \vdots & \vdots & \vdots \\ \psi_n & w_n & \zeta_n \end{pmatrix} : TM \to \mathcal{M}(n \times 3) \quad \textit{and}$$

$$nu = \begin{pmatrix} 0 & 0 & v \\ -v & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : TM \to o_1(3).$$

Writing $df = dFe_{n+2} = -w_1 S_1 + ... + w_n S_n$, and making $\tau_1 = -1$ and $\tau_i = 1$ for $2 \le i \le n$, the first and second fundamental forms are given by:

$$I = \sum_{i=1}^{n} \tau_i w_i^2, \quad II = -(\sum_{i=1}^{n} \tau_i w_i \psi_i) S - (\sum_{i=1}^{n} \tau_i w_i \zeta_i) f - (\sum_{i=1}^{n} \tau_i w_i^2) \widehat{f}.$$

The integrability condition for the existence of such a frame F, the Maurer Cartan equations $d\Phi + \frac{1}{2}[\Phi \wedge \Phi] = 0$, are Gauss-Codazzi-Ricci equations for the immersion f, namely:

The Ricci equation: $d\nu = \eta^* \wedge \eta$, ie:

$$\begin{cases} dv = \sum_i^n \tau_i (\psi_i \wedge \zeta_i) \\ 0 = \sum_1^n \tau_i (\psi_i \wedge w_i) \\ 0 = \sum_1^n \tau_i (\zeta_i \wedge w_i) \end{cases} \tag{4}$$

the Codazzi equation: $d\eta = -(\omega \wedge \eta + \eta \wedge v)$, which in components is:

$$\begin{cases} d\psi_i + \sum_j \tau_i\tau_j\omega_{ij} \wedge \psi_j = w_i \wedge v \\ dw_i + \sum_j \tau_i\tau_j\omega_{ij} \wedge w_j = 0 \\ d\zeta_i + \sum_j \tau_i\tau_j\omega_{ij} \wedge \zeta_j = v \wedge \psi_i, \end{cases} \tag{5}$$

and the Gauss equation $\rho = \eta \wedge \eta^*$ with the curvature form $\rho = d\omega + \omega \wedge \omega$:

$$\rho_{ij} := d\omega_{ij} + \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} = \tau_i(\psi_i \wedge \psi_j + w_i \wedge \zeta_j + \zeta_i \wedge w_j). \tag{6}$$

Then it follows from the second and third Ricci equation, that the second fundamental forms $II_S = -w_1\psi_1 + \sum_2^n w_i\psi_i$ and $II_{\widehat{f}} = -w_1\zeta_1 + \sum_2^n w_i\zeta_i$, where S is being considered as a unit normal field to the immersion $f : M_1^n \to \mathbb{R}_2^{n+3}$, are symmetric forms with respect to the Lorentzian metric.

Proof of Theorem 1.1. Let g the generalized Gauss map associated to $f : M_1^n \to L_1^{n+2}$ immersion in the light cone, defined by

$$g : M_1^n \to \frac{O_2(n+3)}{O_1(n) \times O_1(3)} \qquad p \mapsto d_p f(T_p M_1^n),$$

which takes values in the Grassmannian of Lorentz n-planes in $\mathbb{R}_2^{n+3}$, i.e, into the space of hyperbolas in $\hat{\mathbb{R}}_1^{n+1}$. In fact, using Lorentzian Möbius geometry one can see that those hyperbolas come from the intersection of a one hyperbolic space corresponding to $S_1(p)$, $p \in M_1^n$ and $n-1$ spheres corresponding to $\{S_i(p)\}_{i=1}^n, p \in M_1^n$, so generically it is a hyperbola. In the next we take $F = (S_1, .., S_n, S, f, \widehat{f})$ a f-adapted pseudo-orthonormal frame as in (3). Then according to symmetric decomposition of the Lie algebra, $o_2(n+3) = \mathcal{K} \oplus \mathcal{P}$ where $\mathcal{K} = o_1(n) \times o_1(3)$, and the connection 1-form Φ of F splits into the $\mathcal{K}$ and $\mathcal{P}$-parts, namely: $\Phi = \Phi_{\mathcal{K}} + \Phi_{\mathcal{P}}$ with

$$\Phi_{\mathcal{K}} = \begin{pmatrix} \omega & 0 \\ 0 & \nu \end{pmatrix} : TM \to \mathcal{K}, \quad \Phi_{\mathcal{P}} = \begin{pmatrix} 0 & \eta \\ -\eta^* & 0 \end{pmatrix} : TM \to \mathcal{P}.$$

Hence if the induced metric by the representative is assumed to be flat, one has, from Lemma 2.1, that besides the tangent bundle, the normal bundle of $f : M_1^n \to \mathbb{R}_2^{n+3}$ is also flat. Then it follows from the conditions $d\omega = -\omega \wedge \omega$ and $dv = -v \wedge v$ that $d\Phi_{\mathcal{K}} + \frac{1}{2}[\Phi_{\mathcal{K}} \wedge \Phi_{\mathcal{K}}] = 0$ and $[\Phi_{\mathcal{P}} \wedge \Phi_{\mathcal{P}}] = 2\Phi_{\mathcal{P}} \wedge \Phi_{\mathcal{P}} = 0$. From other hand, since $d\eta = -(\omega \wedge \eta + \eta \wedge v)$ it follows that $d\Phi_{\mathcal{P}} + [\Phi_{\mathcal{K}} \wedge \Phi_{\mathcal{P}}] = 0$. Hence we have checked conditions (2), and so the generalized Gauss map is a curved flat. Conversely, if the generalized Gauss map is a curved flat we have that for all i, j

$$\psi_i \wedge \psi_j + w_i \wedge \zeta_j + \zeta_i \wedge w_j = 0,$$

which implies that $\rho_{ij} = 0$ for all i, j (see equation (6)). Hence the induced metric on M_1^n is flat.

□

It follows that

Corollary 3.1. *Given a conformally flat Lorentzian hypersurface in $\hat{\mathbb{R}}_1^{n+1}$, the generalized Gauss map of any flat light cone lift is a curved flat.*

Remark For the 3-dimensional case with a diagonalizable shape operator with three distinct real eigenvalues a_1, a_2, a_3, one expects that, in similar form as happens in the define positive case, the induced metric by a curved flat was exactly a multiple of the conformal metric $C = -\gamma_1^2 + \gamma_2^2 + \gamma_3^2$, where γ_i are the conformal fundamental forms (1). Now, since we are assuming the three eigenvalues are different, one see that the induce metric by the curved flat is non-degenerate, so the curved flat is regular. Finally, integranting γ_i given by (1), one gets the canonical principal curvature coordinates which come with the regular curved flat. Those give the canonical Guichard‘s net existing on every real diagonalizable conformally flat Lorentzian hypersurface [4].

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