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ON CHARACTERIZATIONS OF UNIFORM DISTRIBUTIONS

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On characterizations of Uniform distributions

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Abstract

In this work we consider random vectors (or matrices) the densities of which can be expressed as functions of the maximum, the minimum and the maximum, or the maximum of sums of components. We show that independence characterize known uniform models (discrete and continuous, univariate and bivariate). That is, we obtain characterizations of univariate and bivariate uniform distributions from suitable conditions of symmetry and independence.

Keywords: Uniform distribution, mixtures, exchangeability, symmetry.

1 Introduction

Characterizations of multivariate spherical distributions have been developed by several authors. The book of Fang et al. (1990) contains a comprehensive review. For example, normality has been characterized within the class of multivariate spherical distributions. In fact (see Fang et al. (1990)) sphericity and independence imply normality. Other conditions as sufficiency and independence provide characterizations of known distributions in the exponential family (see for example Kagan et al. (1973)). Uniformity on an interval has been characterized in the direction of de Finetti-style theorems. See for example Rachev and Ruschendorf (1991), Iglesias, Pereira and Tanaka (1996), Gnedin (1995) and Fortini, Ladelli and Regazzini (2000).

In this work we present characterizations of known uniform distributions (univariate and bivariate) within the class of distributions satisfying certain symmetry conditions (sphericity in the appropriate norm).

In section 2, we deal with the discrete cases. We consider the class of multivariate distributions the probability density of which depends on the maximum, the minimum and the maximum, and on the maximum of sums of coordinates. We prove that the additional assumption of independence characterize the discrete uniform models. Section 3 is devoted to continuous cases.

The following notation will be used throughout the paper. For a non-empty set A, we will denote by A^N the N-fold product of A and by A_+ the subset $\{a \in A : a > 0\}$. Also,

$\mathcal{P}(A)$ will denote the set of parts of A. For the set $\{a_1, a_2, \ldots, a_n\}$, $a_{(1)}$ and $a_{(n)}$ will denote respectively its minimum and maximum. Finally, we will represent Borel σ-field on $\mathbb{R}^N$ by $\mathcal{B}(\mathbb{R}^N)$, the Lebesgue measure on $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$ by λ_N, $N \in \mathbb{N}$, and the Lebesgue measure on $((\mathbb{R}^k)^N, \mathcal{B}((\mathbb{R}^k)^N))$ by $\lambda_k^{(N)}$.

2 Discrete uniform distributions

In this section , we will present characterization theorems for uniform discrete independent and identically distributed random variables and bivariate random vectors. First, we present the results relative to the uniform distribution over $\{0, \ldots .\theta\}$, for some $\theta \in \mathbb{N}$, and second to the uniform distribution over $\{\theta_1, \theta_1 + 1, \ldots, \theta_2\}$, for $\theta_1, \theta_2 \in \mathbb{Z}$, with $\theta_1 < \theta_2$. Posteriorly, we present results for uniform i.i.d. random vectors uniform on the discrete isosceles triangle $\{(a,b) \in \mathbb{N}^2 : a + b \leq \theta\}$, $\theta \in \mathbb{N}$.

2.1 The Univariate case

Let us consider the following subset of $\mathbb{N}^N$, for $N \geq 2$ and $m \in \mathbb{N}$

$$\chi_m^N = \left\{(x_1, \ldots, x_N) \in \mathbb{N}^N : x_{(N)} = m\right\}.$$

The random vector $(X_1, \ldots, X_N)$ is said to be uniformly distributed over χ_m^N (Iglesias, Tanaka and Pereira (1998)), with probability law denoted by Q_{Nm}, if

$$P(X_1 = x_1, \ldots, X_N = x_N) = Q_{Nm}((x_1, \ldots, x_N)) = \frac{I_{\{m\}}(x_{(N)})}{(m+1)^N - m^N}, \tag{1}$$

where the denominator in the right-hand side of the above equality is simply the cardinality of χ_m^N.

Iglesias, Pereira and Tanaka, (1998) proved that if $(X_1, \ldots, X_N)$ is uniformly distributed over χ_m^N, then the n-dimensional marginal $(X_1, \ldots, X_N), n < N$, has law Q_{Nm}^n given by

$$Q_{Nm}^n(x_1, \ldots, x_n) = \begin{cases} \frac{1}{(m+1)^n} \left\{ \frac{1-(\frac{m}{m+1})^{N-n}}{1-(\frac{m}{m+1})^N} \right\}, & \text{if } x_{(N)} < m \\ \\ \frac{1}{(m+1)^n} \left\{ \frac{1}{1-(\frac{m}{m+1})^N} \right\}, & \text{if } x_{(N)} = m. \end{cases}$$

This last result and a little manipulation of the total variation distance properties provided the basic results (Iglesias (1993)) for obtaining both finite and infinite forms of de Finetti's type theorem for the discrete uniform model (the latter being derived from the former by fixing n and taking limits when $N \to \infty$).

The first aim here is to give a characterization of probability measures which are mixtures, in the usual sense, of the elements of the family $\{Q_{Nm} : m \in \mathbb{N}\}$. We then consider the class $\mathcal{C}_N$ of probability measures on $(\mathbb{N}^N, \mathcal{P}(\mathbb{N}^N))$ obtained by mixing the elements of $\{Q_{Nm} : m \in \mathbb{N}\}$ in m, denominated the radial variable.

Proposition 2.1 *A probability measure P belongs to $\mathcal{C}_N$ if, and only if, for each $(x_1, \ldots, x_N) \in \mathbb{N}^N$,*

$$P(\{(x_1, \ldots, x_N)\}) = \varphi_N(x_{(N)}) \tag{2}$$

for an appropriate non-negative function φ_N.

Proof: From definition we have that if P belongs to $\mathcal{C}_N$, then for $(x_1, \ldots x_N) \in \mathbb{N}^N$,

$$P(\{(x_1, \ldots, x_N)\}) = \sum_{m=1}^{\infty} \frac{I_{\{m\}}(x_{(N)})}{(m+1)^N - m^N} . \mu(\{m\}) \ ,$$

where μ is a probability measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. Consequently, by considering the non-negative function $\varphi : \mathbb{N} \to \mathbb{R}_+$ defined by

$$\varphi(t) = \sum_{m=1}^{\infty} \frac{I_{\{m\}}(t)}{(m+1)^N - m^N} . \mu(\{m\}), t \in \mathbb{N},$$

we conclude the first part of the proof.

On other hand, let us suppose that for each $(x_1, \ldots, x_N) \in \mathbb{N}^N$,

$$P(\{(x_1, \ldots, x_N)\}) = \varphi(x_{(N)})$$

Then, by simple calculations, we have that

$$P(X_{(N)} = m) = \varphi(m) |\chi_m^N| \ ,$$

where $|A|$ denotes the cardinality of A. From this we have that

$$P(X_1 = x_1, \ldots, X_N =, x_N | X_{(N)} = m) = \frac{I_{\{m\}}(x_{(N)})}{|\chi_m^N|} = Q_{Nm}(x_1, \ldots, x_N)$$

and consequently $P \in \mathcal{C}_N$. ■

In the following we show that the unique product measure in $\mathcal{C}_N$ is the distribution of N independent and identically distributed random variables uniform on $\{0, \ldots, \theta\}$, for some $\theta \in \mathbb{N}$.

Theorem 2.1 *If $X_1, X_2, \ldots, X_N$ are independent random variables taking values in $\mathbb{N}$ such that*

$$P(X_1 = x_1, \ldots, X_N = x_N) = \varphi_N(x_{(N)})$$

for an appropriate non-negative function φ_N, then X_i is uniformly distributed on $\{0, \ldots, \theta\}$ for some $\theta \in \mathbb{N}$, i=1,...,N.

Proof: If $X_1, \ldots, X_N$ are independent random variables, then

$$Pr(X_1 = x_1, \ldots, X_N = x_N | X_{(N)} = m) = \frac{\Pi_{i=1}^N Pr(X_1 = x_i) I_{\{m\}}(x_{(N)})}{[Pr(X_1 \le m)]^N - [Pr(X_1 \le m-1)]^N}. \quad (3)$$

But, by hypotheses, $X_1, \ldots, X_N$, given $X_{(N)} = m$, is uniformly distributed over χ_m^N, yielding

$$\frac{\Pi_{i=1}^N Pr(X_1 = x_i) I_{\{m\}}(x_{(N)})}{[Pr(X_1 \le m)]^N - [Pr(X_1 \le m-1)]^N} = \frac{I_{\{m\}}(x_{(N)})}{(m+1)^N - m^N}.$$

Thus, by setting $N = 2$ and taking $x_1 = x_2 = m$, it follows that

$$\frac{[Pr(X_1 = m)]^2}{[Pr(X_1 \le m)]^2 - [Pr(X_1 \le m-1)]^2} = \frac{1}{(m+1)^2 - m^2},$$

implying that

$$\frac{Pr(X_1 = m)}{Pr(X_1 \le m)} = \frac{1}{m+1}.$$

Evaluating the above equality for $m = 1, 2, \ldots$, we have that

$$P(X_1 = 0) = P(X_1 = 1) = P(X_1 = 2) = \ldots = P(X_1 = x), \quad x \in \mathbb{N}$$

This implies that there is $\theta \in \mathbb{N}$ such that $P(X_1 > \theta) = 0$, completing the proof. ■

Let us now consider the uniform distribution on the following subset of $\mathbb{N}^N$, for $N \geq 3, m_1, m_2, \in \mathbb{Z}$, with $m_1 < m_2$

$$\chi^N_{m_1,m_2} = \{(x_1, \ldots, x_N) \in \mathbb{Z}^N : x_{(1)} = m_1,\ x_{(N)} = m_2\}$$

The random vector $(X_1, \ldots, X_N)$ is said to be uniformly distributed over $\chi^N_{m_1 m_2}$, with probability law denoted by $Q_N^{m_1,m_2}$, if

$$P(X_1 = x_1, \ldots, X_N = x_N) = Q_N^{m_1,m_2}((x_1, \ldots, x_N)) =$$

$$= \frac{I_{\{m_1\}}(x_{(1)}) I_{\{m_2\}}(x_{(N)})}{(m_2 - m_1 + 1)^N - 2(m_2 - m_1)^N + (m_2 - m_1 - 1)^N}, \tag{4}$$

where the denominator in the right-hand side of the above equality corresponds to the cardinality of $\chi^N_{m_1 m_2}$.

In the following we present a characterization of probability measures which are mixtures of the elements of the family $\{Q_N^{m_1,m_2} : (m_1, m_2) \in \mathbb{Z}^2, m_1 < m_2\}$. We then consider the class $\mathcal{C}_N$ of probability measures on $(\mathbb{Z}^N, \mathcal{P}(\mathbb{Z}^N))$ obtained by mixing the elements of $\{Q_N^{m_1,m_2} : (m_1, m_2) \in \mathbb{Z}^2, m_1 < m_2\}$ in (m_1, m_2).

Proposition 2.2 *A probability measure P belongs to $\mathcal{C}_N$ if, and only if, for all $(x_1, \ldots, x_N) \in \mathbb{Z}^N$,*

$$P((x_1, \ldots, x_N)) = \varphi_N(x_{(1)}, x_{(N)}),$$

for an appropriate non-negative function φ_N.

Proof: As in the previous proposition, we can note that if P belongs to $\mathcal{C}_N$,, then, for $(x_1, \ldots, x_N) \in \mathbb{Z}^N$,

$$P(\{(x_1, \ldots, x_N)\}) = \sum_{(a,b) \in \mathbb{Z}^2} \varphi(x_{(1)}, x_{(N)}) \mu(\{(a, b)\}),$$

where μ is a probability measure on $(\mathbb{Z}^2, \mathcal{P}(\mathbb{Z}^2))$ and $\varphi : \mathbb{Z}^2 \to \mathbf{R}_+$ is defined by

$$\varphi(u,v) = \sum_{(a,b)\in\mathbb{Z}^2} \frac{I_{\{(a,b)\}}(u,v)}{(b-a+1)^N - 2(b-a)^N + (b-a-1)^N}\mu(\{(a,b)\}), (u,v)\in\mathbb{Z}^2,$$

concluding the first part of the proof.

The reciprocal follows in a similar way to proposition 2.1. ∎

Theorem 2.2 *If $X_1, X_2, \ldots, X_N$ are independent random variables taking values in $\mathbb{Z}$ such that*

$$P(X_1 = x_1, \ldots, X_N = x_N) = \varphi_N(x_{(1)}, x_{(N)}) ,$$

for an appropriate non-negative function φ_N, then X_i is uniformly distributed on $\{\theta_1, \theta_1 + 1, \ldots, \theta_2\}$, for some $\theta_1, \theta_2 \in \mathbb{Z}$, $\theta_1 < \theta_2$, i=1,...,N.

Proof: If $X_1, \ldots, X_N$ are independent and identically distributed random variables, we have that

$$P(X_1 = x_1, \ldots, X_N = x_N | X_{(1)} = m_1, X_{(N)} = m_2) =$$

$$= \frac{\Pi_{i=1}^N P(X_1 = x_i) I_{\{(m_1,m_2)\}}(x_{(1)}, x_{(N)})}{DP(m_1, m_2) - DP(m_1, m_2 - 1)} ,$$

where $DP(a,b) = [P(a \leq X_1 \leq b)]^N - [P(a+1 \leq X_1 \leq b)]^N$. But, conditionally on $X_{(1)} = m_1$ and $X_{(N)} = m_2$, $(X_1, \ldots, X_N)$ is uniformly distributed over $\chi^N_{m_1,m_2}$. Then

$$\frac{\Pi_{i=1}^N P(X_1 = x_i) I_{\{(m_1,m_2)\}}(x_{(1)}, x_{(N)})}{DP(m_1, m_2) - DP(m_1, m_2 - 1)} = \frac{I_{\{(m_1,m_2)\}}(x_{(1)}, x_{(N)})}{(m_2 - m_1 + 1)^N - 2(m_2 - m_1)^N + (m_2 - m_1 - 1)^N}.$$

Evaluating the above expression for $N = 3$ with $x_1 = m_1$, $x_2 = m_2$ and $x_3 = x$, $x \in \{a, \ldots, b\}$, we obtain, after some calculations, that

$$P(X_1 = x) = c > 0 \text{ for } x \in \{m_1, \ldots, m_2\}$$

As m_1, m_2 are arbitrary, $\exists\ \theta_1, \theta_2 \in \mathbb{Z}$, $\theta_1 < \theta_2$, such that

$$P(X_1 = x) = \frac{I_{\{\theta_1, \theta_1+1, \ldots, \theta_2\}}(x)}{\theta_2 - \theta_1 + 1} ,$$

that is, X_1 is uniformly distributed on $\{\theta_1, \theta_1+1, \ldots, \theta_2\}$, for some $\theta_1, \theta_2 \in \mathbf{Z}$, with $\theta_1 < \theta_2$. The reciprocal follows by computing the conditional probability. ■

2.2 The Bivariate Case

Let us consider then the following subset of $(\mathbb{N}^2)^N$, for $N \geq 2$ and $m \in \mathbb{N}$

$$\chi_m^N = \{((x_1, y_1), \ldots, (x_N, y_N)) \in (\mathbb{N}^2)^N : (x+y)_{(N)} = m\},$$

where $(x+y)_{(N)} = \max_{1\leq i\leq N}\{x_i+y_i\}$. Then $((X_1,Y_1),\ldots,(X_N,Y_N))$ is uniformly distributed over χ_m^N, with probability law denoted by Q_{Nm}, if

$$P(\{(X_1,Y_1)=(x_1,y_1),\ldots,(X_N,Y_N)=(x_N,y_N)\}) = Q_{Nm}(((x_1,y_1),\ldots(x_N,y_N))) =$$

$$= \frac{I_{\{m\}}((x+y)_{(N)})}{\binom{m+2}{2}^N - \binom{m+1}{2}^N},$$

where the denominator in the right-hand side of the above equality is the cardinality of χ_m^N.

Esteves et al. (2001) verified that if $((X_1,Y_1),\ldots,(X_N,Y_N))$ is uniformly distributed on χ_m^N, then the n-dimensional marginal $((X_1,Y_1),\ldots,(X_n,Y_n))$, $n < N$, has law Q_{Nm}^n given by

$$Q_{Nm}^n(\{((x_1,y_1),\ldots,(x_n,y_n))\}) = \begin{cases} \dfrac{\binom{m+2}{2}^{N-n} - \binom{m+1}{2}^{N-n}}{\binom{m+2}{2}^N - \binom{m+1}{2}^N}, & \text{if}(x+y)_{(n)} < m \\[2ex] \dfrac{\binom{m+2}{2}^{N-n}}{\binom{m+2}{2}^N - \binom{m+1}{2}^N}, & \text{if}(x+y)_{(n)} = m \end{cases}$$

The last result, together with properties of the total distance, provided the basic inequalities (Esteves, Wechsler and Iglesias (2001)) to the obtainment of both finite and infinite version of the Finetti's type theorem for the discrete uniform model on the isosceles triangle $\{(x,y) \in \mathbb{N}^2 : x+y \leq \theta\}, \theta \in \mathbb{N}$ (the later being derived from the former fixing the value of n and taking limits when $N \to \infty$).

As before, we present in the sequel a characterization of probability measures which are mixtures of the elements of the family $\{Q_{Nm} : m \in \mathbb{N}\}$. We then consider the class $\mathcal{C}_N$ of probability measures on $((\mathbb{N}^2)^N, \mathcal{P}((\mathbb{N}^2)^N))$ obtained by mixing the elements of $\{Q_{Nm} : m \in \mathbb{N}\}$ in the radial variable.

Proposition 2.3 *A probability measure P belongs to C_N if, and only if, for all $((x_1, y_1), \ldots, (x_N, y_N)) \in (\mathbb{N}^2)^N$,*

$$P(\{((x_1, y_1), \ldots, (x_N, y_N))\}) = \varphi_N((x + y)_{(N)}) \ , \tag{5}$$

for an appropriate non-negative function φ_N.

Proof: Similar to proposition 2.1. ■

Theorem 2.3 *If $(X_1, Y_1), (X_2, Y_2), \ldots, (X_N, Y_N)$ are independent random vectors taking values in $\mathbb{N}^2$ such that*

$$P((X_1, Y_1) = (x_1, y_1), \ldots, (X_N, Y_N) = (x_N, y_N)) = \varphi_N((x + y)_{(N)}) \ ,$$

for an appropriate function φ_N, then (X_i, Y_i) is uniformly distributed on $\{(a, b) \in \mathbb{N}^2 : a + b \leq \theta\}$, for some $\theta \in \mathbb{N}$, i=1,...,N.

Proof: By definition and considering the independence of the bivariate random vectors, we have

$$P((X_1, Y_1) = (x_1, y_1), \ldots, (X_N, Y_N) = (x_N, y_N) | (X + Y)_{(N)} = m) =$$

$$\frac{\Pi_{i=1}^N Pr((X_1, Y_1) = (x_i, y_i)) I_{\{m\}}((x + y)_{(N)})}{[Pr(X_1 + Y_1 \leq m)]^N - [Pr(X_1 + Y_1 \leq m - 1)]^N}.$$

But, conditional on $(X + Y)_{(N)} = m$, the joint distribution of $((X_1, Y_1), \ldots, (X_N, Y_N))$ is defined by the law Q_{Nm}. Thus,

$$\frac{\Pi_{i=1}^N Pr((X_1, Y_1) = (x_i, y_i)) I_{\{m\}}((x + y)_{(N)})}{[Pr(X_1 + Y_1 \leq m)]^N - [Pr(X_1 + Y_1 \leq m - 1)]^N} = \frac{I_{\{m\}}((x + y)_{(N)})}{\binom{m+2}{2}^N - \binom{m+1}{2}^N}.$$

Taking $N = 1$ and $x_1, y_1 \in \mathbb{N}$ such that $x_1 + y_1 = m$, we have from the previous equality that

$$\frac{P(X_1 = x_1, Y_1 = y_1)}{P(X_1 + Y_1 = m)} = \frac{1}{m + 1}$$

Consequently, we need only obtain the distribution of $X_1 + Y_1$ to determine the law of (X_1, Y_1). Taking now $N = 2$ and $x_1, y_1 \in \mathbb{N}$ such that $x_1 + y_1 = m$, we obtain that

$$\frac{[P((X_1,Y_1)=(x_1,y_1))]^2}{P(X_1+Y_1=m)[P(X_1+Y_1\le m)+P(X_1+Y_1\le m-1)]}=\frac{1}{(m+1)^2}.$$

Using now the fact that $P((X_1,Y_1)=(x_1,y_1))=\frac{P(X_1+Y_1=m)}{m+1}$, we obtain in the above equality that

$$\frac{P(X_1+Y_1\le m)-P(X_1+Y_1\le m-1)}{P(X_1+Y_1\le m)+P(X_1+Y_1\le m-1)}=\frac{1}{m+1}$$

Denoting $P(X_1+Y_1\le m)$ by $f(m)$, it follows that

$$f(m)=\frac{m+2}{m}f(m-1)\ ,$$

implying that

$$f(m)=\binom{m+2}{2}f(0),\quad m\ge 1.$$

Then,

$$P(X_1+Y_1=m)=\binom{m+2}{2}f(0)-\binom{m+1}{2}f(0)=\binom{m+1}{1}f(0).$$

Since $f(0)=P(X_1+Y_1\le 0)=P(X_1+Y_1=0)$, as X_1+Y_1 is a non-negative random variable, it follows that

$$P(X_1+Y_1=m)=(m+1)P(X_1+Y_1=0),$$

From the last equality, we conclude that the probability mass function of X_1+Y_1 is increasing. Therefore, there is $\theta\in\mathbb{N}$ such that for each $m>\theta$, $P(X_1+Y_1=m)=0$. Thus

$$P(X_1+Y_1=m)=\frac{m+1}{\binom{\theta+2}{2}},\quad m\in\{0.\dots,\theta\}.$$

Consequently

$$P((X_1,Y_1)=(x_1,y_1))\ =\ \frac{1}{\binom{\theta+2}{2}},\quad x_1+y_1\in\{0,\dots,\theta\},$$

that is, (X_1, Y_1) is uniformly distributed over $\{(a,b) \in \mathbb{N}^2 : a + b \leq \theta\}$, $\theta \in \mathbb{N}$.

The reciprocal may be obtained calculating conditional probabilities. ■

3 Continuous uniform distributions

In this section, we consider continuous versions of the theorems presented in the previous section.

3.1 The Univariate Case

Here, we consider first absolutely continuous random variables $(X_1, \ldots, X_N)$ with joint probability density function given by

$$f(x_1, \ldots, x_N) = \psi_N \left(\max_{1 \leq i \leq N} \{|X_i|\} \right) , \tag{6}$$

for a non-negative function ψ_N such that

$$\int_0^\infty u^{N-1} \psi_N(u) \mathrm{d}u = \frac{1}{N 2^N} . \tag{7}$$

Using the language of Fang et al. (1990), we call $\psi_N(\cdot)$ the density function generator of the l_∞-spherical distribution. The variable $R = \max_{1 \leq i \leq N}\{|X_i|\}$ is called radial variable. Moreover (see Iglesias et al. (1998)), if $g(\cdot)$ is the density function of R, then

$$g(r) = N 2^N r^{N-1} \psi_N(r) \mathbb{I}_{(0,\infty)}(r) . \tag{8}$$

Note that condition (6) implies that the density of the border of the N-dimensional hipercube centered at the origin is constant and the quantity in the right-hand side of the equality (7) corresponds to the $(N-1)$-dimensional volume of this hipercube with radius 1.

The next result provides a characterization of independent random variables with joint density function of the form (6).

Theorem 3.1 *If $X_1, \ldots, X_N$, $N \geq 2$, are independent random variables with joint density function given by (6), then X_i is uniformly distributed on $(-\theta, \theta)$, for some $\theta > 0$, i=1,...,N.*

Proof: Let $T = \max_{1 \le i \le N}\{|X_i|\}$. Then

$$f_T(t) = N 2^N t^{N-1} \psi_N(t) \mathbb{I}_{(0,\infty)}(t) .$$

On the other hand, if $X_1, \ldots, X_N$ are independent and identically distributed (by exchangeability), then

$$f_T(t) = N[G(t) - G(-t)]^{N-1}[g(t) + g(-t)] ,$$

where $g(G)$ is the probability density function (distribution function) of X_1. Thus, for each $t > 0$

$$\psi_N(t) 2N(2t)^{N-1} = N[G(t) - G(-t)]^{N-1}[g(t) + g(-t)] .$$

As g is symmetric, we obtain that

$$[\psi_N(t)]^N 2N(2t)^{N-1} \; = \; N[G(t) - G(-t)]^{N-1} 2g(t) .$$

But $\psi_N(t) = f(t, \ldots, t) = [g(t)]^N$, which implies that

$$tg(t) \; = \; G(t) - G(0) ,$$

for each $t > 0$ such that $g(t) > 0$. Now, solving the differential equation

$$tG'(t) = G(t) - \frac{1}{2}$$

and considering the fact that $G(t) \to 1$ as $t \to \infty$, we have that there is $\theta > 0$ such that $P(X_1 > \theta) = 0$. Consequently

$$G(t) = \begin{cases} 0, & \text{if } t < -\theta \\ \frac{1}{2} + \frac{t}{2\theta}, & \text{if } -\theta \le t < \theta \\ 1, & \text{if } t \ge \theta. \end{cases}$$

for some $\theta > 0$, concluding the proof. ■

In the following, we consider the uniform distribution depending on two parameters, θ_1, θ_2, $\theta_1 < \theta_2$.

Theorem 3.2 *If $X_1, \ldots, X_N$, $N \geq 3$, are independent random variables taking values in $\mathbb{R}$ with joint density function given by*

$$f(x_1, \ldots, x_N) = \psi_N\left(x_{(1)}, x_{(N)}\right), \tag{9}$$

for some appropriate non-negative function ψ_N such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{y} \psi_N(x,y)N(N-1)(y-x)^{N-2}\mathrm{d}x\mathrm{d}y = 1 ,$$

then X_i is uniformly distributed on (θ_1, θ_2), for some $\theta_1, \theta_2 \in \mathbb{R}$, with $\theta_1 < \theta_2$, i=1,...,N.

Proof: If $X_1, \ldots, X_N$ are random variables satisfying (9), then standard computations lead to the joint density function of $(X_{(1)}, X_{(N)})$, given by

$$h(x,y) = \psi_N(x,y)N(N-1)(y-x)^{N-2} \quad , \quad x \leq y \; . \tag{10}$$

On the other hand, as $X_1, \ldots, X_N$ are independent and identically distributed, we have that

$$h(x,y) = N(N-1)[G(y) - G(x)]^{N-2}g(y)g(x) , \tag{11}$$

where $g(G)$ is the probability density function (distribution function) of X_1. Thus

$$\psi_N(x,y)(y-x)^{N-2} = [G(y) - G(x)]^{N-2}g(y)g(x).$$

Now, for each $t \in [x,y]$ we have that

$$\psi_N(x,y) = f(x,t,\ldots,t,y) = g(x)g(y)[g(t)]^{N-2}.$$

From this we obtain

$$g(t) = \frac{G(y) - G(x)}{y - x} ,$$

for each $t \in [x,y]$ and x, y such that $g(y), g(x) > 0$. The proof is completed from the last equality. ■

Note from (9) that the generator function ψ_N satisfies (Gnedin (1995))

$$\int_{-\infty}^{\infty}\int_{-\infty}^{y} \psi_N(x,y)(y-x)^{N-2}\mathrm{d}x\mathrm{d}y = \frac{1}{N(N-1)} .$$

In the next subsection we extend the previous results to the bivariate case.

3.2 The Bivariate Case

Here we consider bivariate random vectors $(X_1,Y_1),\ldots,(X_N,Y_N)$, $N \geq 2$, taking values in $\mathbb{R}^2_+$, with joint probability density function f satisfying

$$f((x_1,y_1),\ldots,(x_N,y_N)) = \psi_N(\max_{1\leq i\leq N}\{x_i+y_i\}), \tag{12}$$

for some appropriate non-negative function ψ_N on $\mathbb{R}_+$. Computing the total probability integral, we see that such ψ_N must satisfy

$$\int_0^{\infty} u^{2N-2}\psi_N(u)\mathrm{d}u = \frac{2^{N-1}}{N}. \tag{13}$$

Moreover, the density function of $R = \max_{1\leq i\leq N}\{X_i+Y_i\}$ is given by

$$f_R(r) = \psi_N(r)\frac{N}{2^{N-1}}r^{2N-2}\mathbb{I}_{\mathbb{R}_+}(r). \tag{14}$$

Theorem 3.3 *If $(X_1,Y_1),\ldots,(X_N,Y_N)$, $N \geq 2$, are independent absolutely continuous random vectors taking values in $\mathbb{R}^2_+$ with joint probability density function f given by (12), then (X_i,Y_i) is uniformly distributed on $\{(x,y)\in\mathbb{R}^2_+ : x+y\leq\theta\}$, for some $\theta>0$, i=1,...,N.*

Proof: First, we should note that $(X_1,Y_1),\ldots,(X_N,Y_N)$ are exchangeable and that if g is the density function of (X_1,Y_1), then

$$g(a,t-a) = g(0,t) = h(t) ,$$

for each $a\in[0,t]$ and each $t>0$. On the other hand, $(X_1,Y_1),\ldots,(X_N,Y_N)$ are independent, which implies that

$$f_R(t) = N[P(X_1+Y_1\leq t)]^{N-1}f_{X_1+Y_1}(t)$$

and

$$f_{X_1+Y_1}(t) \;=\; \int_{-\infty}^{\infty} g(a,t-a)\mathrm{d}a \;=\; \int_0^t g(a,t-a)\mathrm{d}a \;=\; th(t) \;. \tag{15}$$

Thus, from (14) we have that

$$\frac{N}{2^{N-1}}\psi_N(t)t^{2N-2} \;=\; N[P(X_1+Y_1\le t)]^{N-1}th(t).$$

But, from the hypotheses we have that

$$\psi_N(t) = (g(0,t))^N = [h(t)]^N$$

This yields

$$[P(X_1+Y_1\le t)]^{N-1} \;=\; \frac{[h(t)]^{N-1}t^{2N-2}}{2^{N-1}} \;,$$

for all $t>0$ such that $h(t)>0$. Then,

$$P(X_1+Y_1\le t) \;=\; \frac{h(t)t^2}{2}$$

and, consequently,

$$f_{X_1+Y_1}(t) \;=\; th(t) \;+\; \frac{t^2}{2}h'(t).$$

The last equality and (15) yield

$$\frac{t^2}{2}h'(t) = 0 \;,$$

for each $t>0$ such that $h(t)>0$. That is, $h(t)=c$, for some $c>0$, for each $t>0$ such that $h(t)>0$, implying that

$$P(X_1+Y_1\le t) \;=\; \frac{ct^2}{2}.$$

From this we have that there is $\theta>0$ such that $P(X_1+Y_1\ge\theta)=0$. Noting now that $g(x,y)=h(x+y)=c$, we conclude the proof. ∎

4 Conclusions

In this work. we present characterizations of uniform distributions, discrete and continuous, univariate and bivariate, within classes of multivariate distributions satisfying specific symmetry conditions. In the discrete case, specification of multivariate probability distributions depending only on the maximum, the minimum and the maximum, and on the maximum of sums of components and the additional assumption of independence provide a characterization of uniform models on $\{0,\ldots,\theta\}$, $\theta \in \mathbb{N}$, on $\{\theta_1,\ldots,\theta_2\}$, $\theta_1,\theta_2 \in \mathbb{Z}$, $\theta_1 < \theta_2$, and on $\{(x,y) \in \mathbb{N}^2 : x+y \leq \theta\}$, $\theta \in \mathbb{N}$, respectively. In the continuous case, the consideration of similar conditions on the probability density functions of multivariate symmetric distrubtions in addition to independence characterize the uniform models on $(-\theta,\theta)$, $\theta > 0$, on (θ_1,θ_2), $\theta_1,\theta_2 \in \mathbb{R}$, $\theta_1 < \theta_2$, and on $\{(x,y) \in \mathbb{R}_+^2 : x+y \leq \theta\}$, $\theta > 0$. However, some questions on characterizations of uniform models have not been considered in this paper and, no doubt, may be investigated futurely. Among others, we mention the possibility (or not) of characterizing random vectors $\mathbf{X}$ with distributions belonging to the classes of multivariate symmetric distributions considered here through a stochastic representation of the form $\mathbf{X} = R\mathbf{U}$, for some non-negative random variabel R independent of a random vector $\mathbf{U}$ uniformly distributed over a specific region of $\mathbb{R}^N$ $((\mathbb{R}^2)^N)$ (in the case of n-variate spherical distributions, $n \in \mathbb{N}$, it is well-known that a characterization in this fashion exists, with $\mathbf{U}$ being uniformly distributed over the surface of the n-dimensional sphere of radius 1).

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