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Constrained colourings of random graphs☆

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Abstract

Given graphs G, H_1 and H_2, let $G \xrightarrow{\text{mr}} (H_1, H_2)$ denote the property that in every edge-colouring of G there is a monochromatic copy of H_1 or a rainbow copy of H_2. The *constrained Ramsey number*, defined as the minimum n such that $K_n \xrightarrow{\text{mr}} (H_1, H_2)$, exists if and only if H_1 is a star or H_2 is a forest. We determine the threshold for the property $G(n, p) \xrightarrow{\text{mr}} (H_1, H_2)$ when H_2 is a forest.

Keywords: random graphs; constrained Ramsey; rainbow colourings; monochromatic colourings

1. Introduction

Given graphs G, H_1, and H_2, we write $G \xrightarrow{\text{mr}} (H_1, H_2)$ if in every colouring of $E(G)$ (with no restriction on the number of used colours) there is a *monochromatic* copy of H_1 or a *rainbow* copy of H_2, that is, a copy of H_1 with all edges having the same colour or a copy of H_2 with no two edges of the same colour. Here we investigate the property $G \xrightarrow{\text{mr}} (H_1, H_2)$ when G is the binomial random graph $G(n, p)$.

The *constrained Ramsey number* $\mathrm{r_c}(H_1, H_2)$, sometimes called *rainbow Ramsey number*, is defined as the minimum $n \in \mathbb{N}$ such that $K_n \xrightarrow{\text{mr}} (H_1, H_2)$. In [8] it is proved that the number $\mathrm{r_c}(H_1, H_2)$ exists if and only if H_1 is a star or H_2

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is a forest (for results concerning constrained Ramsey numbers, see [1, 2, 5, 6, 7, 9, 8, 12, 15]). Therefore, assuming that H_1 is a star or H_2 is a forest, since the property $G(n,p) \xrightarrow{\text{mr}} (H_1, H_2)$ is increasing, it admits a threshold function (see [4]). We recall that a function $\hat{p}\colon \mathbb{N} \to [0,1]$ is called a *threshold function* for a graph property $\mathcal{P}$ in $G(n,p)$ if $\lim_{n\to\infty} \mathbb{P}[G(n,p) \in \mathcal{P}] = 1$ for $p \gg \hat{p}$ and $\lim_{n\to\infty} \mathbb{P}[G(n,p) \in \mathcal{P}] = 0$ for $p \ll \hat{p}$. Such limits are called the *1-statement* and *0-statement* respectively. We call any $p' = \Theta(\hat{p})$ 'the threshold' for $\mathcal{P}$.

Definition 1.1. *Given graphs H_1 and H_2 for which $\mathrm{r_c}(H_1, H_2)$ exists, we denote the threshold for $G(n,p) \xrightarrow{mr} (H_1, H_2)$ by $\hat{p}(H_1, H_2)$.*

In this paper we determine $\hat{p}(H_1, H_2)$ when H_2 is a forest. The 2-*density* of a graph G, denoted by $m_2(G)$, is defined as follows, where we denote by $v(G)$ and $e(G)$, respectively, the number of vertices and edges of G.

$$m_2(G) = \begin{cases} \max\left\{\frac{e(J)-1}{v(J)-2} : J \subset G,\ v(J) \geq 3\right\} & \text{if } v(G) \geq 3, \\ 1/2 & \text{if } G = K_2. \end{cases}$$

We will also use the concept of *maximum subgraph density* of a graph G, denoted by $m(G)$, which is defined as

$$m(G) = \max\left\{\frac{e(J)}{v(J)} : J \subset G,\ v(J) \geq 1\right\}.$$

We now discuss the thresholds for $G \xrightarrow{\text{mr}} (H_1, H_2)$, which depend on the structure of H_1 and H_2. Let us first discuss the easy cases. If H_1 or H_2 has only one edge, then the threshold is given by the appearance of an edge in $G(n,p)$, which gives $\hat{p}(H_1, H_2) = n^{-2}$. If H_2 is a forest with $e(H_2) = 2$ and H_1 is any graph with $e(H_1) \geq 2$, then we can easily check that $\hat{p}(H_1, H_2) = n^{-1/m(H_1)}$ (see Proposition 1.3).

In the remainder of the introduction, assume that $e(H_1) \geq 2$ and $e(H_2) \geq 3$. From the celebrated result of Rödl and Ruciński [13], we know that if H_1 is not a star forest, then for $p \ll n^{-1/m_2(H_1)}$ with high probability (that is, with probability tending to 1 as n tends to infinity) there is a colouring χ of the edges of $G(n,p)$ with two colours containing no monochromatic copy of H_1. Clearly, if H_2 is a forest with at least three edges, there is no rainbow copy of H_2 in χ. Therefore, if $p \ll n^{-1/m_2(H_1)}$, then with high probability $G(n,p) \xrightarrow{\text{mr}} (H_1, H_2)$ does not hold. We prove that $n^{-1/m_2(H_1)}$ is indeed the threshold for $G(n,p) \xrightarrow{\text{mr}} (H_1, H_2)$ when H_1 is not a star forest and H_2 is a forest.

Now let H_1 be a star forest. We say that a *pending forest* is a forest composed of a disjoint union of edges and *cherries* (2-edge paths), and a star forest that has at least two components and is not a matching is a *non-trivial disconnected star forest* or simply *disconnected star forest*. If H_1 is a disconnected star forest and H_2 is not a pending forest, then we prove that the threshold for $G(n,p) \xrightarrow{\text{mr}} (H_1, H_2)$ is also given by $n^{-1/m_2(H_1)}$. Summarising, if either H_1 is not a star forest and H_2 is a forest, or H_1 is a disconnected star forest and H_2 is not a pending forest, then H_2 is irrelevant to the threshold, as we show that is given by $n^{-1/m_2(H_1)}$.

For the remaining possibilities for H_1 and H_2, the threshold depends on both H_1 and H_2. We prove that if H_1 is a star, or if H_1 is a disconnected star forest and H_2 is a pending forest, then the threshold for $G(n,p) \xrightarrow{\text{mr}} (H_1, H_2)$ is given by $n^{-1/m_F(H_1,H_2)}$, where the parameter $m_F(H_1, H_2)$ is defined as follows: $m_F(H_1, H_2) = \min\{m(F)\colon F$ is a forest and $F \xrightarrow{\text{mr}} (H_1, H_2)\}$. We show that for such graphs H_1 and H_2, there is a forest F with a fixed number of vertices and components such that $m(F) = m_F(H_1, H_2)$ (see Proposition 4.2).

Our main theorem is as follows.

Theorem 1.2. *Let H_1 be a graph and H_2 be a forest such that $e(H_1) \geq 2$ and $e(H_2) \geq 3$.*

(*i*) *If H_1 is not a star forest; or H_1 is a disconnected star forest and H_2 is not a pending forest, then*

$$\hat{p}(H_1, H_2) = n^{-1/m_2(H_1)}. \tag{1}$$

(*ii*) *If H_1 is a star; or H_1 is a disconnected star forest and H_2 is a pending forest, then*

$$\hat{p}(H_1, H_2) = n^{-1/m_F(H_1,H_2)}. \tag{2}$$

Some particular cases are not covered by Theorem 1.2. The following proposition characterises their behaviour for completeness.

Proposition 1.3. *Let H_1 be a graph and H_2 be a forest.*

(*i*) *If H_1 is a matching with $e(H_1) \geq 2$ and H_2 is a non-pending forest, then*

$$\hat{p}(H_1, H_2) = n^{-1}. \tag{3}$$

(*ii*) *If H_2 is a cherry and H_1 is not a forest, or if H_2 is a 2-edge matching, then*

$$\hat{p}(H_1, H_2) = n^{-1/m(H_1)}. \tag{4}$$

(*iii*) *If H_2 is a cherry and H_1 is a forest, then*

$$\hat{p}(H_1, H_2) = n^{-1-1/(v(H_1)-1)}. \tag{5}$$

Note that when H_2 is a forest with $e(H_2) \geq 2$, Theorem 1.2 and Proposition 1.3 cover all possibilities for graphs H_1. The case $e(H_2) = 2$ is covered by Proposition 1.3 (*ii*)–(*iii*). For $e(H_2) \geq 3$, note first that the case where H_1 is not a star forest is covered in Theorem 1.2 (*i*). On the other hand, if H_1 is a star forest, then H_1 is either a star, a matching, or a disconnected star forest. Items (*i*) and (*ii*) of Theorem 1.2 cover the case where H_1 is a star or a disconnected star forest, and Proposition 1.3 (*i*) deals with the case where H_1 is a matching.

The problem of determining the threshold for $G(n, p) \xrightarrow{\text{mr}} (H_1, H_2)$ when H_1 is a star and H_2 is not a forest is still open. We remark that this problem is a generalisation of the well-known *anti-Ramsey problem*, which aims to determine the threshold for the property that every proper colouring of $E(G(n, p))$ contains a rainbow copy of a given fixed graph H. In fact, an edge-colouring that contains no monochromatic path with 2 edges is a proper colouring. In more generality, an edge-colouring that contains no monochromatic $K_{1,r}$ is an r-bounded colouring. In [10] it is proved that, for every fixed r and every graph H_2, with high probability we have $G(n, p) \xrightarrow{\text{mr}} (K_{1,r}, H_2)$, whenever $p \gg n^{-1/m_2(H_2)}$. This, however, turns out not to be the threshold for some graphs (see [11]).

This paper is organised as follows. In Section 2 we provide some results that will be useful in the proofs of Theorem 1.2 and Proposition 1.3. Section 3 contains the proof of Theorem 1.2 (*i*) and in Section 4 we prove Theorem 1.2 (*ii*). We remark that since Proposition 1.3 follows from simple arguments combined with Theorem 1.2 (*i*) and a construction presented in Subsection 3.2, we choose to omit it in this extended abstract.

2. Random graphs

Given a graph $G = (V, E)$ and a colouring χ of E, for any $X \subset V$, let $d_\chi(v, X)$ be the *colour-degree* of v in X, given by $d_\chi(v, X) = |\{\chi(e) : e \setminus \{v\} \subset X\}|$. We write simply $d_\chi(v)$ for $d_\chi(v, V(G))$. The following definition plays an important rôle in our proof. Let H be a graph, $r \geq 2$ be an integer and let $b > 0$. A graph $G = (V, E)$ satisfies property $Q(b, r, H)$ if every edge colouring χ of G with no monochromatic copy of H is such that every subset $X \subset V$ with $|X| \geq bn$ contains a vertex v with $d_\chi(v, X) > r$.

The aim of this section is to prove the following result, which will be useful in the proof of the one statement of Theorem 1.2 (1).

Theorem 2.1. *Let H be a connected graph and let $r \geq 2$ and $b > 0$. If $p \gg n^{-1/m_2(H)}$, then $G(n, p)$ satisfies property $Q(b, r, H)$ with high probability.*

The following classical result of Bollobás will be useful.

Theorem 2.2 ([3])**.** *Let H be an arbitrary graph with at least one edge. Then, the threshold for H to be a subgraph of $G(n, p)$ is $n^{-1/m(H)}$.*

We write $G \to (H)_r$ for the property that in every r-colouring of the edges of $G(n,p)$ there is a monochromatic copy of H. When dealing with a graph H_1 that is not a star forest, and a forest H_2, we use the following celebrated result proved by Rödl and Ruciński to obtain the 0-statement for the property $G(n,p) \xrightarrow{\text{mr}} (H_1, H_2)$.

Theorem 2.3 (Theorem 1′ of [13]). *For every integer $r \geq 2$ and for every graph H which is not a star forest there exists a constant C such that*

$$\lim_{n\to\infty}(G(n,p) \to (H)_r) = \begin{cases} 0, & \text{if } p \ll n^{-1/m_2(H)} \quad (6)\\ 1, & \text{if } p \geq Cn^{-1/m_2(H)}. \quad (7)\end{cases}$$

We also need the following strengthening of the 1-statement of Theorem 2.3.

Theorem 2.4 (Theorem 3 of [13]). *Let H be a graph with at least one edge and let $r \geq 2$ be an integer. There exist constants n_0, C, b such that if $n \geq n_0$ and $p \geq Cn^{-1/m_2(H)}$, then*

$$\mathbb{P}\left(G(n,p) \to (H)_r\right) \geq 1 - \exp\left(-bn^2 p\right).$$

By using the union bound and Theorem 2.4, we conclude that the random graph $G(n,p)$ satisfies the following property.

Corollary 2.5. *Let H be a graph with at least one edge and let $r \geq 2$ and $\varepsilon > 0$. If $p \gg n^{-1/m_2(H)}$, then the following holds with high probability. For every edge colouring χ of $G(n,p)$ with no monochromatic copy of H and every $X \subset V(G(n,p))$ of size $|X| \geq \varepsilon n$, the colouring $\chi|_{\binom{X}{2}}$ contains more than r colours.*

Proof. It follows directly from the union bound over at most 2^n subsets of vertices and the bound given in Theorem 2.4. □

To prove Theorem 2.1, it suffices to strengthen the conclusion of Corollary 2.5 to ensure that a single vertex is incident to edges of r different colours. For that, the following definition will be useful. We say a colouring χ of a graph F is *r-local* if $d_\chi(v) \leq r$ for every $v \in V(F)$. The following lemma was shown in [14].

Lemma 2.6 (Lemma 2 of [14]). *Let F and H be graphs such that H is connected. If there is an r-local colouring χ of F with no monochromatic copy of H, then there exists $W \subset V(F)$ of size $|W| \geq \frac{r!}{r^r} v(F)$ and an r-colouring χ' of the edges of $F[W]$ with no monochromatic copy of H.*

Combining Corollary 2.5 and Lemma 2.6, we prove Theorem 2.1.

Proof of Theorem 2.1. We want to prove that, with high probability, every edge colouring χ of $G = G(n,p)$ with no monochromatic copy of H is such that every subset $X \subset V$ with $|X| \geq cn$ contains a vertex v with $d_\chi(v, X) > r$.

Take $\varepsilon = cr!/r^r$, let χ be an edge colouring of $G = G(n,p)$ with no monochromatic copy of H and consider a set $X \subset V(G)$ of size $|X| \geq cn$. Suppose for a contradiction that $\chi|_{E(G[X])}$ is r-local. By Lemma 2.6 applied with $F = G[X]$ and H, there exists a set $W \subset V(G[X])$ with $|W| \geq (r!/r^r)|X| \geq \varepsilon n$ and an r-colouring χ' of the edges of $G[W]$ with no monochromatic copy of H. But, from Corollary 2.5, we know that with high probability there is no such r-colouring. □

3. Threshold at $n^{-1/m_2(H_1)}$

Let H_1 be a graph and H_2 be a forest such that $e(H_1) \geq 2$ and $e(H_2) \geq 3$. We prove Theorem 1.2 (i), which gives the threshold $n^{-1/m_2(H_1)}$ for $G(n,p) \xrightarrow{\text{mr}} (H_1, H_2)$ when H_1 is not a star forest, or when H_1 is a disconnected star forest and H_2 is not a pending forest.

3.1. 1-statement

In this section it will be useful to assume that H_1 is connected, a fact that follows easily by induction on the number of components of H_1 together with the following proposition.

Proposition 3.1. *Let G_1 and G_2 be connected graphs such that $m_2(G_1) \geq m_2(G_2)$ and let H be a graph obtained by connecting G_1 and G_2 by a single edge. Then $m_2(H) = \max\{m_2(G_1), 1\}$.*

Proof. If $m_2(G_1) < 1$, then $G_1 = K_2$ and $m_2(H) = 1$, so we may assume $m_2(G_1) \geq 1$. Clearly, it suffices to show that $m_2(H) \leq m_2(G_1)$. Let $J \subset H$ be a subgraph of H with at least three vertices, and let $A_i = V(G_i) \cap V(J)$ for $1 \leq i \leq 2$. We may also assume that A_1 and A_2 are both nonempty. If $|A_1|, |A_2| \geq 3$, then using that $(a+b)/(c+d) \leq \max\{a/c, b/d\}$ for any $a, b \geq 0$ and $c, d > 0$, we have

$$\frac{e(J)-1}{v(J)-2} \leq \frac{(e(H[A_1])-1)+(e(H[A_2])-1)+2}{(|A_1|-2)+(|A_2|-2)+2} \leq \max\{m_2(H[A_1]), m_2(H[A_2]), 1\} \leq m_2(G_1).$$

If $|A_1|, |A_2| \leq 2$, then the graph J has no cycles and therefore $m_2(J) \leq 1 \leq m_2(G_1)$. Moreover, if $|A_i| \geq 3$ and $|A_{3-i}| \leq 2$ for some $1 \leq i \leq 2$, then $e(J) - e(J[A_i]) \leq |A_{3-i}|$, that is, the inclusion of A_{3-i} adds at least as many vertices as edges, implying $m_2(J) \leq \max\{m_2(J[A_i]), 1\}$ by a similar argument as above. This concludes the proof. □

To prove the 1-statements of Theorem 1.2 (*i*), we shall use Theorem 2.1 to prove that for every connected graph H and every fixed tree T, the random graph satisfies $G(n, p) \xrightarrow{\text{mr}} (H, T)$ with high probability as long as $p \gg n^{-1/m_2(H)}$. For this purpose, we will consider the complete d-ary rooted tree of height h, for general h and d, denoted by $T(d, h)$.

Theorem 3.2. *Let H be a connected graph. If $p \gg n^{-1/m_2(H)}$, then $G(n, p) \xrightarrow{mr} (H, T(d, h))$ with high probability.*

Since Theorem 2.1 states that $G(n, p)$ satisfies property $Q(b, r, H)$ with high probability when $p \gg n^{-1/m_2(H)}$, Theorem 3.2 follows directly from the following deterministic lemma.

Lemma 3.3. *Let H be a connected graph. For all positive integers d and h, there exist $0 < b, c < 1$ and an integer $r \geq 1$ with the following property. If G satisfies $Q(b, r, H)$, then in any edge colouring of G there is a monochromatic copy of H or there are $\lfloor c \cdot v(G) \rfloor$ vertex-disjoint copies of $T(d, h)$ in G, each of them rainbow.*

Proof. Let H be a connected graph and fix positive integers d and h. Our proof is by induction on h. Let χ be an edge colouring of G.

We may assume that there are no monochromatic copies of H under χ, as otherwise the proof would be finished. Recall that, in this case, $Q(b, r, H)$ says that every subset $X \subset V(G)$ with $|X| \geq bn$ contains a vertex incident to more than r colours. Therefore, if $h = 1$, we may take $b = 1/2$, $c = 1/2(d+1)$ and $r = d - 1$. Indeed, for such values, the definition of property $Q(b, r, H)$ allows us to iteratively find rainbow copies of $T(d, 1)$ until we have used more than $n/2$ vertices of G. This procedure therefore finds $\lfloor cn \rfloor$ disjoint rainbow copies of $T(d, 1)$, as claimed.

We now show that the result holds for $h \geq 1$. Let b', c' and r' be obtained by applying the base case $h' = 1$ with $d' := 2d^h$. Also, let b'', c'' and r'' be obtained by applying the induction step with $h'' = h - 1$ and $d'' = d$. We will show below that the conclusion of the lemma holds for $b = b''c'/2$, $r = \max\{r', r''\}$ and $c = c'c''/2$.

Let G be a graph satisfying $Q(b, r, H)$, and observe that we may assume that $cv(G) \geq 1$ because the conclusion of the lemma is vacuous otherwise. Since $b' = 1/2 > b$ and $r' \leq r$, the graph G satisfies $Q(b', r', H)$ and by induction hypothesis contains a family $\mathcal{L}$ of $\lfloor c' \cdot v(G) \rfloor \geq (c'/2)v(G)$ rainbow vertex-disjoint copies of $T(2d^h, 1)$. Let $X \subset V(G)$ be the set of roots of such trees. Observe that, since $b''|X| \geq b''(c'/2)v(G) \geq bn$ and $r'' \leq r$, $G[X]$ satisfies property $Q(b'', r'', H)$. Therefore, by induction hypothesis, $G[X]$ contains a family $\mathcal{T}$ of $\lfloor c'' \cdot v(G[X]) \rfloor \geq \lfloor c \cdot v(G) \rfloor$ vertex-disjoint rainbow rooted copies of $T(d, h-1)$ in X.

Notice that, by definition of X, each leaf v of a tree $T \in \mathcal{T}$ is the root of a tree $L_v \in \mathcal{L}$. Since T has at most d^h edges, there are d^h edges in each L_v whose colours do not appear in T. A greedy procedure can then be used to extend T to a tree of height h, concluding the the induction step and the proof of the lemma. □

3.2. 0-*statement*

Let H_2 be a forest with $e(H_2) \geq 3$. In this subsection we prove that if $p \ll n^{-1/m_2(H_1)}$, then the following holds with high probability when H_1 is not a star forest, or when H_1 is a disconnected star forest and H_2 is not a pending forest: there is an edge-colouring of $G(n, p)$ with neither a monochromatic copy of H_1 nor a rainbow copy of H_2. In fact, if H_1 is not a star forest, then this follows directly from the zero statement of Theorem 2.3 with $r = 2$ (recall that H_2 has at least three edges). Thus, we may and shall assume that H_1 is a disconnected star forest and H_2 is not a pending forest.

Since H_1 is a forest, we have $m_2(H_1) = 1$. For $p \ll n^{-1/m_2(H_1)} = n^{-1}$, the expected number of cycles in $G(n, p)$ is $o(1)$, and therefore $G(n, p)$ is a forest with high probability. Therefore, to obtain the aimed 0-statement it is enough to provide an edge-colouring χ of any given forest F avoiding monochromatic copies of H_1 and rainbow copies of H_2.

Proof of the 0-*statement of Theorem 1.2 (i).* Let H_1 be a disconnected star forest and H_2 be a forest with $e(H_2) \geq 3$ that is not a pending forest. Let F be an arbitrary n-vertex forest composed by trees $T_1, \ldots, T_k$, rooted at arbitrary vertices with heights $h_1, \ldots, h_k$.

In what follows, let $V_{\text{ord}} = (v_1, \ldots, v_n)$ be an ordering of $V(F)$ such that every vertex of T_i appears before every vertex of T_j in V_{ord} for all $1 \leq i < j \leq n$, and for each tree T_i, vertices at height h appear before vertices of height $h + 1$, for every $0 \leq h \leq h_i - 1$. We construct a colouring $\chi \colon E(F) \to \mathbb{N}$ that contains no monochromatic copy of H_1 or rainbow copy of H_2. Since H_2 is not a pending forest, either $\Delta(H_2) \geq 3$ or H_2 contains a path with three edges.

If $\Delta(H_2) \geq 3$, then put $\chi(v_i v_j) = i$ to every edge $v_i v_j$ with $i < j$. The colouring χ clearly has no monochromatic copy of H_1 as there are no vertex-disjoint stars with the same colour. Also, since every vertex of F is incident to edges coloured with at most two colours and $\Delta(H_2) \geq 3$, there is no rainbow copy of H_2 in F.

Finally, if H_2 contains a path with 3 edges, then we colour F by setting $\chi(e) = i$, where v_i is the unique vertex of e of odd height in the tree containing e. As before, since there are no vertex-disjoint stars with the same colour, there is no monochromatic copy of H_1. Furthermore, under the colouring χ, there is no rainbow path $v_0 v_1 v_2 v_3$, since either v_1 or v_2 would have odd heights and therefore its two incident edges would have the same colour. □

4. Threshold below n^{-1}

Let H_2 be a forest with $e(H_2) \geq 3$. Here we prove Theorem 1.2 (ii), which gives a threshold smaller than n^{-1} for $G(n, p) \xrightarrow{\text{mr}} (H_1, H_2)$ when H_1 is a star, or when H_1 is a disconnected star forest and H_2 is a pending forest.

Recall that $m_F(H_1, H_2) = \min\{m(F) \colon F \text{ is a forest and } F \xrightarrow{\text{mr}} (H_1, H_2)\}$. Propositions 4.1 and 4.2 below imply that the parameter $m_F(H_1, H_2)$ is well defined for these particular graphs H_1 and H_2 (Corollary 4.3), from which it follows that the threshold for $G(n, p) \xrightarrow{\text{mr}} (H_1, H_2)$ is given by $n^{-1/m_F(H_1, H_2)}$. Note that the 1-statement follows from the fact that $F \xrightarrow{\text{mr}} (H_1, H_2)$ and the fact that $F \subset G(n, p)$ with high probability.

Proposition 4.1. *If H_1 is a star and H_2 is a forest, then there exists a tree T such that $T \xrightarrow{mr} (H_1, H_2)$.*

Proof. Let H_1 be a star with s edges. Consider a tree H_2' such that $V(H_2') = V(H_2)$ and $E(H_2') \subset E(H_2)$, rooted at some arbitrary vertex v. Let $e(H_2') = \ell$, and let h be the height of H_2'. We will show that an $((s - 1)(\ell - 1) + 1)$-ary tree T of height h satisfies $T \xrightarrow{\text{mr}} (H_1, H_2')$, which implies $T \xrightarrow{\text{mr}} (H_1, H_2)$.

Note that in any edge-colouring avoiding a monochromatic copy of H_1 there are at most $s - 1$ edges with any given colour at each vertex of T. Thus, the edges from every non-leaf of T to its children must be coloured with at least ℓ different colours. Therefore, a greedy embedding that starts by assigning v to the root of T and always chooses edges of previously unused colours will succeed in finding a rainbow copy of H_2' in any colouring of T that avoids a monochromatic copy of H_1. □

In the next proposition we consider the case where H_1 is a disconnected star forest and H_2 is a pending forest. Given a graph G with edges coloured by χ and $v \in V(G)$, recall that $d_{G,\chi}(v)$ denotes the numbers of colour used at edges incident to v.

Proposition 4.2. *If H_1 is a disconnected star forest and H_2 is a pending forest, then there exists a tree T such that $T \xrightarrow{mr} (H_1, H_2)$.*

Proof. Since H_2 is a pending forest, we may assume (by extending isolated edges to cherries if necessary) that H_2 is a disjoint union of cherries, as clearly an edge-colouring of a tree with a rainbow copy of a graph that contains H_2 also contains a rainbow copy of H_2. We may also assume, to simplify notation, that H_1 has s stars with s edges each and that H_2 is composed by s cherries.

We prove that a complete d-ary tree T with height $h(T) = 16s^3 + 3s$ and $d = 2s^2 + s$ satisfies $T \xrightarrow{\mathrm{mr}} (H_1, H_2)$. Let χ be an arbitrary edge-colouring of such a tree T. If there are $3s$ vertices of T with colour-degree at least $2s$, then there is a rainbow copy of the forest of cherries H_2. Otherwise, by removing these vertices we obtain a subgraph (a forest) F of T with at least $v(T) - 3s$ vertices such that for all vertices v of F we have $d_{F,\chi}(v) < 2s$. Therefore, for every internal $v \in V(F)$, since $d_F(v) \geq d_T(v) - 3s \geq 2s(s-1)$ there is a monochromatic star in F centred at v with s edges.

Note that after removing a vertex from a complete d-ary tree T', we obtain at least one complete d-ary tree of height $h(T') - 1$ and same internal degree. Then, since at most $3s$ vertices were removed from T to obtain F, we know that F contains a complete d-ary tree T' of height $16s^3$ with $d = 2s^2 + s$ with the property that for every internal $v \in V(F)$ there is a monochromatic star centred at v with s edges.

Since T' is a complete d-ary tree, there is a path P with $16s^3$ vertices such that every edge of P belongs to a monochromatic star with s edges. Supposing there is no monochromatic H_1 in T', every subpath of P with $2s$ vertices contains a rainbow cherry, and therefore P contains at least $8s^2$ rainbow cherries in total. To finish the proof, we greedily pick a rainbow cherry and delete from P all edges whose colour appear in the chosen cherry. Since every colour appears at most $2s$ times in P, this deletes at most $4s$ edges and therefore destroys at most $8s$ rainbow cherries. Since P originally contained $8s^2$ cherries, it is possible to repeat this procedure s times and find a rainbow copy of H_2, as desired. □

Corollary 4.3. *Let H_1 be a star and H_2 be a forest, or let H_1 be a disconnected star forest and H_2 be a pending forest. There is a forest F_m with a fixed number of vertices and components such that $F_m \xrightarrow{mr} (H_1, H_2)$ and for any forest F' with $m(F') < m(F_m)$ we do not have $F' \xrightarrow{mr} (H_1, H_2)$.*

Proof. Let $v(H_1, H_2) = \min\{k \in \mathbb{N} \colon$ there is a forest F with components of size at most k such that $F \xrightarrow{\mathrm{mr}} (H_1, H_2)\}$. Note that in view of Propositions 4.1 and 4.2, the parameter $v(H_1, H_2)$ is well defined. Let $\mathcal{F}(H_1, H_2)$ be the family of all forests F with components of size at most $v(H_1, H_2)$ such that $F \xrightarrow{\mathrm{mr}} (H_1, H_2)$ and consider a forest F_m with minimum number of vertices among all forests of $\mathcal{F}(H_1, H_2)$. □

We now prove the main result of this section.

Proof of Theorem 1.2 (ii). Let H_1 be a star and H_2 be a forest, or let H_1 be a disconnected star forest and H_2 a pending forest. From Corollary 4.3, there is a forest F with a fixed number of vertices and components such that $m(F) = m_F(H_1, H_2)$.

For $p \gg n^{-1/m_F(H_1,H_2)}$, the random graph $G(n, p)$ contains a copy of F with high probability by Theorem 2.2. Therefore, any colouring of $G(n, p)$ contains either a monochromatic copy of H_1 or a rainbow copy of H_2. On the other hand, if $p \ll n^{-1/m_F(H_1,H_2)}$, with high probability $G(n, p)$ is a forest with $m(G(n, p)) < m_F(H_1, H_2)$, which from the definition of $m_F(H_1, H_2)$, implies the existence of a colouring of $G(n, p)$ containing neither a monochromatic copy of H_1 nor a rainbow copy of H_2. This concludes the proof. □

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