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Anderson t-motives and abelian varieties with MIQF: Results coming from an analogy

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Analogy between Anderson t-motives and abelian varieties with multiplication by an imaginary quadratic field (MIQF) is a source of 2 results:

(1) A description of abelian varieties with MIQF of dimension r and signature $(n, r-n)$ in terms of "lattices" of dimension r in $\mathbb{C}^n$;

(2) A construction of exterior powers of abelian varieties with MIQF having $n = 1$.

Keywords: Anderson t-motives; abelian varieties with multiplication by an imaginary quadratic field; exterior power of abelian varieties.

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0. Introduction

The origin of this paper is an analogy (see Sec. 1.2) between the following two objects corresponding to the function field case and the number field case,

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respectively:

(A) An Anderson t-motive M of rank r and dimension n, pure, uniformizable, having the nilpotent operator N equal to 0 (see Sec. 1 for the exact definitions), and

(B) An abelian variety A over $\mathbb{C}$ with multiplication by an imaginary quadratic field K, of dimension r and of signature $(n, r-n)$.

We abbreviate this variety as an abelian variety with MIQF K. We do not require End $(A) = O_K$, but only End $(A) \otimes_{\mathbb{Z}} \mathbb{Q} = K$.

Using well-known constructions for Anderson t-motives, we get analogous constructions for abelian varieties with MIQF. First, let us consider the lattice $L(M)$ associated to a t-motive M of type (A) (notations: q is a power of a prime p, $\mathbb{F}_q$ the finite field of order q, θ a transcendental element, $\mathbb{C}_\infty \supset \mathbb{F}_q[\theta]$ the functional analog of $\mathbb{C}$, see details in Sec. 1):

(C) $L(M)$ is a $\mathbb{F}_q[\theta]$-lattice of rank r in $\mathbb{C}_\infty^n$.

Let us consider the lattice $L(A)$ associated to an abelian variety A with MIQF (notations of (B)):

(D) $L(A) \subset \mathbb{C}^r$; $L(A)$ is isomorphic to $\mathbb{Z}^{2r}$ and $L(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is K^r with respect to the action of K on $\mathbb{C}^r$ (this action is not an ordinary multiplication by elements of K, because of a nontrivial signature).

Objects of types (C) and (D) are not quite analogous, is not it? The first result of this paper is Theorem 2.3 which gives us — roughly speaking — that

(E) A defines an r-dimensional K-submodule (a "lattice") of $\mathbb{C}^n$ (not of $\mathbb{C}^r$!). Moreover, there is a 1–1 correspondence between A with MIQF K and these "lattices", together with some additional data.

(E) is an analog of (C).

Second, it is known that if M of type (A) has $n = 1$ then its k-th exterior power $\lambda^k(M)$ is also an object of type (A). By analogy, we can expect that if A is an abelian variety with MIQF having $n = 1$ then $\lambda^k(A)$ is defined, and is also an abelian variety with MIQF. We really give this construction (Sec. 3).

Both these results are of elementary nature, they could be known to Riemann.[a] Probably this analogy will be a source of more results. For example, it would be interesting to define analogs of t-motives having $N \neq 0$.

This paper is organized as follows. In Sec 1, we recall definitions of t-motives and we formulate the results which are starting points (using the analogy 1.2) of the results of Secs. 2 and 3. Section 2 contains the exact statement and the proof of (E). In Sec. 3, we apply this result to construct exterior powers of abelian varieties with MIQF having $n = 1$. Formally, these sections are independent of Sec. 1, i.e. they do not require any knowledge of the theory of Anderson t-motives.

[a]Claire Voisin told to the second author that she knew the construction of Sec. 3 (not published), but she did not know its relation with exterior power of Drinfeld modules.

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Remark 0.1. Many constructions/results of the theory of Anderson t-motives are analogs of the corresponding constructions/results of the theory of abelian varieties. The direction of analogies of this paper is the opposite: from Anderson t-motives to abelian varieties.

1. Origin of Construction: Anderson t-Motives

A standard reference for Anderson t-motives is [2, Sec. 5]. Let p, q, θ, $\mathbb{F}_q$ be as above. The field $\mathbb{F}_q(\theta)$ is the function field analog of $\mathbb{Q}$. There is a valuation ord on $\mathbb{F}_q(\theta)$ defined by the condition ord $\theta = -1$.

The field of the Laurent series $\mathbb{F}_q((1/\theta))$ is the completion of $\mathbb{F}_q(\theta)$ in the topology defined by ord. It is the function field analog of $\mathbb{R}$. Let $\mathbb{C}_\infty$ be the completion of an algebraic closure of $\mathbb{F}_q((1/\theta))$. $\mathbb{C}_\infty$ is the function field analog of $\mathbb{C}$. By definition, $\mathbb{C}_\infty$ is complete. It is also algebraically closed [2, Proposition 2.1].

The definition of a t-motive M is given in [2, Definitions 5.4.2, 5.4.12] (Goss uses another terminology: "abelian t-motive" of [2] = "t-motive" of this paper; L of Goss should be considered as $\mathbb{C}_\infty$). Particularly, M is a free $\mathbb{C}_\infty[T]$-module of dimension r (this number r is called the rank of M) endowed by a $\mathbb{C}_\infty$-skew-linear operator τ satisfying some properties. A nilpotent operator $N = N(M)$ associated to a t-motive is defined in [2, Remark 5.4.3.2]. We shall consider only pure [2, Definition 5.5.2] uniformizable [2, Theorem 5.9.14(3)], t-motives. Its dimension n is defined in [2, Remark 5.4.13.2] (Goss denotes the dimension by ρ). Condition $N = 0$ implies $n \leq r$. A t-motive of dimension 1 is called a Drinfeld module,[b] they are all pure, uniformizable, and their N is 0.

If $N(M) = 0$ then attached to such t-motive is a lattice $L = L(M)$ which is a free r-dimensional $\mathbb{F}_q[\theta]$-module in $\mathbb{C}_\infty^n$, and if $N \neq 0$ then $L(M)$ is a slightly more complicated object, we do not need to consider details for this case. Inclusion of L in $\mathbb{C}_\infty^n$ defines a surjective map

$$\alpha = \alpha(M) : L \bigotimes_{\mathbb{F}_q[\theta]} \mathbb{C}_\infty \to \mathbb{C}_\infty^n \tag{1.1}$$

having the property

$$\text{Restriction of } \alpha \text{ to } L \bigotimes_{\mathbb{F}_q[\theta]} \mathbb{F}_q((\theta^{-1})) \text{ is injective.} \tag{1.2}$$

Tensor product of t-motives M_1, M_2 is simply their tensor product over $\mathbb{C}_\infty[T]$, where the action of τ is defined by the formula $\tau(m_1 \otimes m_2) = \tau(m_1) \otimes \tau(m_2)$. If both M_1, M_2 are pure uniformizable then $M_1 \otimes M_2$ is pure uniformizable as well. The same definition is valid for exterior (respectively, symmetric) powers of M.

[b] Terminological discrepancy "module" — "motive" occurs because of existence of different, but equivalent, definitions of these objects.

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1.1.

It is easy to check that even if $N(M_1)$, $N(M_2)$ are 0 then $N(M_1 \otimes M_2)$, $N(\lambda^k(M_1))$, $N(S^k(M_1))$ are not 0. The only exception: if M is a Drinfeld module ($\Leftrightarrow n = 1$) then $N(\lambda^k(M)) = 0$, this is an elementary calculation.

There is a natural problem to describe $L(M_1 \otimes M_2)$, $L(\lambda^k(M_1))$, $L(S^k(M_1))$ in terms of $L(M_1)$, $L(M_2)$. It was solved by Anderson (non-published), a formula which is equivalent to this description is stated without proof in [10, p. 3]. See [3, Sec. 6], for more details and for the proof of the formula of Pink in the case when $N(M_1)$, $N(M_2)$ are 0. A final version of the formula for L of tensor operations of M_* is given in [9, Sec. 7.4]. The same formula for the case of Hom (namely, for the dual t-motive $M' := \mathrm{Hom}(M, \mathfrak{C})$, where $\mathfrak{C}$ is the only Drinfeld module of rank 1)[c] is given in [3, Sec. 5] for the case $N = 0$ and in [5] for the case $N \neq 0$.

Remark 1.1. More exactly, for the duals of Drinfeld modules (they are pure t-motives M having $N = 0$, $n = r - 1$, see [3, Corollary 8.4]) we have the same property $N(\lambda^k(M)) = 0$. Since the situation for them is the same as for Drinfeld modules themselves, we do not consider this case.

Let us state the theorem describing the lattice of an exterior power for the only case when all N are 0, i.e. the case where M is a Drinfeld module of rank r. We need a definition:

Definition 1.2. For a short exact sequence of vector spaces over a field

$$0 \to B_1 \xrightarrow{i} B_2 \to C \to 0$$

we define its k-th exterior power as the following exact sequence:

$$0 \to \lambda^k(B_1) \xrightarrow{\lambda^k(i)} \lambda^k(B_2) \to C_k \to 0.$$

Now, let us consider a Drinfeld module M of rank r and a short exact sequence whose epimorphism is (1.1) for M:

$$0 \to \mathrm{Ker}\ \alpha(M) \to L(M) \bigotimes_{\mathbb{F}_q[\theta]} \mathbb{C}_\infty \xrightarrow{\alpha(M)} \mathbb{C}_\infty \to 0. \tag{1.3}$$

Now, let us consider $\lambda^k(M)$ — the k-th exterior power of M. It is an Anderson t-motive having the rank $\binom{r}{k}$ and the dimension $n(r,k) := \binom{r-1}{k-1}$. Let us consider a short exact sequence whose epimorphism is (1.1) for $\lambda^k(M)$:

$$0 \to \mathrm{Ker}\ \alpha(\lambda^k(M)) \to L(\lambda^k(M)) \bigotimes_{\mathbb{F}_q[\theta]} \mathbb{C}_\infty \xrightarrow{\alpha(\lambda^k(M))} \mathbb{C}_\infty^{n(r,k)} \to 0. \tag{1.4}$$

Theorem 1.3. *There exists a canonical isomorphism from the k-th exterior power of* (1.3) *in the meaning of Definition* 1.2, *to* (1.4).

[c]There are various versions of the notion of the dual t-motive. See [6, (1.8)] for definitions of other versions of the notion of duality.

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Proof. It is completely analogous to the proofs of the analogical theorems for other tensor and Hom operations listed in 1.1 above, namely, to the proofs of [3, Sec. 5] (case of $M' := \mathrm{Hom}(M, \mathfrak{C})$, $N = 0$), [3, Sec. 6] (case of $M_1 \otimes M_2$, both $N = 0$), [9, Sec. 7.4] (general case) and [5] (case of $M' := \mathrm{Hom}(M, \mathfrak{C})$, $N \neq 0$). Hence, we omit it. □

1.2. *Why there is an analogy*

The simplest fact giving evidence to the analogy between Anderson t-motives and abelian varieties with MIQF is a similar form of their Siegel matrices. Really, a Siegel matrix of an abelian variety with MIQF is a $(n \times r - n)$-matrix with entries in $\mathbb{C}$ satisfying (2.8). Analogously, let M be an uniformizable Anderson t-motive of rank r, dimension n, having $N = 0$, and $L(M) \subset \mathbb{C}_\infty^n$ its lattice. It is isomorphic to $\mathbb{F}_q[\theta]^r$. Let $e_1, \dots, e_r$ be a basis of $L(M)$ over $\mathbb{F}_q[\theta]$ such that $e_1, \dots, e_n$ is a basis of the ambient space $\mathbb{C}_\infty^n$ over $\mathbb{C}_\infty$. Hence, there exists a matrix $Z \in M_{r-n,n}(\mathbb{C}_\infty)$, $Z = \{z_{ij}\}$ such that

$$e_{n+i} = \sum_{j=1}^{n} z_{ij} e_j, \quad \text{i.e.} \quad \begin{pmatrix} e_{n+1} \\ \cdots \\ e_r \end{pmatrix} = Z \begin{pmatrix} e_1 \\ \cdots \\ e_n \end{pmatrix}. \tag{1.5}$$

Remark 1.4. The fact that Siegel matrices for two cases are transposed is not important, this is a consequence of definitions. Condition of positivity of $I_n - Z\bar{Z}^t$ clearly has no analogy in the function field case. "Almost all" $Z \in M_{r-n,n}(\mathbb{C}_\infty)$ are Siegel matrices of some lattices: the only restriction imposed on Z is a condition that $e_{n+1}, \dots, e_r$ defined by (1.5) together with $e_1, \dots, e_n$ are linearly independent over $\mathbb{F}_q[\theta]$.

The fact that the dimension of the moduli set of pure t-motives of dimension n and rank r is $n(r-n)$ ([8, Theorem 3.2]) gives, first, one more evidence to the analogy, and second, suggests that we have a near 1–1 correspondence between pure uniformizable Anderson t-motives and the set of Siegel matrices factorized by an arithmetic subgroup. See [4] for details and for a particular case.

Further, the next fact giving evidence to the analogy is the theory of reduction of Anderson t-motives, from one side, and of abelian varieties with MIQF, from another side. Recall that if A is an abelian variety over $\mathbb{Q}$ of dimension g having a good ordinary reduction $\tilde{A}$ at a prime p (this is a generic case) then we have a reduction map on p-division points: $\pi : A_p \to \tilde{A}_p$ (here $\tilde{A}_p$ is the group of closed points of order p of $\tilde{A}$), we have Ker π is isomorphic to ${\mathbb{F}_p}^g$.

Further, if A is a generic abelian variety with MIQF of signature $(n, r-n)$ then (case $n \geq r-n$), we have $\dim \operatorname{Ker} \pi = n$.

The same formula holds for an Anderson t-motive M of rank r, dimension n. Let M be defined over $\mathbb{F}_q(\theta)$, $\mathfrak{p}$ a prime ideal of $\mathbb{F}_q[\theta]$ such that M has a good ordinary reduction $\tilde{M}$ at $\mathfrak{p}$. Again let $\pi : M_\mathfrak{p} \to \tilde{M}_\mathfrak{p}$ be the reduction map on $\mathfrak{p}$ -division points. Like for abelian varieties with MIQF, we have $\dim \operatorname{Ker} \pi = n$.

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Finally, the theories of reductions of Hecke correspondences of Shimura varieties parameterizing abelian varieties with MIQF (see, for example [12]), and of objects parameterizing Anderson t-motives (see, for example [7]), also are similar (it is necessary to take into consideration that the results of [7] are preliminary. At the moment we do not dispose even an exact definition of objects parameterizing Anderson t-motives).

Let us give definitions for the number field case (we give here only a sketch of definitions; see [12] for the exact statements). Let X be a Shimura variety and G a reductive group over $\mathbb{Q}$ associated to X according Deligne [1]. Let p be a prime of good reduction of X. $\mathbb{H}_p(X)$ — the p-part of the Hecke algebra of X — is isomorphic to $\mathcal{H}(G(\mathbb{Q}_p))$ — the algebra of double cosets $G(\mathbb{Z}_p)gG(\mathbb{Z}_p)$, $g \in G(\mathbb{Q}_p)$. There exists a Levi subgroup M of G having the following property:

The p-part of $\mathbb{H}_p(\tilde{X})$ — the Hecke algebra of $\tilde{X}$ (the reduction of X at p) is isomorphic to $\mathcal{H}(M(\mathbb{Q}_p))$. More exactly, there exists a commutative diagram:

$$\begin{array}{cccc} S_M^G : & \mathcal{H}(G(\mathbb{Q}_p)) & \to & \mathcal{H}(M(\mathbb{Q}_p)) \\ & \beta_1 \downarrow & & \beta_2 \downarrow \\ \gamma : & \mathbb{H}_p(X) & \to & \mathbb{H}_p(\tilde{X}), \end{array}$$

where S_M^G is the Satake map of Hecke algebras, β_1, β_2 are isomorphisms, and γ is the map of the reduction of a correspondence.

See, for example [12, p. 44, (*) and p. 49, (1.10)] for the definition and properties of M for the case of any Shimura variety of PEL-type.

Case of abelian varieties with MIQF. If X is a Shimura variety parameterizing abelian varieties with MIQF K, of dimension r and of signature $(n, r-n)$, then $G(\mathbb{Q}) = GU(n, r-n)(K)$. If p splits in $K/\mathbb{Q}$ then $G(\mathbb{Q}_p) = GL_r(\mathbb{Q}_p) \times GL_1(\mathbb{Q}_p)$. Let $i_{n,r-n} : GL_n(\mathbb{Q}_p) \times GL_{r-n}(\mathbb{Q}_p) \hookrightarrow GL_r(\mathbb{Q}_p)$ be the inclusion of the subgroup of $(n \times r-n)$-block diagonal matrices. Then the inclusion $M(\mathbb{Q}_p) \hookrightarrow G(\mathbb{Q}_p)$ is the inclusion

$$i_{n,r-n} \times GL_1(\mathbb{Q}_p) : GL_n(\mathbb{Q}_p) \times GL_{r-n}(\mathbb{Q}_p) \times GL_1(\mathbb{Q}_p) \hookrightarrow GL_r(\mathbb{Q}_p) \times GL_1(\mathbb{Q}_p).$$

Case of Anderson t-motives. The results (without proofs) are given in [7]. We see that for Anderson t-motives of rank r, dimension n we have the same groups G, M (up to the factor GL_1): $G = GL_r$, $M = GL_n \times GL_{r-n} \subset GL_r$, and hence the same descriptions of Satake maps and fibers of reductions of $\mathfrak{p}$-torsion points.

Sections 2 and 3 contain constructions of the number field case analogs of the map α of (1.1), and of Theorem 1.3 respectively. From one side, finding of these constructions was inspired by the analogy; from another side, their existence is a support to the analogy.

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2. Abelian Varieties with MIQF

We fix an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-\Delta})$. MIQF means that there exists an inclusion $K \overset{\iota}{\hookrightarrow} \text{End }(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. Particularly, End $(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ can be greater than K, so we need to fix K and an inclusion ι. For simplicity, we impose some non-essential restrictions. First, an abelian variety A is treated up to isogeny (hence, we are not interested in End (A); it can be O_K or its subring). Second, we consider a category of triples:

{Abelian variety A with MIQF K, up to isogeny, inclusion ι, and a fixed polarization form on the tangent space of A at 0.}

Third, we impose a restriction (2.2), see below.

Now, we give definitions concerning such abelian varieties and formulate a theorem on their descriptions in terms of linear algebra. We use notations of [11] whenever possible.

Definitions for abelian varieties with MIQF. Let $A = V/D$, where $V = \mathbb{C}^r$ and $D = \mathbb{Z}^{2r}$, be an abelian variety with MIQF, $\mathfrak{D} := D \otimes_{\mathbb{Z}} \mathbb{Q}$ ($\mathfrak{D}$ is $\mathbb{Q}D$ of [11]). Since we consider A up to isogeny, we shall deal only with $\mathfrak{D}$ and not with D. We fix an inclusion $\iota : K \hookrightarrow \text{End }(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ defining multiplication, and we fix an Hermitian polarization form $H = B + iE$ of A on V, where B and E are, respectively, its real and imaginary parts. There are two structures of K-module on V: the ordinary one which is the restriction of the $\mathbb{C}$-module structure, and the *-structure (multiplication is denoted by $k * v$, $k \in K$, $v \in V$) coming from ι (i.e. $k * v := (\iota(k))(v)$). $\mathfrak{D}$ is a K-*-module. We choose a basis $\mathfrak{x}_1, \ldots, \mathfrak{x}_r$ of $\mathfrak{D}$ over K [11, p. 157, (9)]. According to [11, p. 157, (11)] there exists a matrix $T = \{t_{ij}\} \in M_r(K)$ such that

$$\forall\, k_1, k_2 \in K, i, j = 1, \ldots, r \quad E(k_1 * \mathfrak{x}_i, k_2 * \mathfrak{x}_j) = \text{Tr}_{K/\mathbb{Q}}(k_1 t_{ij} \bar{k}_2). \tag{2.1}$$

T has properties

(a) $\bar{T}^t = -T$, i.e. iT is Hermitian [11, p. 157, (12)] and
(b) Signature of iT is $(n, r - n)$ [11, p. 160, (25)].

We restrict ourselves by abelian varieties A for which there exist $\mathfrak{x}_1, \ldots, \mathfrak{x}_r$ such its T is (here $J_{n,r-n} := \begin{pmatrix} I_n & 0 \\ 0 & -I_{r-n} \end{pmatrix}$):

$$T = -\sqrt{-\Delta} J_{n,r-n}. \tag{2.2}$$

This condition is an analog of the condition that A is a principally polarized (PP) abelian variety, for the case of abelian varieties without nontrivial multiplications. Hence, we shall call abelian varieties with MIQF having T satisfying (2.2) by varieties with MIQF-PP.

Conversely, the above objects define A uniquely. More exactly, let V be $\mathbb{C}^r$, $\mathfrak{D} \subset V$ a full $\mathbb{Q}$-lattice, $\iota : K \to \text{End } V$ an action of K on V such that $\mathfrak{D}$ is stable with respect of this action, $H = B + iE$ a positive non-degenerate Hermitian form on V such that there exists a basis $\mathfrak{x}_1, \dots, \mathfrak{x}_r$ of $\mathfrak{D}$ over K such that E on $\mathfrak{D}$ is given by the formula (2.1) and T is from (2.2).

Theorem 2.1. *There is a* 1–1 *correspondence between these* $V, \mathfrak{D}, K, \iota, H$ *and abelian varieties with MIQF-PP, up to isogeny.*

This is a classical result from the theory of abelian varieties.

The main Theorem 2.3 establishes an equivalence of objects of Theorem 2.1, and the below objects defining the type (E).

2.1. *Definitions for the type (E)*

We consider the set of triples (L, H_L, α), where

(1) L is a K-vector space of dimension r;
(2) H_L is a K-valued Hermitian form on L of signature $(n, r-n)$ such that there exists a basis of L over K satisfying the condition:

$$\text{the matrix of } H_L \text{ in this basis is } J_{n,r-n}. \tag{2.3}$$

We denote by $H_{L,\mathbb{C}}$ the extension of H_L to $L \otimes_K \mathbb{C}$.

(3) $\alpha : L \otimes_K \mathbb{C} \to \mathbb{C}^n$ is a $\mathbb{C}$-linear map such that α is surjective, and

$$\text{The restriction of } -H_{L,\mathbb{C}} \text{ to Ker } \alpha \text{ is a positive definite form.} \tag{2.4}$$

Remark 2.2. This α is clearly an analog of α of 1.1. We see that conditions of surjectivity of α hold in both cases, while the property 1.2 apparently has no analog in the number field case.

Theorem 2.3. *There is a* 1–1 *correspondence between* $V, \mathfrak{D}, K, \iota, H$ *from Theorem* 2.1 *and the above triples* (L, H_L, α) — *objects of type* (E).

Remark 2.4. If the restriction of α to $L \subset L \otimes_K \mathbb{C}$ is injective (this holds for almost all triples (L, H_L, α)) then the K-vector space in $\mathbb{C}^n$ mentioned in (E) is $\alpha(L)$.

Proof. Let the set $V, \mathfrak{D}, K, \iota, H$ from Theorem 2.1 be given. There is a canonical decomposition $V = V^+ \oplus V^-$ where

$$V^+ := \{v \in V \mid k * v = kv\}. \tag{2.5}$$

$$V^- : = \{v \in V \mid k * v = \bar{k}v\}. \tag{2.6}$$

We denote by $\pi_+ : V \to V^+$ the projection along V^-. Let $\beta : \mathfrak{D} \to V$ be the tautological inclusion.

The triple (L, H_L, α) corresponding to $(V, \mathfrak{D}, K, \iota, H)$ is constructed as follows. We define $L := \mathfrak{D}$, the structure of a K-vector space on L comes from the multiplication $k * d$ on $\mathfrak{D}$ (considering this multiplication of K on L we omit $*$ in $k * l$). We can consider β as a map from L to V.

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We define

$$\alpha : L \otimes_K \mathbb{C} \to V^+ = \mathbb{C}^n$$

by the formula $\alpha(l \otimes z) = z \cdot \pi_+(\beta(l))$. Formula (2.5) shows that it is well-defined. The form H_L is defined by the equality

$$\forall\, l_1, l_2 \in L \quad \mathrm{Tr}_{K/\mathbb{Q}}(\sqrt{-\Delta} H_L(l_1, l_2)) = E(\beta(l_1), \beta(l_2)). \tag{2.7}$$

Lemma 2.5. *Formula* (2.7) *really defines* H_L *uniquely.*

Proof. Unicity: we fix $l_1, l_2 \in L$, and we consider a $\mathbb{Q}$-linear form $\gamma : K \to \mathbb{Q}$ defined by the formula

$$\gamma(k) = E(\beta(k * l_1), \beta(l_2)).$$

There exists the only $k_0 \in K$ such that $\gamma(k) = \mathrm{Tr}_{K/\mathbb{Q}}(k_0 k)$. We see that if (2.7) holds for all pairs $k * l_1, l_2$, then necessarily $\sqrt{-\Delta} H_L(l_1, l_2) = k_0$.

Existence: Let $\mathfrak{x}_1, \ldots, \mathfrak{x}_r$ be a basis of $\mathfrak{D}$ over K such that the corresponding T has the form (2.2). We define H_L by the condition that the matrix of H_L in $\mathfrak{x}_1, \ldots, \mathfrak{x}_r$ is $J_{n,r-n}$. Equation (2.1) implies that H_L satisfies (2.7). $\square$

To prove (2.4) we recall a well-known coordinate description.

The Siegel domain for abelian varieties with MIQF is the following (here $Z = \{z_{ij}\}$ is a Siegel matrix):

$$\mathcal{H}^3_{n,r-n} := \{Z \in M_{n,r-n}(\mathbb{C}) \mid I_n - Z\bar{Z}^t \text{ is positive Hermitian}\}$$
$$(\Leftrightarrow I_{r-n} - \bar{Z}^t Z \text{ is positive Hermitian}). \tag{2.8}$$

[11, p. 162, 2.6]. For any $Z \in \mathcal{H}^3_{n,r-n}$ we can construct an above abelian variety A_Z (satisfying (2.2)) as follows ([11]). We fix a basis $e_1, \ldots, e_r$ of V over $\mathbb{C}$ such that $e_1, \ldots, e_n$ (respectively, $e_{n+1}, \ldots, e_r$) is a basis of V^+ (respectively, V^-) over $\mathbb{C}$. It defines the K-*-action on V. Let $\mathfrak{x}_*$ (respectively, e_*) be the matrix column of $\mathfrak{x}_1, \ldots, \mathfrak{x}_r$ (respectively, $e_1, \ldots, e_r$). They satisfy

$$\mathfrak{x}_* = Y e_* \quad \text{where } Y = \begin{pmatrix} I_n & Z \\ \bar{Z}^t & I_{r-n} \end{pmatrix} \tag{2.9}$$

[11, p. 162, (35), Type IV]. E is defined by (2.1) and (2.2). These conditions define A_Z.

Proof. Equation (2.9) implies that elements

$$\lambda_i := \mathfrak{x}_{n+i} \otimes 1 - \sum_{k=1}^{n} \mathfrak{x}_k \otimes \bar{z}_{ki}, \tag{2.10}$$

$i = 1, \ldots, r-n$, form a basis of Ker α, we have

$$H_L(\lambda_i, \lambda_j) = \sum_{k=1}^{n} \bar{z}_{ki} z_{kj} - \delta_i^j = \{\bar{Z}^t Z - I_{r-n}\}_{ij}, \tag{2.11}$$

hence (2.8) implies (2.4).

So, we have constructed a well-defined map from the set of objects $V, \mathfrak{D}, K, \iota, H$ (of type (B)) to the set of objects (L, H_L, α) (of type (E)).

To construct the inverse map we need a definition. Let W be a $\mathbb{C}$-vector space. We denote by $\mathfrak{i}(W)$ the complex conjugate space together with a map $\mathfrak{i}: W \to \mathfrak{i}(W)$ which is an isomorphism of $\mathbb{R}$-vector spaces and satisfies $\mathfrak{i}(zw) = \bar{z}\mathfrak{i}(w)$, $z \in \mathbb{C}$, $w \in W$.

Let a triple (L, H_L, α) be given. Let $(\mathrm{Ker}\ \alpha)^\perp \subset L \otimes_K \mathbb{C}$ be the $H_{L,\mathbb{C}}$-orthogonal space of Ker α, $\pi_\alpha : L \otimes_K \mathbb{C} \to \mathrm{Ker}\ \alpha$ the projection along $(\mathrm{Ker}\ \alpha)^\perp$, and let us consider the composition $\mathfrak{i} \circ \pi_\alpha : L \otimes_K \mathbb{C} \to \mathfrak{i}(\mathrm{Ker}\ \alpha)$.

We let $V = \mathbb{C}^n \oplus \mathfrak{i}(\mathrm{Ker}\ \alpha)$ (here $\mathbb{C}^n$ is the target of α), and we define the inclusion $\gamma : L = \mathfrak{D} \hookrightarrow V$ by the formula

$$\gamma(l) = (\alpha(l), \mathfrak{i} \circ \pi_\alpha(l)).$$

The polarization $H = B + iE$ on V is defined by the same formula (2.7); here H_L is given, so (2.7) defines E for any pair $(\beta(l_1), \beta(l_2))$. Further, H is defined uniquely by E (but clearly we must prove that this H really is a positive Hermitian form). The action of K on V is defined by the property that $k * v = kv$ for $v \in \mathbb{C}^n$ and $k * v = \bar{k}v$ for $v \in \mathfrak{i}(\mathrm{Ker}\ \alpha)$ (or, the same, $k * (v, \mathfrak{i}(y)) = (kv, \mathfrak{i}(ky))$), i.e. $\mathbb{C}^n = V^+$, $\mathfrak{i}(\mathrm{Ker}\ \alpha) = V^-$.

Let us prove that the above V, H, $\gamma(L)$ and the action of K really define objects of Theorem 2.1. Obviously, the signature of the action of K on V is $(n, r-n)$. Further, for $k \in K$, $l \in L$ we have $\gamma(kl) = k * \gamma(l)$, i.e. $\mathfrak{D}$ is a K-*-lattice. Let $\mathfrak{x}_1, \dots, \mathfrak{x}_r$ be a basis of L over K satisfying (2.3). It defines a $n \times (r-n)$-matrix $Z = \{z_{ij}\}$ as follows (this is the same as (2.10)):

$$\alpha(\mathfrak{x}_{n+i}) = \sum_{k=1}^{n} \bar{z}_{ki}\alpha(\mathfrak{x}_k). \tag{2.12}$$

(It is easy to prove that $\alpha(\mathfrak{x}_k)$, $k = 1, \dots, n$ form a basis of $\mathbb{C}^n$). Condition (2.4) implies that $Z \in \mathcal{H}^3_{n,r-n}$ (calculations coincide with the ones of (2.11)). This means that λ_i defined by (2.10) with z_{**} from (2.12) form a basis of Ker α, we define

$$\mu_i := \mathfrak{x}_i \otimes 1 - \sum_{k=1}^{r-n} \mathfrak{x}_{n+k} \otimes z_{ik}, \tag{2.13}$$

$i = 1, \dots, n$. Obviously, $H_{L,\mathbb{C}}(\lambda_i, \mu_j) = 0$, this means that $\mu_1, \dots, \mu_n$ is a basis of $(\mathrm{Ker}\ \alpha)^\perp$.

Let Y be defined by (2.9) with z_{**} from (2.12).

We have

$$Y \begin{pmatrix} I_n & -Z \\ -\bar{Z}^t & I_{r-n} \end{pmatrix} = \begin{pmatrix} I_n - Z\bar{Z}^t & 0 \\ 0 & -\bar{Z}^t Z + I_{r-n} \end{pmatrix},$$

hence (2.8) implies that Y is invertible. We define a vector column e_* by the formula $e_* := Y^{-1} \cdot \gamma(\mathfrak{x}_*)$. We get immediately using (2.12), (2.13) that $e_1, \dots, e_n$ from a

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basis of V^+, $e_{n+1}, \dots, e_r$ from a basis of V^-. So, results of [11] imply that V, H, $\gamma(L)$ and the action of K really are objects from Theorem 2.1.

The proof of the affirmation that the above two constructions are converse is practically a tautology. □

3. Exterior Powers of Abelian Varieties with MIQF having $n = 1$

Let $V, \mathfrak{D}, K, \iota, H$ be from Theorem 2.1. The associated triple (L, H_L, α) defines an exact sequence of $\mathbb{C}$-vector spaces

$$0 \to \text{Ker } \alpha \overset{i}{\hookrightarrow} L \otimes_K \mathbb{C} \overset{\alpha}{\to} \mathbb{C}^n \to 0.$$

Let us take the k-th exterior power of this exact sequence (Definition 1.2):

$$0 \to \lambda^k(\text{Ker } \alpha) \overset{\lambda^k(i)}{\hookrightarrow} \lambda^k(L) \otimes_K \mathbb{C} \overset{\alpha_k}{\to} W_k \to 0.$$

There exists an Hermitian form $\lambda^k(H_L)$ on $\lambda^k(L)$; recall that

$$\lambda^k(H_L)(l_1 \wedge \dots \wedge l_k, l'_1 \wedge \dots \wedge l'_k) = \det\{H_L(l_i \wedge l'_j)\}.$$

It is obvious that if $n = 1$ then the restriction of $(-1)^k \lambda^k(H_L)$ to $\lambda^k(\text{Ker } \alpha)$ is positive definite. This means that the triple $\lambda^k(L)$, $(-1)^{k-1}\lambda^k(H_L)$, α_k satisfies conditions of Theorem 2.3 and hence, according Theorem 2.1, defines an abelian variety with MIQF-PP (up to isogeny) $\lambda^k(A)$ which is called the k-th exterior power of A. Its signature is $\binom{r-1}{k-1}$, $\binom{r-1}{k}$.

This construction can be applied not only to abelian varieties up to isogeny, but to abelian varieties themselves (with MIQF-PP, of signature $(1, r-1)$). This is an exercise for the reader.

Remark 3.1. It is easy to see that if $n \neq 1, r-1$ then this construction cannot be applied to abelian varieties of signature $(n, r-n)$; we cannot also define symmetric powers of abelian varieties with MIQF, as well as their tensor products. This is clearly an analog of 1.1.

Remark 3.2. For the function field case there exists a natural definition of the exterior power of t-motives and 1.3 is a theorem, while for the number field case there is no such definition hence we must consider the above construction — an analog of the Theorem 1.3 — as the definition of the exterior powers of abelian varieties with MIQF.

Remark 3.3. This construction of exterior power can be continued to the corresponding Shimura varieties. Let us recall some definitions of [1]. Attached to a Shimura variety X is an algebraic group $G = G_X$ over $\mathbb{Q}$ and a map $h = h_X : \mathbb{S} \to G$ over $\mathbb{R}$, where $\mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}}(G_m)$, satisfying some conditions. A map $\varphi : X \to Y$ of Shimura varieties comes from a map $G_\varphi : G_X \to G_Y$ satisfying $G_\varphi \circ h_X = h_Y$. If X is a Shimura variety parameterizing abelian varieties with MIQF of signature

(r,s) then $G_X = GU(r,s)$. The map $G_\varphi : GU(1,r-1) \to GU(\binom{r-1}{k-1},\binom{r-1}{k})$ corresponding to the above construction is obvious. Really, let L, H_L be of 2.1 ($n=1$); then $\gamma \in GU(1,r-1)$ is an endomorphism of L (denoted by γ as well) satisfying $H_L(\gamma(l_1),\gamma(l_2)) = \lambda(\gamma)H_L(l_1,l_2)$. Clearly, $G_\varphi(\gamma)$ is $\lambda^k(\gamma) : \lambda^k(L) \to \lambda^k(L)$. It is an elementary exercise to check all compatibilities.

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