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Partial Actions and Galois Theory

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Abstract

In this article, among other results, we develop a Galois theory of commutative rings under partial actions of finite groups, extending the well-known results by S. U. Chase, D. K. Harrison and A. Rosenberg.

Introduction

In the celebrated paper by S. U. Chase, D. K. Harrison and A. Rosenberg [3] the authors developed a Galois theory for commutative ring extensions $S \supset R$, under the assumptions that S is separable over R, finitely generated and projective as an R-module, and the elements of the Galois group G are pairwise strongly distinct R-automorphisms of S. In particular, Theorem 1.3 of that paper gives several equivalent conditions for the definition of a Galois extension and Theorem 2.3 states a one-to-one correspondence between the subgroups of G and the R-subalgebras of S which are separable and G-strong.

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On the other hand, partial actions of groups have been introduced in the theory of operator algebras giving powerful tools of their study (see, in particular, [6], [7], [10], [15] and [17]). A related concept, that of a partial representation of a group on a Hilbert space, has been defined independently by R. Exel [7], and J. C. Quigg and I. Raeburn [17]. Several relevant classes of C^*-algebras, were deeply investigated in [8], [9], [10] from the point of view of partial actions and partial representations of groups, including the Cuntz-Krieger algebras introduced in [4].

Given a partial action of a group on an object it is natural to ask whether it is a restriction of a global action defined on a bigger object. Such global action is called a globalization or an enveloping action, provided that certain minimality condition is satisfied which guarantees its uniqueness. Globalizations of partial actions where first considered by F. Abadie in his PhD Thesis of 1999 (see also [1]).

Partial actions in a pure algebraic context were first studied in [5]. A partial action α of a group G on a unital algebra S is a collection of ideals S_σ together with isomorphisms $\alpha_\sigma : S_{\sigma^{-1}} \to S_\sigma$, $\sigma \in G$, which satisfy some additional conditions of compatibility with the group. From the categorical point of view it seems to be reasonable supposing that the S_σ's and S are objects of the same category, i.e., each S_σ is a unital algebra. This idea is confirmed when dealing with globalizations: a partial action on a unital algebra possesses an enveloping action (which is necessarily unique) if and only if every S_σ is an algebra with identity element [5]. That this situation is natural in one more sense follows from the results of this article: assuming this condition, a complete generalization of the results on Galois Theory of commutative rings by Chase-Harrison-Rosenberg [3] can be obtained in the context of partial actions.

Recently R. Exel wrote a preprint on Hecke algebras, which can be found on his homepage [11]. Among a number of results he proves that given a Hecke pair (G, L) such that the normalizer of L in G is normal in G, the corresponding Hecke algebra is isomorphic to a crossed product by a twisted partial action. It is known that an H-extension of algebras $R \subset S$, where H is a Hopf algebra, is Galois with normal basis property if and only if S is a crossed product of R by a Hopf algebra ([16], Corollary 8.2.5). This also suggests that there may be applications of the ideas in this paper to Hecke algebras and a theory of partial Hopf Galois extensions.

The purpose of this paper is to introduce the notion of a partial Galois extension and to develop a Galois theory for a commutative ring extension

$S \supset R$, when G is a group acting partially on S by R-linear maps. Some of our results are proved using similar ideas of [3], but it is necessary to check carefully many details which come from the fact that the action of the Galois group is partial instead of being global.

We shall deal with a partial action α of G on S which has an enveloping action, i.e., there exist a ring S' and a global action of G on S' such that S is an ideal of S' and the restriction of the global action to the ideals S_σ gives the partial action α [5]. Roughly speaking, a partial Galois extension can be considered as a direct summand of a Galois extension. If T is a (non-necessarily commutative) ring which is a (global) Galois extension of B with Galois group G and e is a central idempotent of T, then G acts partially on $S = Te$. Then we define the invariant subring S^α of S under α and the extension $S \supset S^\alpha$ is called a partial Galois extension. It follows from our results that any partial Galois extension is of this type.

In the first sections of the paper rings are not necessarily commutative. Section 1 is a preliminary section. In Section 2 we define the trace map of a partial action of a finite group on an algebra and fixed subrings and we obtain some relations. Partial Galois extensions are defined in Section 3. The main result of this section proves that an algebra S with a partial action α is a partial Galois extension of its fixed subring if and only if the enveloping action is a (global) Galois extension of its fixed subring.

In Section 4 we consider partial Galois extensions of fixed subrings which are contained in the center of the extension. We prove a theorem giving several equivalent conditions for S to be a partial Galois extension of a central subring R, extending to our case Theorem 1.3 of [3]. Section 5 is devoted to prove the Fundamental Galois Theorem for partial actions of commutative rings, giving an extension of Theorem 2.3 of [3]. Finally, in Section 6 we include some examples and additional remarks.

Throughout the paper rings are always with identity element and algebras are assumed to be unital and associative algebras. Unadorned $\otimes$ means $\otimes_R$.

1. Prerequisites

We first recall the notion of a partial action of a group on an algebra [5].

Let G be a group and S a unital algebra over a commutative ring k. A *partial action* α of G on S is a collection of ideals S_σ, $\sigma \in G$, of S and isomorphisms of (non-necessarily unital) k-algebras $\alpha_\sigma : S_{\sigma^{-1}} \to S_\sigma$ such that:

(i) $S_1 = S$ and α_1 is the identity automorphism of S;

(ii) $S_{(\sigma\tau)^{-1}} \supseteq \alpha_\tau^{-1}(S_\tau \cap S_{\sigma^{-1}})$;

(iii) $\alpha_\sigma \circ \alpha_\tau(x) = \alpha_{\sigma\tau}(x)$, for every $x \in \alpha_\tau^{-1}(S_\tau \cap S_{\sigma^{-1}})$ and $\sigma, \tau \in G$.

In case we have a partial action of G on S as above we simply say that α is a partial action of G on S, where the ideals associated with the action will be denoted by S_σ, unless otherwise stated.

We recall some facts which are already known (see [5]). The property (ii) of the definition easily implies that $\alpha_\sigma(S_{\sigma^{-1}} \cap S_\tau) = S_\sigma \cap S_{\sigma\tau}$, for all $\sigma, \tau \in G$. Also $\alpha_\sigma^{-1} = \alpha_{\sigma^{-1}}$, for every $\sigma \in G$.

In this paper, unless otherwise stated, we assume that any S_σ is generated by a central idempotent $1_\sigma \neq 0$, i.e., S_σ is an k-algebra with identity. It is clear that in this case $S_\sigma \cap S_\tau = 1_\sigma 1_\tau S$. In the particular case $S_\sigma = S$, for all $\sigma \in G$, we have a usual (global) action of the group G on the k-algebra S.

Assume that α is a partial action of G on S. By Theorem 4.5 of [5], α possesses an *enveloping action*, which means that there exist a ring S' and a (global) action of G by automorphisms of S' such that S can be considered as an ideal of S' generated by a central idempotent 1_S of S' and the following properties hold:

(i) the subalgebra of S' generated by $\bigcup_{\sigma\in G} \sigma(S)$ coincides with S' and we have $S' = \sum_{\sigma\in G} \sigma(S)$;

(ii) $S_\sigma = S \bigcap \sigma(S)$, for every $\sigma \in G$;

(iii) $\alpha_\sigma(x) = \sigma(x)$, for all $\sigma \in G$ and $x \in S_{\sigma^{-1}}$.

Note that the authors in [5] allow some S_σ to be equal zero, i.e., they consider the zero algebra as a ring with identity element. As we said above we assume here that all the ideals S_σ are non-zero. Hereafter we will denote by (S', G) the enveloping action of α.

Note that $S'1_\sigma = S1_\sigma = S \bigcap \sigma(S) = S'1_S \bigcap S'\sigma(1_S) = S'1_S\sigma(1_S)$, and this implies that $1_\sigma = 1_S\sigma(1_S)$.

Assume that α is a partial action having an enveloping action. Then for any $x \in S$ we have $\alpha_\sigma(x1_{\sigma^{-1}}) = \sigma(x1_S\sigma^{-1}(1_S)) = \sigma(x)1_S$, where $\sigma(x) \in S'$. We give an example as how to use this relation: we have $\alpha_\sigma(1_\tau 1_{\sigma^{-1}}) = \sigma(1_\tau)1_S = \sigma(1_S)\sigma(\tau(1_S))1_S = 1_\sigma 1_{\sigma\tau}$.

All the above remarks will be used freely throughout the sequel.

Natural examples of partial actions can be easily given:

Example 1.1 Assume that T is an algebra over k and G acts on T by k-automorphisms, and let S be an ideal of T. For any $\sigma \in G$ put $S_\sigma = S \cap \sigma(S)$. Define $\alpha_\sigma : S_{\sigma^{-1}} \to S_\sigma$ as the restriction of the automorphism σ to $S_{\sigma^{-1}}$. Then it is easy to see that α is a partial action of G on S, called the restriction of the global action of G on T to S. In addition, if S is generated by a central idempotent of T we have that any S_σ has an identity element.

2. The trace map and fixed subrings

Throughout the rest of the paper G is a finite group. Assume that α is a partial action of G on a k-algebra S, where k is a commutative ring. The subring of invariants of S under α is defined as

$$S^\alpha = \{x \in S \,|\, \alpha_\sigma(x 1_{\sigma^{-1}}) = 1_\sigma x, \text{ for all } \sigma \in G\}.$$

Note that $x \in S^\alpha$ is equivalent to $\alpha_\sigma(xa) = x\alpha_\sigma(a)$, for every $a \in S_{\sigma^{-1}}$, $\sigma \in G$. Denote by (S', G) the enveloping action of α, $R' = S'^G$ and $R = S^\alpha$.

Since $S' = \sum_{\sigma \in G} \sigma(S)$, it is easy to see that the identity element of S' can be written as a boolean sum of the idempotents $\sigma(1_S)$, $\sigma \in G$, of S'. Denote the elements of G by $\{\sigma_1 = 1, \sigma_2, \ldots, \sigma_n\}$. Thus we can write $1_{S'}$ as an orthogonal sum $1_{S'} = e_1 \oplus e_2 \oplus \cdots \oplus e_n$, where $e_1 = 1_S$, $e_2 = (1_{S'} - 1_S)\sigma_2(1_S)$ and $e_j = (1_{S'} - 1_S) \ldots (1_{S'} - \sigma_{j-1}(1_S))\sigma_j(1_S)$, for $2 \leq j \leq n$. It is clear that

$$1_{S'} = \sum_{1 \leq l \leq n} \sum_{i_1 < \cdots < i_l} (-1)^{l+1} \sigma_{i_1}(1_S) \ldots \sigma_{i_l}(1_S).$$

We will use a map defined as follows: for any $x \in S'$ put

$$\psi(x) = \sum_{1 \leq l \leq n} \sum_{i_1 < \cdots < i_l} (-1)^{l+1} \sigma_{i_l}(x) \sigma_{i_1}(1_S) \ldots \sigma_{i_l}(1_S).$$

It is clear that ψ is a (right and left) R'-linear map and we can write $\psi(x) = \sum\limits_{1 \leq i \leq n} \sigma_i(x) e_i$. By the above $\psi(1_S) = 1_{S'}$.

The trace map plays an important role when having actions of finite groups on algebras. In our case we define it by $\mathrm{tr}_{S/R}(x) = \sum_{\sigma \in G} \alpha_\sigma(x 1_{\sigma^{-1}})$, for any $x \in S$. The usual trace map from S' to R' will be denoted by $\mathrm{tr}_{S'/R'}$. We have

Lemma 2.1

(i) $tr_{S/R} : S \to R$ *is a (left and right) R-linear map.*

(ii) $tr_{S/R}(x) = tr_{S'/R'}(x)1_S$, *for any* $x \in S$.

(iii) $tr_{S'/R'}(S') = tr_{S'/R'}(S)$.

Proof. (i) If $x \in S$ and $\tau \in G$,

$$\alpha_\tau(\mathrm{tr}_{S/R}(x)1_{\tau^{-1}}) = \sum_{\sigma\in G}\tau(\sigma(x)1_S)1_S = \sum_{\sigma\in G}\tau(\sigma(x))\tau(1_S)1_S = \mathrm{tr}_{S/R}(x)1_\tau.$$

Hence $\mathrm{tr}_{S/R}(x) \in R$.

(ii) For any $x \in S$ we have

$$\mathrm{tr}_{S/R}(x) = \sum_{\sigma\in G}\alpha_\sigma(x1_{\sigma^{-1}}) = \sum_{\sigma\in G}\sigma(x)1_S = \mathrm{tr}_{S'/R'}(x)1_S.$$

(iii) Assume that $y \in \mathrm{tr}_{S'/R'}(S')$ and take $x \in S'$ with $\mathrm{tr}_{S'/R'}(x) = y$. We can write $x = \sum_{\sigma\in G}\sigma(x_\sigma)$ with $x_\sigma \in S$, since $S' = \sum_{\sigma\in G}\sigma(S)$. Thus we have

$$\begin{aligned} y &= \sum_{\rho\in G}\rho(x) = \sum_{\sigma\in G}\Big(\sum_{\rho\in G}\rho\sigma(x_\sigma)\Big) \\ &= \sum_{\sigma\in G}\mathrm{tr}_{S'/R'}(x_\sigma) \in \mathrm{tr}_{S'/R'}(S). \end{aligned}$$

Thus $\mathrm{tr}_{S'/R'}(S') \subseteq \mathrm{tr}_{S'/R'}(S)$ and so (iii) clearly follows. □

Corollary 2.2 *Under the same assumptions as above* $tr_{S'/R'}$ *is onto* R' *if and only if* $tr_{S/R}$ *is onto* R.

Proof If there is $c \in S'$ with $\mathrm{tr}_{S'/R'}(c) = 1_{S'}$, then there exist an element $d \in S$ such that $\mathrm{tr}_{S/R}(d) = 1_S$, by Lemma 2.1. Conversely, assume that there exists $c \in S$ with $\mathrm{tr}_{S/R}(c) = 1_S$. Thus $\mathrm{tr}_{S'/R'}(c)1_S = 1_S$ and we have $1_{S'} = \psi(\mathrm{tr}_{S'/R'}(c)1_S) = \mathrm{tr}_{S'/R'}(c)\psi(1_S) = \mathrm{tr}_{S'/R'}(c)1_{S'} = \mathrm{tr}_{S'/R'}(c)$. The result follows from Lemma 2.1(i). □

Proposition 2.3 *Assume that* R, R', S, S', G *and* α *are as above and that* $tr_{S'/R'}$ *is onto. Then the restriction of the map* ψ *to* R *is a ring isomorphism from* R *onto* R' *whose inverse is the mapping* ϕ *sending* r' *to* $r'1_S$, *for any* $r' \in R'$.

Proof It is clear that ψ is a homomorphism of rings. If $r \in R$ there exists $c \in S$ such that $\mathrm{tr}_{S/R}(c) = r$, by Corollary 2.2. Thus $\mathrm{tr}_{S'/R'}(c)1_S = r$ and hence $\psi(r) = \psi(\mathrm{tr}_{S'/R'}(c)1_S) = \mathrm{tr}_{S'/R'}(c)\psi(1_S) = \mathrm{tr}_{S'/R'}(c) \in R'$. Therefore $\psi|_R : R \to R'$. Also $\psi(r)1_S = \mathrm{tr}_{S'/R'}(c)1_S = r$. Finally $\psi(r'1_S) = r'\psi(1_S) = r'$, for every $r' \in R'$. □

3. Partial Galois Extensions

The definition of Galois extensions used in [3] was used later on also for non-commutative rings in many papers. In this section we do not need the commutativity of the rings and so we will use the same definition of Galois extensions as given in ([3], Theorem 1.3) for arbitrary ring extensions.

Assume that T is a ring, B is a subring of T and G is a group acting on T by B-automorphisms. Recall that T is said to be a Galois extension of B with group G if B is equal to the fixed subring T^G of T under the action of G and there exist x_i, y_i in T, $1 \leq i \leq n$, such that $\sum_{1\leq i\leq n} x_i\sigma(y_i) = \delta_{1,\sigma}$, for every $\sigma \in G$. The elements x_i, y_i are called a Galois coordinate system of T.

Now assume that α is a partial action of a group G on a k-algebra S. The above definition induces the following:

Definition 3.1 S is said to be a *partial Galois extension of R with partial action α* (an α-partial Galois extension, for short) if $S^\alpha = R$ and there exist elements $x_i, y_i \in S$, $1 \leq i \leq n$, such that $\sum_{1\leq i\leq n} x_i\alpha_\sigma(y_i 1_{\sigma^{-1}}) = \delta_{1,\sigma}$, for each $\sigma \in G$.

As in the global case we say that the elements x_i, y_i are partial Galois coordinates of S over R. The following proposition shows that examples of partial Galois extensions appear naturally.

Proposition 3.2 *Assume that T is a Galois extension of a ring B with Galois group G, $S = Te$ is the ideal of T generated by a central idempotent e and suppose that $S \cap \sigma(S) \neq 0$, for any $\sigma \in G$. Then S is an α-partial Galois extension of S^α, where α is the partial action induced on S by the global action of G on T.*

Proof. If x_i, y_i are Galois coordinates of T over B, then it can easily be checked that ex_i, ey_i are partial Galois coordinates of S over S^α. □

The main purpose of this section is to show that any partial Galois extension can be obtained in the way given in Proposition 3.2. For this we will use the enveloping action of α which we denote as above by (S', G). We prove the following main result:

Theorem 3.3 *Assume that α is a partial action of G on S and (S', G) is its enveloping action, and as above $R = S^{\alpha}$ and $S'^G = R'$. Then the following statements are equivalent:*

(i) *S' is a (global) Galois extension of R' with Galois group G.*

(ii) *S is an α-partial Galois extension of R.*

Proof. (i)$\Rightarrow$(ii) holds by Proposition 3.2. Conversely, assume that (ii) holds. Then there exist $x_i, y_i \in S$, $1 \le i \le m$, such that $\sum_{1\le i\le m} x_i\alpha_\sigma(y_i 1_{\sigma^{-1}}) = \delta_{1,\sigma}$, for every $\sigma \in G$. Using the same notation as in the former section, consider in S' the elements $x'_{ij} = \sigma_j(x_i)e_j$ and $y'_{ij} = \sigma_j(y_i)e_j$, $1 \le i \le m$, $1 \le j \le n$. Then

$$\begin{aligned}\sum_{i,j} x'_{ij}y'_{ij} &= \sum_{i,j}\sigma_j(x_i)\sigma_j(y_i)e_j = \sum_j \sigma_j(\sum_i x_i y_i)e_j\\ &= \sum_j \sigma_j(1_S)e_j = \psi(1_S) = 1_{S'}.\end{aligned}$$

Also, putting $g_j = (1_{S'} - 1_S)\dots(1_{S'} - \sigma_{j-1}(1_S))$ and $\sigma_{jl} = \sigma_j^{-1}\sigma_l\sigma_j$ we have

$$\begin{aligned}\sum_{i,j} x'_{ij}\sigma_l(y'_{ij}) &= \sum_{i,j}\sigma_j(x_i)\sigma_l(\sigma_j(y_i))e_j\sigma_l(e_j)\\ &= \sum_j \sigma_j(\sum_i x_i\sigma_{jl}(y_i)1_S)g_j\sigma_l(e_j)\\ &= \sum_j \sigma_j(\sum_i x_i\alpha_{\sigma_{jl}}(y_i 1_{\sigma_{jl}^{-1}}))g_j\sigma_l(e_j) = 0,\end{aligned}$$

for $2 \le l \le n$, and the proof is complete. □

Remark 3.4 Under the same conditions as above, if S is a commutative ring and an α-partial Galois extension of R, then by a well-known result in commutative Galois theory, Theorem 3.3 and Corollary 2.2, it follows immediately that there exists $c \in S$ such that $\mathrm{tr}_{S/R}(c) = 1_S$.

Remark 3.5 Some authors who have worked on Galois theory of non-commutative rings used the definition of Galois extensions as a ring extension $A \supset B$ such that there exist Galois coordinates in A and furthermore $\mathrm{tr}_{A/B}(A) = B$ (see, for example [14], p. 310). We can use also the corresponding to this definition for partial actions in non-commutative rings. So a partial Galois extension in this sense would be a ring extension $S \supset R$ together with a partial action of G on R such that the conditions of Definition 3.1 are satisfied and $\mathrm{tr}_{S/R}$ is onto R. We can conclude from Corollary 2.2 and Theorem 3.3 that also in this case S is a partial Galois extension of R if and only if S' is a Galois extension of R'.

4. Galois Extensions of Commutative Rings.

In this section we consider partial Galois extensions whose fixed subrings are contained in the center. The main purpose here is to prove a result corresponding to Theorem 1.3 of [3]. The proof follows the lines of the proof of that theorem and so we will omit many details.

Throughout the section we assume that S is a ring, R is a subring of the center of S and α is a partial action of G on the R-algebra S. The skew group ring $S\star_\alpha G$ is defined as the set of all formal sums $\sum_{\sigma\in G} x_\sigma u_\sigma$, $x_\sigma \in S_\sigma$, with the usual addition and the multiplication determined by $(x_\sigma u_\sigma)(y_\tau u_\tau) = \alpha_\sigma(\alpha_{\sigma^{-1}}(x_\sigma)y_\tau)u_{\sigma\tau}$. Since each S_σ is an algebra with identity, then $S\star_\alpha G$ is an associative R-algebra ([5], Corollary 3.2).

A natural map $j : S\star_\alpha G \to \mathrm{End}_R(S)$ is defined by $j(\sum_{\sigma\in G} x_\sigma u_\sigma)(z) = \sum_{\sigma\in G} x_\sigma \alpha_\sigma(z1_{\sigma^{-1}})$, for every $z \in S$. It can easily be seen that j is a homomorphism of left S-modules and R-algebras.

Let M be a left $S\star_\alpha G$-module. We put

$$M^G = \{m \in M : (1_\sigma u_\sigma)m = 1_\sigma m, \text{for all } \sigma \in G\},$$

the R-submodule of invariants of M under G. Note that M is a left S-module via the embedding $x \mapsto xu_1$ from S into $S\star_\alpha G$.

The algebra S can be considered as a left $S\star_\alpha G$-module via j, that is, $(x_\sigma u_\sigma)y = j(x_\sigma u_\sigma)(y)$, for all $y \in S$ and $\sigma \in G$. Then the subring of invariants as defined above is $S^G = \{x \in S \,|\, \alpha_\sigma(x1_{\sigma^{-1}}) = x1_\sigma, \text{ for all}\, \sigma \in G\}$ which coincides with the definition of S^α given in Section 2.

Now we give the sketch of the proof of the following result, corresponding to Theorem 1.3 of [3].

Theorem 4.1 *Let α be a partial action of a (finite) group G on an R-algebra S. Then the following statements are equivalent:*

(i) *S is an α-partial Galois extension of R.*

(ii) S is a finitely generated projective R-module and $j : S \star_\alpha G \to \mathrm{End}_R(S)$ is an isomorphism of left S-modules and R-algebras.

(iii) S is a finitely generated projective R-module and for every left $S \star_\alpha G$-module M the map $\mu : S \otimes M^G \to M$, given by $\mu(x \otimes m) = xm$, is an isomorphism of left S-modules.

(iv) S is a finitely generated projective R-module and the map $\psi : S \otimes S \to \prod_{\sigma \in G} S_\sigma$, defined by $\psi(x \otimes y) = (x\alpha_\sigma(y1_{\sigma^{-1}}))_{\sigma \in G}$, is an isomorphism of left S-modules.

Proof. (i)$\Rightarrow$(ii) Take $x_i, y_i \in S$, $1 \le i \le n$, with $\sum_{1 \le i \le n} x_i \alpha_\sigma(y_i 1_{\sigma^{-1}}) = \delta_{1,\sigma}$ and define $f_i \in \mathrm{Hom}_R(S, R)$ by $f_i(x) = \mathrm{tr}_{S/R}(y_i x)$, for all $x \in S$. It follows easily that $\sum\limits_{1 \le i \le n} x_i f_i(x) = x$, for all $x \in S$. Hence S is a finitely generated projective R-module.

To show that j is an isomorphism, for $h \in \mathrm{End}_R(S)$ take $w \in S \star_\alpha G$ given by $w = \sum_{\sigma \in G} \sum_{1 \le i \le n} h(x_i) \alpha_\sigma(y_i 1_{\sigma^{-1}}) u_\sigma$. We can see that $j(w)(x) = h(x)$, for any $x \in S$. Thus j is surjective. Finally, suppose that $v = \sum_{\sigma \in G} x_\sigma u_\sigma \in \mathrm{Ker}(j)$. Then $j(v)(x_i) = 0$, for all $1 \le i \le n$. An easy computation gives $0 = \sum_{\tau \in G} \sum_{1 \le i \le n} j(v)(x_i) \alpha_\tau(y_i 1_{\tau^{-1}}) u_\tau = v$.

(ii)$\Rightarrow$(iii) Since S is a finitely generated projective R-module there exist $x_i \in S$ and $f_i \in \mathrm{Hom}_R(S, R)$, $1 \le i \le l$, such that $x = \sum_{1 \le i \le l} f_i(x) x_i$, for all $x \in S$. We define $\nu : M \to S \otimes M^G$ by $\nu(m) = \sum_{1 \le i \le l} x_i \otimes v_i m$, where $v_i = j^{-1}(f_i) \in S \star_\alpha G$. It follows that ν is a well-defined homomorphism which is the inverse of μ.

(iii)$\Rightarrow$(iv) Put $\mathfrak{F} = \{f : G \to S \mid f(\sigma) \in S_\sigma, \text{ for all } \sigma \in G\}$. Then it is clear that $\mathfrak{F}$ is isomorphic to $\prod_{\sigma \in G} S_\sigma$ as left S-modules. Also, $\mathfrak{F}$ has a structure of a left $S \star_\alpha G$-module given by $((x_\sigma u_\sigma) f)(\tau) = x_\sigma \alpha_\sigma(f(\sigma^{-1}\tau) 1_{\sigma^{-1}})$, for every $f \in \mathfrak{F}$ and $\sigma, \tau \in G$. It follows from the assumption (iii) that the map $\mu : S \otimes \mathfrak{F}^G \to \mathfrak{F} \simeq \prod_{\sigma \in G} S_\sigma$ defined by $\mu(x \otimes f) = (x f(\sigma))_{\sigma \in G}$ is an isomorphism of left S-modules.

Also the map $S \to \mathfrak{F}^G$, $x \mapsto f_x$, where $f_x : G \to S$ is defined by $f_x(\tau) = \alpha_\tau(x 1_{\tau^{-1}})$, $\tau \in G$, is an isomorphism of left R-modules. Consequently the composition $S \otimes S \to S \otimes \mathfrak{F}^G \to \prod_{\sigma \in G} S_\sigma$ is an isomorphism of left S-modules, which is clearly equal to ψ.

(iv)$\Rightarrow$(i) Take $x \in S^\alpha$. It follows from (iv) that $x \otimes 1 = 1 \otimes x$. On the other hand, since S is a faithfully projective R-module, R is a direct summand of S. Hence $x \in R$ and so $S^\alpha = R$ follows.

To obtain the partial Galois coordinates take $(1,0,\ldots,0) \in \prod_{\sigma\in G} S_\sigma$, whose first entry corresponds to $\sigma = 1$. Since ψ is an isomorphism, there exists $w = \sum_{1\le i\le n} x_i \otimes y_i \in S \otimes S$ such that $\psi(w) = (1,0,\ldots,0)$ and (i) follows. The proof is complete. □

We say that two elements σ and τ of G are *strongly distinct*, with respect to the partial action α of G on S (α-strongly distinct, for short), if for any non-zero idempotent $e \in S_\sigma \cup S_\tau$ there exists $x \in S$ such that $\alpha_\sigma(x1_{\sigma^{-1}})e \neq \alpha_\tau(x1_{\tau^{-1}})e$

Theorem 4.2 *Let α be a partial action of a group G on an R-algebra S. If S is an α-partial Galois extension of R, then S is separable over R and the elements of G are pairwise α-strongly distinct. If, in addition, S is commutative and $S^\alpha = R$, the converse also holds.*

Proof. Since S is a direct summand of its enveloping algebra S' and S' is a Galois extension of $R' = S'^G$, by Theorem 3.3, we have that S is R'-separable. It follows easily that S is separable over $R \simeq R'$.

Now take $\sigma, \tau \in G$ and suppose that $e \in S_\sigma \cup S_\tau$ is a non-zero idempotent. If $\alpha_\sigma(x1_{\sigma^{-1}})e = \alpha_\tau(x1_{\tau^{-1}})e$, $x \in S$, we have $\sigma(x)1_S e = \tau(x)1_S e$ and so $x\sigma^{-1}(e) = \sigma^{-1}\tau(x)\sigma^{-1}(e)$ in S'. Using Galois coordinates x_i, y_i, $1 \le i \le n$, of S' over R' and the last relation for $x = y_i$, for every i, we easily obtain $\sigma = \tau$. So the elements of G are pairwise α-strongly distinct.

Conversely, assume that S is a commutative separable algebra over $R = S^\alpha$ and the elements of G are pairwise α-strongly distinct. For $\sigma \in G$ consider the homomorphism of S-algebras (S acting on the left) $\theta_\sigma : S \otimes S \to S \otimes S_\sigma$ defined by $\theta_\sigma(x \otimes y) = x \otimes \alpha_\sigma(y1_{\sigma^{-1}})$. Denote by $e = \sum_{1\le i\le n} x_i \otimes y_i \in S \otimes S$ the idempotent of the separability of S over R and by $\mu : S \otimes S \to S$ the multiplication map. It is easy to verify that for any $\sigma \in G$, the element $e_\sigma = \mu(\theta_\sigma(e)) = \sum x_i\alpha_\sigma(y_i1_{\sigma^{-1}}) \in S_\sigma$ is an idempotent. Also, $xe_\sigma = \alpha_\sigma(x1_{\sigma^{-1}})e_\sigma$, for any $x \in S$. Since the elements of G are pairwise α-strongly distinct, for $\sigma \neq 1$ we obtain $e_\sigma = 0$ and so $\sum_{1\le i\le n} x_i\alpha_\sigma(y_i1_{\sigma^{-1}}) = 0$. The proof is complete. □

Remark 4.3 Note that if S is an α-partial Galois extension of a central subring R, (S', G) is the enveloping action and $R' = S'^G$ we can consider $S \otimes_R S^0$ as contained in $S' \otimes_{R'} S'^0$. In fact, it is easy to see that $R' \simeq R$ is a subring of the center of S' and the remark follows since S is a direct summand of S'.

Corollary 4.4 *Let S be an α-partial Galois extension of R. If $1_G = \prod_{\sigma\in G} 1_\sigma \neq 0$, then $S1_G$ is a (global) Galois extension of $R1_G$ with Galois group G.*

Proof. It is easy to see that $\alpha_\sigma|_{S1_G} \in \mathrm{Aut}_{R1_G}(S1_G)$, for all $\sigma \in G$. Also $(S1_G)^G = S^\alpha 1_G = R1_G$. Finally if $x_i, y_i \in S$, $1 \leq i \leq n$, are the partial Galois coordinates of S over R, then $x_i 1_G, y_i 1_G$, $1 \leq i \leq n$, are Galois coordinates of $S1_G$ over $R1_G$. □

Corollary 4.5 *Let S be an α-partial Galois extension of R and T be a commutative R-algebra. Then $T \otimes S$ is a $1 \otimes \alpha$-partial Galois extension of $T \simeq T \otimes R$, where the partial action of G on $T \otimes S$ is given by the maps $1 \otimes \alpha_\sigma : T \otimes S_{\sigma^{-1}} \to T \otimes S_\sigma$, for any $\sigma \in G$.*

Proof. It is easy to see that $T\otimes S_\sigma$, $\sigma \in G$, are non-zero ideals of $T\otimes S$ and that $(T\otimes S_\sigma, 1\otimes\alpha_\sigma)$ defines a partial action on $T\otimes S$. Also T can be identified with $T \otimes R \subseteq T \otimes S$. Furthermore, a Galois coordinate system for S over R easily gives a Galois coordinate system for $T \otimes S$. Finally, $T \otimes R = (T \otimes S)^\alpha$ follows as in the global case using the partial trace map. □

To finish this section we prove the following

Corollary 4.6 *Let S be an α-partial Galois extension of R. Then, for any prime ideal $\wp$ of R, $rank_{R_\wp}(S_\wp) \leq |G|$. Moreover, $rank_{R_\wp}(S_\wp) = |G|$, for any $\wp$, if and only if S is a (global) Galois extension of R with group G.*

Proof. By Corollary 3.4 we can assume that R is a local ring. So each S_σ is a finitely generated free R-module. Since $S \otimes S \simeq \prod_{\sigma\in G} S_\sigma$, we have

$$(rank_R(S))^2 = rank_R(S \otimes S) = \sum_{\sigma\in G} rank_R(S_\sigma) \leq |G| rank_R(S)$$

and the first part follows.

For the second assertion assume that $rank_R(S) = |G|$. Then $rank_R(S_\sigma) = rank_R(S)$, for every $\sigma \in G$, and since S_σ is a direct summand of S it follows that $S = S_\sigma$. □

5. The Galois Correspondence

In this section we assume that S is a commutative partial Galois extension of R with partial action α of G on S. For a subalgebra T of S we set $H_T = \{\sigma \in G \,|\, \alpha_\sigma(x1_{\sigma^{-1}}) = x1_\sigma, \text{ for all } x \in T\}$. We say that T is α-strong if for every $\sigma, \tau \in G$, with $\sigma^{-1}\tau \notin H_T$, and any non-zero idempotent $e \in S_\sigma \cup S_\tau$ there exists an element $t \in T$ such that $\alpha_\sigma(t1_{\sigma^{-1}})e \neq \alpha_\tau(t1_{\tau^{-1}})e$. If the action of G on S is a global action, then we have the well-known notion of a G-strong subalgebra [3].

We will see in Example 6.3 that the set H_T is not always a subgroup of G, even when T is R-separable and α-strong. The fundamental Theorem of Galois theory we will prove here is the following result which extends Theorem 2.3 of [3].

Theorem 5.1 *Let S be a partial Galois extension of R with partial action α of G on S. Then there is a one-to-one correspondence between the subgroups of G and the separable subalgebras T of S which are α-strong and such that H_T is a subgroup of G.*

We begin with the following

Theorem 5.2 *Let S be an α-partial Galois extension of R and H a subgroup of G. Then $\alpha_H = \{\alpha_\sigma : S_{\sigma^{-1}} \to S_\sigma \,|\, \sigma \in H\}$ is a partial action of H on S and S is an α_H-partial Galois extension of $T = S^{\alpha_H}$. Also T is R-separable and α-strong and $H_T = H$.*

Proof. Obviously the restriction α_H of α to H is a partial action of H on S and the first part follows directly from the definition.

Denote by (S', G) the enveloping action of α and put $S'^G = R'$. Since $S' = \sum_{\sigma \in G} \sigma(S)$ it is easy to see that there exists a global action of H on $\widetilde{S} = \sum_{\sigma \in H} \sigma(S)$ which is the enveloping action of α_H.

The global action of G on S' induces a partial action β of G on $\widetilde{S}$ for which it is also enveloping. Then Proposition 2.3 implies that $R'1_{\widetilde{S}} = (\widetilde{S})^\beta$ and $R'1_S = R$. Consequently $(\widetilde{S})^\beta 1_S = R'1_{\widetilde{S}}1_S = R'1_S = R$. Also, $(\widetilde{S})^H 1_S = S^{\alpha_H}$.

On the other hand, $S' = S'1_{\widetilde{S}} \oplus S'(1_{S'} - 1_{\widetilde{S}}) = \widetilde{S} \oplus S'(1_{S'} - 1_{\widetilde{S}})$. Clearly, $S'(1_{S'} - 1_{\widetilde{S}})$ is H-invariant, hence $S'^H = (\widetilde{S})^H \oplus (S'(1_{S'} - 1_{\widetilde{S}}))^H$ and $S'^H 1_{\widetilde{S}} = (\widetilde{S})^H 1_{\widetilde{S}} = (\widetilde{S})^H$. Thus $T = S^{\alpha_H} = (\widetilde{S})^H 1_S = S'^H 1_{\widetilde{S}} 1_S = S'^H 1_S$.

By Theorem 3.3 S' is a Galois extension of R' with Galois group G. Hence by the results of [3] the R'-algebra S'^H is separable, G-strong and $H = H_{S'^H}$. It is easy to verify that if $e' \in S'^H \otimes_{R'} S'^H$ is the separability idempotent of S'^H over R', then $e = e'(1_S \otimes 1_S) \in S'^H 1_S \otimes_{R' 1_S} S'^H 1_S = S^{\alpha_H} \otimes S^{\alpha_H} = T \otimes T$ is the separability idempotent of T over R. Thus T is R-separable.

Now we prove that T is α-strong. Take any $\sigma, \tau \in G$ with $\sigma^{-1}\tau \notin H_T$ and a non-zero idempotent $e \in S_\sigma \bigcup S_\tau$. Since S'^H is G-strong and $\sigma^{-1}\tau \notin H_T \supseteq H = H_{S'^H}$, if $e1_\sigma 1_\tau \neq 0$ there exists $x \in S'^H$ such that $\sigma(x)e1_\sigma 1_\tau \neq \tau(x)e1_\sigma 1_\tau$. Consequently $x1_S \in T$ and $\alpha_\sigma(x1_S 1_{\sigma^{-1}})e1_\sigma 1_\tau = \alpha_\sigma(x1_{\sigma^{-1}})e1_\sigma 1_\tau = \sigma(x)e1_\sigma 1_\tau \neq \tau(x)e1_\sigma 1_\tau = \alpha_\tau(x1_S 1_{\tau^{-1}})e1_\sigma 1_\tau$, and therefore $\alpha_\sigma(x1_S 1_{\sigma^{-1}})e \neq \alpha_\tau(x1_S 1_{\tau^{-1}})e$. Finally, if $e1_\sigma 1_\tau = 0$ and $e \in S_\sigma$ we have $\alpha_\sigma(1_S 1_{\sigma^{-1}})e = 1_\sigma e = e \neq 0 = e1_\sigma 1_\tau = e1_\tau = \alpha_\tau(1_S 1_{\tau^{-1}})e$. The case $e \in S_\tau$ is similar.

Finally, clearly $H_T \supseteq H$. Conversely, assume that $\sigma \in H_T \backslash H$, $H = H_{S'^H}$. Since S'^H is G-strong it follows that there is $x \in S'^H$ with $\sigma(x)1_\sigma \neq x1_\sigma$ (recall that $1_\sigma \neq 0$). Hence $\alpha_\sigma(x1_S 1_{\sigma^{-1}}) = \sigma(x)1_\sigma \neq x1_\sigma = x1_S 1_\sigma$ which is a contradiction because $\sigma \in H_T$ and $x1_S \in S'^H 1_S = T$. Thus $H_T = H$ and the proof is complete. □

For the next result we need the following lemma. We omit its proof because it is similar to the last part of the proof of Theorem 4.2.

Lemma 5.3 *Let S be an α-partial Galois extension of R and T a separable and α-strong R-subalgebra of S. Then there exist $x_i, y_i \in T$, $1 \leq i \leq m$, such that $\sum_{1 \leq i \leq m} x_i y_i = 1$ and $\sum_{1 \leq i \leq m} x_i \alpha_\sigma(y_i 1_{\sigma^{-1}}) = 0$, for all $\sigma \in G \setminus H_T$.*

The next theorem completes the proof of Theorem 5.1.

Theorem 5.4 *Let S be an α-partial Galois extension of R and T be a separable and α-strong R-subalgebra of S such that H_T is a subgroup of G. Then $S^{\alpha_H} = T$ for $H = H_T$, where α_H is the partial action of H on S defined above.*

Proof. Clearly $T \subseteq S^{\alpha_H}$. Now we prove the converse inclusion. Let (S', G) be the enveloping action of α and $R' = S'^G$. As in the proof of Theorem 5.2 we consider the subalgebra $\widetilde{S} = \sum_{\sigma \in H} \sigma(S)$ on which H acts as a group of automorphisms and $(\widetilde{S}, H)$ is the enveloping action of the partial action $\alpha_H = \{\alpha_\sigma : S_{\sigma^{-1}} \to S_\sigma \,|\, \sigma \in H\}$ of H on S.

By Proposition 2.3 there exists a ring isomorphism $\psi_H : S^{\alpha_H} \to (\widetilde{S})^H$

such that $\psi_H(x)1_S = x$, for all $x \in S^{\alpha_H}$. Write $\widetilde{T} = \psi_H(T)$.

Claim 1 There are elements $\widetilde{x}_i, \widetilde{y}_i \in \widetilde{T}$, $1 \le i \le m$, such that $\sum_{1\le i\le m} \widetilde{x}_i\widetilde{y}_i = 1_{\widetilde{S}}$ and $\sum_{1\le i\le m} \widetilde{x}_i\sigma(\widetilde{y}_i) = 0$, for every $\sigma \in G \setminus H$.

Indeed, by Lemma 5.3 there exist elements $x_i, y_i \in T$, $1 \le i \le m$, such that $\sum_{1\le i\le m} x_iy_i = 1_S$ and $\sum_{1\le i\le m} x_i\alpha_\sigma(y_i1_{\sigma^{-1}}) = 0$, for all $\sigma \in G \setminus H$. Since $\sum_{1\le i\le m} x_i\sigma(y_i) \in S \cap \sigma(S) = S_\sigma$ we have that $\sum_{1\le i\le m} x_i\sigma(y_i) = \sum_{1\le i\le m} x_i\sigma(y_i)1_\sigma = \sum_{1\le i\le m} x_i\alpha_\sigma(y_i1_{\sigma^{-1}}) = 0$.

Write $\widetilde{x}_i = \psi_H(x_i)$ and $\widetilde{y}_i = \psi_H(y_i)$ in $\widetilde{T}$, $1 \le i \le m$, and denote by $\tau_1 = 1, \tau_2, \ldots, \tau_l$ the elements of H. As we saw in Section 2, for $x \in S^H$ we have $\psi_H(x) = \sum_{1\le i\le l} \tau_i(x)e_i$, where $e_1, \ldots, e_l \in \widetilde{S}$ are pairwise orthogonal idempotents. Therefore $\sum_{1\le i\le m} \widetilde{x}_i\widetilde{y}_i = \psi_H(\sum_{1\le i\le m} x_iy_i) = \psi_H(1_S) = 1_{\widetilde{S}}$ and for each $\sigma \in G \setminus H$ we have

$$\begin{aligned}\textstyle\sum_{1\le i\le m} \widetilde{x}_i\sigma(\widetilde{y}_i) &= \sum_{1\le i\le m}\sum_{1\le j,j'\le l} \tau_j(x_i)e_j\sigma(\tau_{j'}(y_i)e_{j'}) \\ &= \sum_{1\le j,j'\le l} e_j\sigma(e_{j'})\tau_j(\sum_{1\le i\le m} x_i\tau_j^{-1}\sigma\tau_{j'}(y_i)) = 0,\end{aligned}$$

which completes the proof of Claim 1.

Note that since $H \subseteq H_{\widetilde{T}}$ and the elements $\widetilde{x}_i, \widetilde{y}_i$ of Claim 1 are in $\widetilde{T}$, this claim implies, in particular, that $H_{\widetilde{T}} = H$.

As we pointed out in Theorem 5.2, the restriction of (S', G) to $\widetilde{S}$ gives a partial action β of G on $\widetilde{S}$ for which it is also enveloping. Then it follows from Proposition 2.3 that there is an isomorphism of rings $S'^G \to (\widetilde{S})^\beta$ sending x to $x1_{\widetilde{S}}$. Also, the map $x \mapsto x1_S$ is an isomorphism from S'^G onto R. Thus we have an isomorphism $(\widetilde{S})^\beta \to R$ defined by $y \mapsto y1_S$, for any $y \in (\widetilde{S})^\beta$, whose inverse is ψ_H restricted to R. Hence $\psi_H(R) = (\widetilde{S})^\beta$ and consequently $\widetilde{T} = \psi_H(T)$ is separable over $(\widetilde{S})^\beta$.

Note that $S'^H = S'^H1_{\widetilde{S}} \oplus S'^H(1_{S'} - 1_{\widetilde{S}}) = (S'1_{\widetilde{S}})^H \oplus S'^H(1_{S'} - 1_{\widetilde{S}}) = (\widetilde{S})^H \oplus S'^H(1_{S'} - 1_{\widetilde{S}})$. In particular $S'^H1_S = (\widetilde{S})^H1_S = S^{\alpha_H}$. Consider the subalgebra $T' = \widetilde{T} \oplus S'^H(1_{S'} - 1_{\widetilde{S}})$.

Claim 2 T' is separable over R' and G-strong.

In fact, S'^H is separable over R' and so $S'^H(1_{S'} - 1_{\widetilde{S}})$ is separable over $R'(1_{S'} - 1_{\widetilde{S}})$. Since $\widetilde{T}$ is $(\widetilde{S})^\beta$-separable and $(\widetilde{S})^\beta = S'^G1_{\widetilde{S}} = R'1_{\widetilde{S}}$, it follows that T' is separable over $(\widetilde{S})^{\beta} \oplus R'(1_{S'} - 1_{\widetilde{S}}) = R'$.

To prove that T' is G-strong assume that $\sigma \in G \setminus H$ and $e \in S'$ is an idempotent, and suppose that $\sigma(x+y)e = (x+y)e$, for all $x \in \widetilde{T}$ and $y \in S'^H(1_{S'} - 1_{\widetilde{S}})$.

Put $e = e_1 + e_2$ with $e_1 = e1_{\widetilde{S}}$ and $e_2 = e(1_{S'} - 1_{\widetilde{S}})$. Then multiplying $\sigma(x+y)e = (x+y)e$ by $1_{\widetilde{S}}$ we obtain $\sigma(x+y)e_1 = (x+y)e_1 = xe_1$, for all $x \in \widetilde{T}$ and $y \in S'^H(1_{S'} - 1_{\widetilde{S}})$. In particular, taking $y = 0$ we see that $\sigma(x)e_1 = xe_1$, for every $x \in \widetilde{T}$.

By Claim 1 there exist $\widetilde{x}_i, \widetilde{y}_i \in \widetilde{T}$, $1 \le i \le m$, with $\sum_{1\le i\le m} \widetilde{x}_i\widetilde{y}_i = 1_{\widetilde{S}}$ and $\sum_{1\le i\le m} \widetilde{x}_i\sigma(\widetilde{y}_i) = 0$. Hence $0 = \sum_{1\le i\le m} \widetilde{x}_i\sigma(\widetilde{y}_i)e_1 = \sum_{1\le i\le m} \widetilde{x}_i\widetilde{y}_ie_1 = 1_{\widetilde{S}}e_1 = e_1$.

Thus $e = e_2$ and $\sigma(x+y)e_2 = (x+y)e_2 = ye_2$, for all $x \in \widetilde{T}$ and $y \in S'^H(1_{S'} - 1_{\widetilde{S}})$. Taking $x = 0$ and any y we obtain $\sigma(y)e_2 = ye_2$. Since S'^H is G-strong and separable over R', by Lemma 5.3 there exist $u_j, v_j \in S'^H$, $1 \le j \le l$, with $\sum_{1\le j\le l} u_jv_j = 1_{S'}$ and $\sum_{1\le j\le l} u_j\sigma(v_j) = 0$. Consequently $0 = \sum_{1\le j\le l} u_j\sigma(v_j)\sigma(1_{S'} - 1_{\widetilde{S}})e_2 = \sum_{1\le j\le l} u_jv_j(1_{S'} - 1_{\widetilde{S}})e_2 = 1_{S'}(1_{S'} - 1_{\widetilde{S}})e_2 = e_2$, which completes the proof of Claim 2.

Now we are able to complete the proof of the theorem. By Claim 2 and the results in [3] $T' = S'^{H'}$ for $H' = \{\sigma \in G \mid \sigma(x) = x, \text{for all}\, x \in T'\}$. Also, by definition of T', an element $\sigma \in G$ is in $H_{T'} = H'$ if and only if $\sigma \in H_{\widetilde{T}} = H$ and thus $H' = H$. It follows that $T' = S'^H$ and $S^{\alpha_H} = (\widetilde{S})^H 1_S = S'^H 1_S = T'1_S = \widetilde{T}1_S = T$. The proof is complete. □

6. Examples and Remarks

In this section first we give some examples which illustrate our results.

Example 6.1 Let R be a commutative ring and put $S = Re_1 \oplus Re_2 \oplus Re_3$, where $\{e_1, e_2, e_3\}$ is a set of non-zero orthogonal idempotents whose sum is one. We denote by G the cyclic group of order 4 generated by σ, and define a partial action of G on S taking $S_1 = S$, $S_\sigma = Re_1 \oplus Re_2$, $S_{\sigma^2} = Re_1 \oplus Re_3$ and $S_{\sigma^3} = Re_2 \oplus Re_3$, and defining $\alpha_1 = id_S$,

$\alpha_\sigma : S_{\sigma^3} \to S_\sigma$ by $\alpha_\sigma(e_2) = e_1$ and $\alpha_\sigma(e_3) = e_2$,

$\alpha_{\sigma^2} : S_{\sigma^2} \to S_{\sigma^2}$ by $\alpha_{\sigma^2}(e_1) = e_3$ and $\alpha_{\sigma^2}(e_3) = e_1$, and

$\alpha_{\sigma^3} : S_\sigma \to S_{\sigma^3}$ by $\alpha_{\sigma^3}(e_1) = e_2$ and $\alpha_{\sigma^3}(e_2) = e_3$.

Then it is easy to verify that S is an α-partial Galois extension of R. In this case it is also clear that the enveloping action of α is the trivial extension $S' = S \oplus Re_4$ of R, where the global action is given by $\sigma^i(e_j) = e_{j-i(mod4)}$.

Example 6.2 Let R be a commutative ring and put $S = \sum_{1\leq i\leq 4} \oplus Re_i$, where $\{e_1, e_2, e_3, e_4\}$ is a set of non-zero orthogonal idempotents whose sum is one. Denote by G the cyclic group generated by σ of order 5. The mappings defined on the ideals $\{S, S_\sigma = Re_2, S_{\sigma^2} = Re_4, S_{\sigma^3} = Re_3, S_{\sigma^4} = Re_1\}$ by $\alpha_1 = id_S$, $\alpha_\sigma(e_1) = e_2$, $\alpha_{\sigma^2}(e_3) = e_4$, $\alpha_{\sigma^3}(e_4) = e_3$ and $\alpha_{\sigma^4}(e_2) = e_1$, give a partial action α of G on S.

It can easily be verified that $S^\alpha = R(e_1 + e_2) \oplus R(e_3 + e_4)$ and taking $x_i = y_i = e_i$, $1 \leq i \leq 4$, we have a Galois coordinate system for S over S^α. Thus S is an α-partial Galois extension of S^α. Also, it is not difficult to show that the enveloping action is given by $S' = S \oplus \sum_{1\leq j\leq 6} \oplus Rv_j$, where the set $\{v_j | 1 \leq j \leq 6\}$ is a set of orthogonal idempotents which are also orthogonal with the e_i's and such that $\sum_i e_i + \sum_j v_j = 1_{S'}$. The action of σ is given by $e_1 \to e_2 \to v_1 \to v_2 \to v_3 \to e_1$ and $e_3 \to v_4 \to e_4 \to v_5 \to v_6 \to e_3$. Here we have $S'^G = R(e_1 + e_2 + v_1 + v_2 + v_3) \oplus R(e_3 + e_4 + v_4 + v_5 + v_6)$.

Example 6.3 Let A be a cyclic (global) Galois extension of a commutative ring R with Galois group G generated by σ of order 6. Set $S = \sum_{1\leq i\leq 5} \oplus Ae_i$, where $\{e_i \,|\, 1 \leq i \leq 5\}$ is a set of non-zero orthogonal idempotents whose sum is one. Define the partial action α of G on S taking $A_{\sigma^i} = Ae_{6-i}$ and $\alpha_{\sigma^i}(ae_i) = \sigma^i(a)e_{6-i}$, $1 \leq i \leq 5$. Thus we have a partial action of G on S and $S^\alpha = \{ae_1 + be_2 + ce_3 + \sigma^2(b)e_4 + \sigma(a)e_5 \,|\, a, b \in A, c \in A^{\sigma^3}\}$.

Let $a_i, b_i \in A, 1 \leq i \leq m$, a Galois coordinate system for A over R and consider the elements $x_j = y_je_j, j = 1, 2, 4, 5$ together with the elements $x_{i3} = a_ie_3, y_{i3} = b_ie_3$. It is easy to see that this gives a Galois coordinate system for S over S^α. Hence S is an α-partial Galois extension of S^α.

We have two non-trivial separable α-strong subalgebras T of S with H_T a subgroup of G: $T_1 = \{x_1e_1 + x_2e_2 + x_3e_3 + \sigma^2(x_2)e_4 + x_5e_5 \,|\, x_i \in A\}$ and $T_2 = \{x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 \,|\, x_3 \in A^{\sigma^3}, x_i \in A \text{ for } i \neq 3\}$. Furthermore the subalgebra $T = \{x_1e_1 + x_2e_2 + x_3e_3 + \sigma^2(x_2)e_4 + \sigma(x_1)e_5 \,|\, x_i \in A\}$ is S^α-separable and α-strong but $H_T = \{id, \sigma, \sigma^2, \sigma^4, \sigma^5\}$ is not a subgroup of G.

We can apply the results to reduced rings and its rings of quotients. Assume that S is reduced and denote by Q the complete ring of quotients of

S. Recall that Q can be obtained considering the filter $\mathcal{F}$ of essential ideals of S and the set of all S-homomorphisms $f : H \rightarrow S$, where $H \in \mathcal{F}$. Two homomorphisms $f : H \rightarrow R$ and $f' : H' \rightarrow R$ are said to be equivalent if the restrictions of f and f' to $H \cap H'$ are equal. Then Q is defined as the set of all the equivalence classes of this homomorphisms with natural operations. Also, S can be considered as a subring of Q.

Let α be a partial action of the group G on S and we denote, as in the former sections, by $(S_\sigma)_{\sigma \in G}$ the ideals involved in the action. There is a one-to-one correspondence, via contraction, between the closed ideals of S and the closed ideals of Q. So for any $\sigma \in G$ there exists a closed ideal S^*_σ of Q such that $S^*_\sigma \cap S = [S_\sigma]$, where $[S_\sigma]$ denotes the closure of S_σ in S. Also, S^*_σ is the complete ring of quotients of the reduced ring (without identity) S_σ see ([12], Section 1).

Thus the isomorphisms $\alpha_\sigma : S_{\sigma^{-1}} \rightarrow S_\sigma$ can be extended to isomorphisms $\alpha^*_\sigma : S^*_{\sigma^{-1}} \rightarrow S^*_\sigma$, which we will denote by α_σ again, and it can easily be seen that this defines a partial action α of G on Q [13]. Note that this extended action satisfies the assumption we used in the former sections, i.e., the ideals S^*_σ have identity elements.

Let S be a reduced ring and α be a partial action of G on S. Denote by Q the complete ring of quotients of S and again by α the extended partial action of α to Q. As a consequence of Theorem 5.1 we obtain the following

Corollary 6.4 *Under the above notation, assume that there exist elements $x_i, y_i \in S$, $1 \leq i \leq n$, such that $b = \sum_{1 \leq i \leq n} x_i y_i$ is a non-zero divisor of S and $\sum_{1 \leq i \leq n} x_i \alpha_\sigma(y_i a) = 0$, for any $a \in S_{\sigma^{-1}}$ and $id \neq \sigma \in G$. Then Q is an α-partial Galois extension of Q^α. In particular, Theorem 5.1 can be applied to the extension $Q \supset Q^\alpha$.*

Proof. As b is a non-zero divisor, bS is an essential ideal of S and so b is invertible in Q. Also, since S_σ is essential in S^*_σ, for any $\sigma \in G$, it follows easily that the elements $b^{-1}x_i, y_i, 1 \leq i \leq n$, give a Galois coordinate system in Q. Consequently Q is an α-partial Galois extension of Q^α and the result follows. □

Remark 6.5 After a first version of this paper was finished Caenepeel and Groot [2] extended some of our results to the context of Galois corings. On the other hand, it can easily be seen that the algebra $S \star_\alpha G$ has a structure of a bialgebra which do not seem to be a Hopf algebra. This bialgebra acts naturally on S, as we have seen in Section 3. So we think that it is possible to extend the results also to the context of bialgebras acting on algebras.

References

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