A SCHEME TO SOLVE APPROXIMATELY SOME LINEAR ORDINARY DIFFERENTIAL EQUATIONS IN BANACH SPACES AND DELAYED PDE WITH REGULATED SOLUTIONS

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Introduction: In this paper we propose an approximation scheme to the solutions of some linear PDE, following the scheme proposed in [1] for the integral equation of type (K) and on general spaces, that works in the space of the regulated functions (i.e., functions that have discontinuities only of the first kind).

In the first part we repropose the theorems of approximation done in [1] into the context of the regulated right-continuous functions, now in the frame of the left-continuous one. Essentially, we need the left-continuity here, to be able to transform freely integral equations of the first kind into one of the second kind and vice-versa.

In the second part we apply the results of the first part to approximate the solutions of a linear O.D.E. on B-spaces (actually a P.D.E.) and in the third part, we apply another time the results obtained in the first one, to a delayed PDE.

To develop the arguments in the second part, we will roughly follow Pazy [2] (concerning the theory of semigroups), together with a theorem that gives an answer to the following question on linear forced PDE: how much one must have to restrict the forcing function, to gets the solution in the system kept by the usual variation-of-constant formula, being strong? In this direction we are using a Hönig result [3] that extends and amplify one by Travis [4], connecting semi-variation and semigroup.

Finally: the equations whose solutions we are approximating here, are interesting by themselves, and I hope it would be a nice reason to do it.

I. The Scheme of Approximation to the integral equation of type (K)

In this section we keep the notations and definitions of [5] and [2]: X is a B-space and for $a, b \in \mathbb{R} \cup \{+\infty, -\infty\}$, the set $G([a,b], X)$ of the regulated functions from $[a,b]$ into X, is a B-space, when endowed with the *sup* norm (if $b = \infty$ [or respect.: $a = -\infty$], we define $f \in G([a,b], X)$ satisfying $\lim_{t\to\infty} f(t) = f(+\infty) < \infty$ [or respect.: $\lim_{t\to-\infty} f(t) = f(-\infty) < \infty$]). The integral equation which we are dealing with is the:

$$(K) \qquad x(t) - x(a) + \int_a^t d_s K(t,s)x(s) = u(t) - u(a)$$

where x is regulated and $u \in \Omega \subset G([a,b],x)$. The integral considered here is the interior (or Dushinik type) integral and $K \in G_0^\sigma \cdot SV^u([a,b],L(X))$, with $L(X)$ denoting the usual space of all linear and bounded operators on X.

Under quite general conditions (see [5]) the equation (K) has an unique resolvent $R \in G_I^\sigma \cdot SV^u([a,b],L(X))$ with the solution done by:

$$(\rho) \qquad x(t) = u(t) - R(t,a)[u(a) - x(a)] - \int_a^t \cdot d_s R(t,s)u(s).$$

In [1] it was proposed a scheme of approximation to x and u connected by (K) or (ρ), in the case in which they are right-continuous regulated functions. Now we translate the scheme to the left-continuous case, and no surprises arise in such translation. We only recall that $G^-([a,b],X)$ denotes the closed subspace of the left-continuous regulated functions. Moved by the definition of

$$G^-(d,Z,\epsilon) = \{f \in G^-([a,b],X);\ \|f(t) - \Sigma r_i \chi_i^-(t)\| < \epsilon;\ r_i \in Z\},$$

[where $Z \subset X$, and χ_i^- is the usual characteristic function on the interval $(d_{i-1},d_i]$, of the partition d of $[a,b]$, and $\epsilon > 0$] we locate our analysis on $(\delta_{n-1},\delta_n] \subset (d_{m-1},d_m]$.

Theorem 1^-. *Given Z a bounded subset of $X, d \in D$ and the operator K in (K), set for every $\epsilon > 0$ and the function $x \in G^-(d,Z,\epsilon)$ its i-th ϵ-approximation $\xi_i (1 \le i \le p)$.*

For every $\eta > 0$, let us consider the correspondent division $\delta \gg d$ such that $\eta \geq \dot{w}_\delta(K_{di})$ $(0 \le i \le m-1)$.

Fixed $(\delta_{n-1},\delta_n] \subset (d_{m-1},d_m]$, and $u(a) \in X$, we define the operator $A_\epsilon : X^{m+1} \longrightarrow X$

$$A_\epsilon(x(a),\xi_1,\xi_2,\dots,\xi_m) = u(a) - x(a) + \xi_m - K(\delta_n,d_0)\xi_1 - \\ - \sum_{l=1}^{m-1} K(\delta_n,d_l)[\xi_{l+1} - \xi_l]$$

such that, for all $t \in (\delta_{n-1},\delta_n]$ there exists a $\gamma = \gamma(\epsilon,\eta)$, with

$$(1) \qquad \|u(t) - A_\epsilon(x(a),\xi_1,\xi_2,\dots,\xi_m)\| \le \gamma(\epsilon,\eta)$$

and γ decreasing faster then ϵ. The value γ is

$$\gamma(\eta,\epsilon) = \epsilon(1 + SV^u_{[a,d_m]}(K)) + \eta \cdot Var(\sum_{i=1}^m \xi_i \chi_i^-)$$

THEOREM 2$^-$. *Given R the mapping in (ρ), $d \in D_{[a,b]}$, and for $\epsilon > 0$ let be $u \in G^-(d, Z, \epsilon)$ with i-th ϵ-approximation $\tau_i (1 \leq i \leq m)$. For $\eta > 0$ set δ the partition such that $\eta \geq \dot{w}_\delta(R_{di})$, $(0 \leq i \leq m-1)$. Then for all $t \in (\delta_{n-1}, \delta_n] \subset (d_{m-1}, d_m]$, if $\delta_n \neq d_m$, and for all $t \in (\delta_{n-1}, \delta_n) \subset (d_{m-1}, d_m]$, if $\delta_n = \delta_m$ in which case we extend the result by continuity to $t = \delta_n$, we have for a fixed initial $x(a)$ and $u(a)$:*

$$\|x(t) - B_\epsilon(x(a), u(a), \tau_1, ..., \tau_m)\| \leq$$

$$\leq \epsilon(1 + SV_{[a,d_m]}(R)) + \eta(\|x(a) - u(a)\| + Var(\sum_{i=1}^{m} \tau_i \chi_i^-)) \tag{2}$$

where

$$B_\epsilon(x(a), u(a), \tau_1, ..., \tau_m) = \tau_m + R(\delta_n, d_0)[x(a) - u(a) + \tau_1] + \\ + \sum_{l=1}^{m-1} R(\delta_n, d_l)[\tau_{l+1} - \tau_l].$$

In the next section we will consider the interval $[a, b]$ being the $[0, \infty)$ one.

II. The PDE $\dot{x} = Ax + f$

II.1. Let be A the generator of a C^0-semigroup $\{T(t) \in L(X); t \geq 0\}$ and $x, f \in G^-([0, b], X)$. Hönig in [2], reaches the following result:

PROPOSITION. *([2], Corol. 8). Defining the space $D_{Gr}(A)$ being the set domain of A endowed with the graph-norm (i.e.: $\|x\| = \|x\|_X + \|Ax\|_X$, $x \in D(A)$), then for every $f \in G^-([0, b], D_{Gr}(A))$, the strong solution (in the sense that the derivative is taken in X) of*

$$\dot{x} = Ax + f \qquad (x(0) = x_0) \tag{3}$$

is done by

$$x(t) = T(t)x_0 + \int_0^t T(t - s)f(s)ds. \tag{4}$$

This result is fundamentally due to the fact that the function $T : [0, b] \to L(D_{Gr}(A), X)$ is absolutely continuous, hence of bounded semivariation, and so we can apply the results of the theory of bounded semivariations functions to the theory of the C^0-semigroups.

Using the representation theorem for linear operator on the space $G^{-}([0,b],X)$ (see [5]) we get that (4) is equal to

$$x(t) = f(t) + R(t,0)x_0 - \int_0^t \cdot d_s R(t,s) f(s), \tag{5}$$

where

$$-R(t,s)x = \int_s^t T(t-\sigma)x\,d\sigma - H(t,s)x + x \qquad (x \in X), \tag{6}$$
$$H(t,s) = \begin{cases} 1 \text{ if } s \quad t \\ 0 \text{ if } s > t \quad o \end{cases}$$

and $R(t,0) = T(t)$. This last equality arises when we compare the equations (4) and (5) in the homogeneous (i.e., with $f \equiv 0$) case.

Observe that if f would not be left-continuous at $t > 0$, we cannot allways separate $f(t)$ from the integral term in (5). For look at this, take the example even in the scalar case, for $\epsilon > 0$ small:

$$f_\epsilon(z) = \begin{cases} 1 \text{ if } \delta \in \mathbb{Q} \cap (t-\epsilon, t) \\ 10 \text{ if } \delta = t \\ 0 \text{ in } (t-\epsilon, t) \setminus \mathbb{Q} \cap (t-\epsilon, t) \end{cases}$$

where $\mathbb{Q}$ is the set of the rational numbers in $\mathbb{R}$.

Observe also, that the equality (4) in the context that we are moving on, is the equation (ρ) associated to (K'). But, what about the kernel K'? We get K' by renormalizing (4) and then "solving" R in (ρ) with an operator resolvent S. Having in mind that the relation among the kernel and the resolvent of the (K') equation is bicontinuous (see [5], Rem. 2, p.13), then $K' \in G_0^{\sigma} \cdot SV^{u}$ is exactly the operator S renormalized. However, our main interest is by now to show an example in which we get approximations of the solution of a PDE in the form (3) by applying the scheme get in the Theorem 2^{-} above.

To do it, we state the:

Proposition 1. *Let be R as in (5). Then given $d = \{d_0, d_1, \ldots, d_p\}$ a finite division of $[0,b]$, we get for each $\eta > 0$, that the division δ induced by η and d in the Theorem 2^{-}, do not depends on s.*

Proof: First of all, we see that for each $\eta > 0$ we have always $\dot{w}_\delta(H(\cdot, d_i)) = 0$, $(i \geq 0)$, because $\delta \gg d$.

To end the proof we take under consideration that for each $t_1, t_2 > 0$, it's true:

$$\int_s^{t_1} T(t_1 - \sigma)d\sigma - \int_s^{t_2} T(t_2 - \sigma)d\sigma = \int_{t_1}^{t_2} T(\tau)d\tau. \blacksquare$$

PROPOSITION 2. *Suppose the semigroup $(T(t))_{t\geq 0}$ being generated by A. Given $\eta > 0$ and $d \in D_{[0,b]}$, then the division δ of $[0, b]$ in the Theorem 2^- can be taken as $\delta = d \cup \delta^*$, where the elements of d^* are recursively done by: $\delta_0^* = 0$ and for $n \in \mathbb{N}$:*

$$\delta_{n+1}^* = \frac{1}{|\beta|} ln(e^{\beta\delta_n^* + \frac{\eta\cdot\beta}{M}})$$

for computable real numbers β and $M \geq 1$.

If, moreover, $(T(t))_{t\geq 0}$ is a contraction C^0-semigroup, then we can take δ as $d \cup \hat{\delta}$, where the elements $\hat{\delta}_n (n \geq 0)$ of $\hat{\delta}$, are $\hat{\delta}_n = n\eta$.

PROOF: Choose some $l > 0$ such that $\sup_{0\leq t\leq l} ||T(t)|| = M < \infty$. Let be $\beta \geq \frac{1}{l} ln||T(l)||$, (i.e. $||T(l)|| \leq e^{\beta l}$). Then using the result of the Proposition 1 above, we end the proof. $\blacksquare$

In the following, we give an example of a linear PDE (actually an O.D.E.) in two different kinds of B-spaces, with the sake of to be more specific in the application of the above results.

II. 2. Example: Suppose $a : [a, \infty) \to \mathbb{R}$ a function of class C^1, with $a(x) > 0$ for $x \geq 0$ and such that $\int_0^x a(\xi)d\xi \to \infty$ as $x \to \infty$ (in the part II.2.3 below, these condition on the mapping a, as we will see, can be weakened).

(7) $$\frac{\partial u}{\partial t}(t, x) + a(x)\frac{\partial u}{\partial x}(t, x) = f(t, x) \qquad (t \in [0, b], x \geq 0)$$

with initial condition $u(0, x) = \phi(x)$ for $x > 0$ [where ϕ is a given smooth function with $\phi(0) = 0$].

Making $U, F \in G^-([0, b], X)$:

(8) $$U(t)x = u(t, x) \text{ and } F(t)x = f(t, x),$$

let us take $F \in \Omega \subset G^-([0, \infty], X)$. [We note that we can enclose some controllability particularities (e.g. the controls being applied only on the boundary of a region) when choosing such Ω]. Then, according (8), we can transform (7) into the problem:

(9) $$\begin{cases} \dot{U}(t) = A[U(t)] + F(t) \\ U(0) = \phi \end{cases} \qquad (F \in \Omega,\ t \geq 0)$$

applied on $x \geq 0$.

Here we have A being the operator

(10) $$Av = a.v' \quad (v \text{ in the domain of } A).$$

Let us study the problem (9) into two different kinds of X.

II.2.1 - Consider

$$X = \{v : (0,\infty) \longrightarrow \mathbb{R}; v \text{ is } C^0 \text{ and } v(0) = 0; v(x) \to \infty \text{ as } x \to \infty\}$$

with the *sup* norm.

In this case the domain of A, is

$$D(A) = \{v \in X; v \in C^1 \text{ and } a.v' \in X\},$$

and A is the generator of the contraction C^0-semigroup (see [2]).

(11) $$T(t) = e^{tA}$$

Let us go now to apply the Theorem 2^- in the specific case in which $F(0) = 0$

We remember that for each $v \in X$, we have

$$\lim_{r\to\infty}(I - \frac{t}{r}A)^{-(r+1)}v = T(t)v \quad (t \geq 0).$$

Fixing a finite division d of $[0,\infty)$ and the m elements in X. $0 = r_1, r_2, ..., r_m$, and keeping for each $\epsilon > 0$, $\eta > 0$:

$$\delta = \{p\eta; p = 0, 1, 2, ..., \hat{n}\} \cup d,$$

the result (2) yields: if for all $F \in G^-([0,\infty), D_{Gr}(A))$ in (9) we have

$$\|F(t) - \sum_{i=1}^{m} r_i\chi_i^-(t)\|_{Gr} = \sup_{0\leq x\leq d_m} \|F(t)(x) - \sum_{i=1}^{m} r_i(x)\cdot\chi_i^-(t)\| + \\ + \sup_{0\leq x\leq d_m} \|[F(t)'(x)\| < \epsilon$$

and for all $t \in (\delta_{n-1}, \delta_m] \subset (d_{m-1}, d_m]$, we define $\xi_n \in X$ as:

(12)
$$\begin{aligned}\xi_n = \tau_m + \lim_{r\to\infty}\Big(I - \frac{(\delta_n - d_0)}{r}A\Big)^{-r} \cdot a(\cdot) \cdot \int_0^{\cdot} \phi(\sigma)d\sigma + \\ + \lim_{r\to\infty}\sum_{l=1}\Big(I - \frac{(\delta_n - d_l)}{r}A\Big)^{-r} \cdot a(\cdot) \cdot \int_0^{\cdot} (\tau_{l+1} - \tau_l)(\sigma)d\sigma,\end{aligned}$$

we get

(13)
$$||v(t) - \xi_n|| < \epsilon(1 + SV_{[0,d_m]}(R)) + \eta(||\phi|| + ||\tau_2|| + \sum_{l=2}^{m-1}||\tau_{l+1} - \tau_l||).$$

Observe that we can write the second term of the inequality (13) more explicitly, looking at the definition of the semivariation of R:

$$SV_{[0,d_m]}(R) \le c,$$

where c is taken as the constant for which

$$||A\int_0^{d_m} T(d_m - \sigma)f(\sigma)d\sigma|| \le c||f||$$

for all step-function $f : [0, d_m] \longrightarrow X$.

II.2.2. $X = L_p(\mathbb{R}^+), (1 \le p < \infty)$

Now we define the domain of the infinitesimal generator $A = A_p$:

$D(A_p) = \{v : L_p(\mathbb{R}^+); v$ has an absolutely continuous element in its equivalent class, and $v(0) = 0$, and $av' \in L_p(\mathbb{R}^+)\}$.

By a well-known application of the Lumer-Phillips theorem for the dissipative operator $A_p - \frac{M}{p}$ (with $sup\ a'(x) \le M < \infty$) we have an estimate on the norm of $T(t) = T_p(t)$ $(t \ge 0)$:

(14)
$$||T_p(t)\phi||_{L_p(\mathbb{R}^+)} \le e^{t\frac{m}{p}}||\phi||_{L_p(\mathbb{R}^+)}$$

where $m = sup\ a'(x) < \infty$.

II.2.3. Finally, if we treat the problem by direct integration along the characteristic directions of the P.D.E. in the problem (7), (see [6]), it is possible to weaken the conditions on the function a, improving the estimates in (14).

In fact, the function a can have a finite number of zeros on $\mathbb{R}^+, x_1, ..., x_N$ in such a way that for every $\mu > \mu_0 = max\{l_+, a'(x_K)$ where $K = 1, ..., N\}$ there is $M_\mu < \infty$ such that:

$$\|T(t)\|_{L(L_p(\mathbb{R}^+))} \leq M_\mu e^{\mu \frac{t}{p}}. \tag{15}$$

The number l_+ that appears in the definition of μ_0 above is: $l_+ = \lim_{x \to +\infty} a'(x)$ for ultimately positive a and $l_+ = -\infty$ if a it is ultimately negative.

The estimate (15) is a substancial improvement over the (14) one, and this is reflected in the approximation of the solution of (9), when transferring the results of the part 2.1 above, to the L_p context.

III. The delayed PDE $\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + u(t-r,x) + v(t,x)$.

Here we consider $r > 0$ and the $u(\cdot, x)$ and $v(\cdot, x)$ as regulated left-continuous functions in the variable t.

Our goal in this part is to approximate the regulated left-continuous solutions of the system in the L_2 context, for $t \in [-r, b]$, $(b \geq 0)$, and $x \in [0, \pi]$ and $\theta \in [-r, 0]$:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + u(t-r,x) + v(t,x), \\ \exists \phi \in G^-([-r,0], L_2([0,\pi], \mathbb{R})) \text{ such that } \phi(-r) = 0 \text{ and } u(\theta, x) = \phi(\theta)(x), \\ v(\cdot, x) \in \Omega \subset G^-([-r,b], L_2([0,\pi])) \text{ for every } x. \end{cases} \tag{16}$$

If we consider

$$U(t), V(t) : [0, \pi] \longrightarrow \mathbb{R} \qquad (t \geq -r)$$

defined by

$$U(t)x = u(t,x) \text{ and } V(t)x = v(t,x),$$

where

$$U \in G^-([-r,b], L_2([0,\pi])$$

and

$$V \in \Omega \subset G^-([-r,b], L_2([0,\pi])$$

then formally we can transform (31) in an integral equation applicable to each $x \in [0, \pi]$, as the next result shows:

THEOREM 4. *Carrying on the above conditions and notations, we have that (16) is equivalent to*

$$(17)\qquad \begin{cases} U(t) - \int_{-r}^{t} T(t-s)U(s-r)ds = \int_{-r}^{t} T(t-s)V(s)ds \\ U(-r) = 0 \end{cases}$$

where $(T(t))_{t\geq 0}$ is the C^0-semigroup generated by the operator A, that is the minimal extension to a closed densely defined linear operator in L_2, of the map

$$\Psi \longrightarrow \frac{d^2\Psi}{dx^2}$$

defined for all smooth Ψ which vanishes at 0 and π.

A is a dissipative operator, then $(T(t))_{t\geq 0}$ is of contraction.

Actually, for all $t \geq 0, T(t) \in L(L_2([0,\pi]))$ and $T(t)\Psi(x) = \sum_{n=1}^{\infty} e^{-n^2 t}\Psi_n \cdot \sin(nx)$ where Ψ_n is the usual hilbertian coefficient of Ψ in the (sine) orthonormal basis of $L_2([0,\pi])$,

$$\Psi_n = \sqrt{\frac{2}{\pi}} \int_0^{\pi} \Psi(s)\sin(ns)ds.$$

PROOF: From (17), we have:

$$(18)\qquad \dot{U}(t) = AU(t) + U(t-r) + V(t)$$

Defining $U_n(t)$ and $V_n(t)$ the ususal hilbertian coefficients of $U(t)$ and $V(t)$, respectively, in the (sine) basis of $L_2([0,\pi])$, formally processing the derivative, (18) yields, for all $n = 1, 2, \ldots$,

$$\dot{U}(t) = -n^2 U_n(t) + U_n(t-r) + V_n(t).$$

Then:

$$U_n(t)e^{n^2 t} - U_n(-r)e^{n^2 r} = \int_{-r}^{t} e^{n^2 s} U_n(s-r)ds + \int_{-r}^{t} e^{n^2 s} V_n(s)ds$$

and:

$$(19)\qquad U(t) - \int_{-r}^{t} T(t-s)U(s-r)ds = \int_{-r}^{t} T(t-s)V(s)ds.$$

For all the properties of $(T(t))_{t\geq 0}$ and A we could have directly calculate the semigroup and the duality map of A (that is a very nice application of the divergence theorem), but to follow closely the aims of this work we refer the eventual readers to [6] or ([8], Ex. 4). ∎

Corollary 1. *(The exact problem which we are dealing with). The equality (19) is equivalent to the integral equation of type (K):*

$$U(t)+\int_{-r}^{t}\cdot d_sK(t,s)U(s)=\int_{r}^{t}\cdot d_sZ(t,s)V(s) \tag{20}$$

with initial condition:

$$U(\theta)=\int_{-r}^{\theta}\cdot d_sZ(\theta,s)V(s),$$

where for all $t,s\geq -r$ we define:

$$K(t,s)=\begin{cases}\int_s^{t-r}T(t-s)d\sigma & if\, t>s+r\\ 0 & if\, t\leq s+r\end{cases}$$

$$Z(t,s)=\begin{cases}-\int_s^{t}T(t-\sigma)d\sigma & if\, t>s\\ 0 & if\, t\leq s\end{cases}$$

The operators K,Z belong to $G_0^{\sigma}\cdot SV^u([-r,b],L_2([0,\pi]))$.

Proof: It follows easily from the representation theorem for linear operators on $G^-([-r,b],X),X$ being a B-space, (see [7], Th.I.5.1), and on the fact that $K(t,s)=0$ if $-r\leq t\leq 0$, because in this case $s\geq t-r$ for $t\geq s$.∎

At this point, fixing the forcing term V in (20) we are able to represent U approximately by a step function on $L_2([0,\pi])$ (in the sense of the Theorem 1^-) and hence, applying $U(t)$ on each $x\in[0,\pi]$, we are able to represent the approximate solution of the process (16) which we are dealing with. This will be done in the following:

Theorem 6. *Let be $b>0$ and a division d of $[-r,b]$ with p elements, and with $0\in d$ and suppose that there exists $V_1,...,V_p$ in $L_2([0,\pi])$ with*

$$\|\int_{-r}^{t}\cdot d_sZ(t,s)V(s)-\sum_{i=1}^{p}V_i\chi_i^-(t)\|<\mu$$

for a fixed $\mu>0$ on d (χ^- is the usual characteristic function on the intervals $(d_{i-1},d_i]$, $1\leq r\leq p$, of d).

Then there exist a division d^K of $[-r,b]$ with $d^K\gg d$ and a computable step function G on the intervals generating by d^K in such a way that G solves (20) and G can be kept so close to U as wanted.

PROOF: Let be the division of $[-r,b]$

$$d^K = \{-r = d_0^K, d_1^K, \ldots, d_q^K = 0, d_{q+1}^K, \ldots, d_{n_0}^K = b\}$$

containing the points of $\{lr; l \in \mathbb{N}\} \cap [0,b]$, and for a given $\eta > 0$ the points of the division $\{l\eta; l \in \mathbb{N}\} \cap [0,b]$ ordered by the usual order in $\mathbb{R}$.

Define for all $i, 1 \le i \le q, U_i = V_i$, on each interval $(d_{i-1}, d_i]$ and set U_j, for each interval $(d_{j-1}, d_j], q \le j \le n_0$, as the following.

To reach our aim we drop the proof in three essential cases depending on the relative position of $\Delta_j = (d_{j-1}, d_j]$ and $\Delta_{j+1} = (d_j, d_{j+1}]$. The analysis of the general case will be a straightforward extension of these three:

(1st case): $\Delta_j, \Delta_{j+1} \subset (0,r]$ and the approximation to $F(t)$ is constant (say V_α) in $(\delta_{j-1}, \delta_{j+1}]$,

(2nd case): $\Delta_j, \Delta_{j+1} \subset (0,r]$ and the approximations to $F(t)$ in Δ_j and Δ_{j+1} are different (respectively V_α and $V_{\alpha+1}$),

(3rd case): $\Delta_j, \Delta_{j+1} \subset (r,2r]$ and the approximation to $F(t)$ is V_α in $(\delta_{j-1}, \delta_{j+1}]$.

Let us denote now the elements $d_0^K, \ldots, d_{p_j}^K$, of d^K for which $d_i^K < d_j^K - r, (i \in \mathbb{N})$.

We define (following the theorem 1^- above):

$$U_j = V_1 + V_\alpha + K(d_j^K, d_0^K)V_1 + \sum_{l=1}^{p_j} K(d_j^K, d_l^K)[V_{l+1} - V_l]. \tag{21}$$

Then we can see that:

(in the 1st case): $||U_{j+1} - U_j|| \le ||[K(d_{j+1}, d_0^K) - K(d_j^K, d_0^K)]V_1||+$

$$+ \sum_{l=1}^{p_j} ||[K(d_{j+1}^K, d_l^K)] \cdot (V_{l+1} - V_l)||+ \tag{22}$$

$$+ \sum_{l=p_j+1}^{p_{j+1}} ||K(d_{j+1}^K, d_l^K)V_l|| = (\sim)$$

where the symbol $(\sim)$ denotes the formal statement in the second part of the inequality, and (in the 2nd case):

$$||U_{j+1} - U_j|| \le ||V_{\alpha+1} - V_\alpha|| + (\sim) \tag{23}$$

and

(<u>in the 3rd case</u>):

(24) $$\|U_{j+1} - U_j\| \leq (\sim)$$

Remembering that for every $U \in L^2([0,\pi])$ and $b > a$, and $c > 0$:

$$\|\int_a^{b+c} T(b+c-\sigma)U\, d\tau - \int_a^b T(b-\sigma)U\, d\sigma\|_2 =$$
$$\|\int_{b+c-a}^{b-a} T(\tau)U\, d\tau\|_2 \leq c\|U\|_2,$$

where the last inequality follows for $(T(t))_{t\geq 0}$ is of contraction, then (22) - (24), respectively, yields:

(25) $$\|U_{j+1} - U_j\| \leq \eta(\|V_1\| + \sum_{l=1}^{p_j}\|V_{l+1} - V_l\|)$$
$$+ \sum_{l=p_j+1}^{p_{j+1}} [1 + (d_{j+1}^K - r - d_l^K)] \cdot \|V_l\| = (\Delta) \qquad \text{(note that } d_{p_{j+1}}^K \leq 0),$$

(26) $$\|U_{j+1} - U_j\| \leq \|V_{\alpha+1} - V_\alpha\| + (\Delta) \qquad \text{(note that } \alpha > q),$$

(27) $$\|U_{j+1} - U_j\| \leq (\Delta) \ : \qquad \text{(note that in this case } p_j > q).$$

Then we can see by means of (25) - (27) that it is possible to compute a step function $U : [-r,b] \longrightarrow L_2([0,\pi])$ namely:

$$U(t) = \sum_{i=1}^{q+m_0} U_i \chi_i^-(t) \qquad \text{(on } d^K)$$

in such a way, that, for every $\delta > 0$:

$$U(t) + \int_{-r}^{t} \cdot d_s K(t,s)U(s) = G(t) \qquad (b \geq t \geq -r)$$

has a solution $U(t)$, satisfying for all $t \in (d_{n-1}^K, d_n^K] \subset [-r,0]$:

$$U(t) = G(t) = U_n$$

and for all $t \in (d_{n-1}^K, d_n^K] \subset (0,b]$, the following statement: if $\|U(t) - U_n\| < \epsilon, (\epsilon > 0)$, then

$$\|G(t) - \sum_{l=q+1}^{q+m_0} V_i \chi_i^-(t)\| \leq (b+r)\epsilon + \lambda(\eta, V_1, ..., V_{p_n}).$$

So, for $t \in [-r, b]$ we keep:

$$\|G(t) - F(t)\| \leq \delta + (b + r)\epsilon + \lambda(\eta, V_1, ..., V_{r_*})$$

where $\lambda(\eta) \to 0$ as $\eta \to 0$, and $F(t) = \int_{-r}^{t} \cdot d_s Z(t,s) V(s)$.

In particular if one wants, $U(t)$ can be taken as:

$$U(t) = \sum_{i=1}^{n_*} U_i \chi_i^-(t). \blacksquare$$

Note that this theorem assures us the possibility in to keep a result in the sinthesys of a system of the (K) type what is not so easy (see [1], 3.).

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