RT-MAT 2006 - 14

# GALOIS ALGEBRAS II: REPRESENTATION THEORY

Vyacheslav Futorny and
Serge Ovsienko

**Agosto 2006**

Esta é uma publicação preliminar ("preprint").

# GALOIS ALGEBRAS II: REPRESENTATION THEORY

VYACHESLAV FUTORNY AND SERGE OVSIENKO

ABSTRACT. Representation theory is developed for the class of Galois algebras introduced recently by the authors. In particular, categories of Harish-Chandra modules are studied for integral Galois algebras which include generalized Weyl algebras, the universal enveloping algebra of $\mathrm{gl}_n$, the quantization and Yangians for $\mathrm{gl}_2$.

CONTENTS



2000 *Mathematics Subject Classification.* Primary: 16D60, 16D90, 16D70, 17B65.

## 1. Introduction

The important tools in the representation theory of algebras are the restriction of representations onto subalgebras and the induction from subalgebras. The choice of a subalgebra is essential in order to have an effective representation theory. Commutative algebra provides the following classical example. An integral extension $A \subset B$ of two commutative rings induces a surjective map $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$, i.e. the fiber of $\varphi$ is non-empty for every point of $\operatorname{Spec} A$. For example, this is the case when $A = B^G$, where $G$ is a finite subgroup of the automorphism group of $B$. Moreover, if $B$ is finite over $A$ then all fibers $\varphi^{-1}(I)$, $I \in \operatorname{Spec} A$ are finite. In particular, $\varphi$ induces a surjection from the maximal spectrum of $B$ to the maximal spectrum of $A$. Hence every character of $A$, i.e. a homomorphism into a field, can be extended to a character of any integral extension of $A$, and the number of different extensions is finite if $B$ is finite over $A$. The Hilbert-Noether theorem provides an example of such situation with $B$ being the symmetric algebra on a finite-dimensional vector space $V$ and $A$ being the $G$-invariants of $B$, where $G$ is a finite subgroup of $GL(V)$.

The primary goal of this paper is to generalize these results to the "semi-commutative" case $\Gamma \subset U$ where $U$ is an associative non-commutative Galois algebra with respect to an integral domain $\Gamma$. The canonical embedding $\Gamma \subset U$ induces a multi-valued "function" from the set $L \operatorname{Specm} U$ of left maximal ideals of $U$ to $\operatorname{Specm} \Gamma$. The goal is to find natural sufficient conditions for the fibers of this map to be non-empty and finite for any point in $\operatorname{Specm} \Gamma$. Essential techniques in the development of such approach are based on the theory of categories of Harish-Chandra $U$-modules with respect to $\Gamma$, developed in [DFO].

Let $K$ its field of fractions, $K \subset L$ a finite Galois extension, $G = G(L/K)$ the corresponding Galois group, $\mathcal{M} \subset \operatorname{Aut} L$ a separating (cf. Definition 2) submonoid. Assume that the group $G$ acts on $\mathcal{M}$ by conjugation and this action skew commutes with the action on $L$. Then $G$ acts on the skew group algebra $L * \mathcal{M}$ by isomorphisms. Denote by $L * \mathcal{M}^G$ the subalgebra of $G$-invariants in $L * \mathcal{M}$. A finitely generated $\Gamma$-subalgebra $U \subset L * \mathcal{M}^G$ is called a *Galois algebra with respect to* $\Gamma$ if $KU = UK = L * \mathcal{M}^G$ [FO]. Hence, a Galois algebra $U$ with respect to $\Gamma$ is simply a $\Gamma$-order in $L * \mathcal{M}^G$. The properties and the structure theory of Galois algebras have been studied in [FO]. Well known examples of Galois algebras include generalized Weyl algebras over integral domains with infinite order automorphisms, such as $n$-th Weyl algebra $A_n$, quantum plane, $q$-deformed Heisenberg algebra, quantized Weyl algebras, Witten-Woronowicz algebra among the others [Ba], [BavO]; the universal enveloping algebra $U(\mathrm{gl}_n)$ with respect to the Gelfand-Tsetlin subalgebra; quantized enveloping algebra $\check{U}_q(\mathrm{gl}_2)$ with respect to Gelfand-Tsetlin subalgebra [KS]; restricted Yangians with respect to Gelfand-Tsetlin subalgebras for $\mathrm{gl}_2$ [FMO].

Note that the algebra $L * \mathcal{M}^G$ has the canonical decomposition into the sum of pairwise non-isomorphic finite dimensional left or right $K$-modules (cf. (2)). For a class of Galois algebras the algebra $U$ itself decomposes into the sum of pairwise non-isomorphic finitely generated $\Gamma$-bimodules (Corollary 3.3). After the localization this decomposition coincides with the decomposition of $U[S^{-1}]$ (or $[S^{-1}]U$). These algebras satisfy some local finiteness condition and they are defined as follows.

**Definition 1.** *A Galois algebra $U$ with respect to $\Gamma$ is called* right (respectively left) integral *if for any finite dimensional right (respectively left) $K$-subspace $W \subset U[S^{-1}]$ (respectively $W \subset [S^{-1}]U$), $W \cap U$ is finitely generated right (respectively left) $\Gamma$-module. A Galois algebra is* integral *if it is both right and left integral.*

If $\Gamma \subset U \subset K \subset L$ and $U$ is finitely generated over $\Gamma$, then $U$ is clearly Galois algebra with respect to $\Gamma$. Moreover, $U$ is integral if and only if $U$ is an integral extension of $\Gamma$. All Galois algebras listed above are also examples of integral Galois algebras with respect to corresponding subalgebras. If $U$ is a Galois algebra with respect to $\Gamma$, which is free as a right (left) $\Gamma$-module then $U$ is right (left) integral (cf. Proposition 3.1).

The properties of integral Galois algebras are studied in Section 3. Their representations are discussed in Section 6.

Our first main result is the following

**Theorem A.** Let $U$ be a right integral Galois algebra with respect to an integral domain $\Gamma$, $\varphi : \Gamma \to U$ a canonical embedding and $\varphi^* : L\,\mathrm{Specm}\,U \to \mathrm{Specm}\,\Gamma$ the induced multivalued function. Then the fibers of $\varphi^*$ are non-empty for any point of $\mathrm{Specm}\,\Gamma$.

Our second main result gives sufficient conditions for the the fibers of $\varphi^*$ to be finite. Consider an induced action of $\mathcal{M}$ on $\mathrm{Specm}\,\Gamma$ and for $\mathsf{m} \in \mathrm{Specm}\,\Gamma$ denote by $\mathrm{St}_{\mathcal{M}}(\mathsf{m})$ the stabilizer of $\mathsf{m}$ in $\mathcal{M}$.

**Theorem B.** Let $\Gamma$ be an integral domain which is finitely generated as a $\Bbbk$-algebra, $U$ an integral Galois algebra with respect to $\Gamma$. If $\mathrm{St}_{\mathcal{M}}(\mathsf{m})$ is finite then the fiber $(\varphi^*)^{-1}(\mathsf{m})$ is finite.

These two theorems guarantee that an integral Galois algebra with respect to $\Gamma$ has a nice theory of Harish-Chandra modules with respect to $\Gamma$ (cf. Section 6.4 ). Moreover, integral Galois algebras allow to study effectively the whole category of modules. We are going to address this question in a subsequent paper.

The following result shows that generic maximal ideals of $\Gamma$ parametrize simple Harish-Chandra modules.

**Theorem C.** Let $\mathcal{M}$ be a group, $\Gamma$ a noetherian normal $\Bbbk$-algebra, $U$ an integral Galois $\Gamma$-algebra. Then there exists a massive subset $W \subset \mathrm{Specm}\,\Gamma$ such that for any $\mathsf{m} \in W$, $|(\varphi^*)^{-1}(\mathsf{m})| = 1$ and hence there exists a unique simple $U$-module $L_{\mathsf{m}}$ whose support contains $\mathsf{m}$. Moreover, the extension category generated by $L_{\mathsf{m}}$ contains all indecomposable modules whose support contains $\mathsf{m}$ and is equivalent to the category $\hat{\Gamma}_{\mathsf{m}} - \mathrm{mod}$ of modules over the completion of $\Gamma$ with respect to $\mathsf{m}$.

As an application of a developed theory we obtain the following generalized version of the Harish-Chandra theorem.

**Theorem D.** Let $\mathcal{M}$ be a group, $\Gamma$ a noetherian normal $\Bbbk$-algebra, $U$ an integral Galois $\Gamma$-algebra. Then for any nonzero $u \in U$ there exists a massive set of non-isomorphic simple Harish-Chandra $U$-modules on which $u$ acts nontrivially.

## 2. Preliminaries

*All fields in the paper contain the base algebraically closed field $\Bbbk$ of characteristic 0. All the algebras in the paper are $\Bbbk$-algebras. If $K$ is a field then $\bar{K}$ will denote the algebraic closure of $K$.*

2.1. **Categorical setup.** If $A$ is an associative ring then by $A - \text{mod}$ we denote the category of finitely generated left $A$-modules. Let $\mathcal{C}$ be a category, $\mathtt{i}, \mathtt{j} \in \operatorname{Ob}\mathcal{C}$. Sometimes we will write $\mathcal{C}(\mathtt{i}, \mathtt{j})$ instead of $\operatorname{Hom}_{\mathcal{C}}(\mathtt{i}, \mathtt{j})$.

Recall, that a category $\mathcal{C}$ is called the category over $\Bbbk$, provided that all $\operatorname{Hom}_{\mathcal{C}}$-sets are endowed with a structure of a $\Bbbk$-vector space and all the compositions are $\Bbbk$-bilinear.

The category of $\mathcal{C}$-modules $\mathcal{C} - \text{Mod}$ is defined as the category of $\Bbbk$-linear functors $M : \mathcal{C} \longrightarrow \Bbbk - \text{Mod}$, where $\Bbbk - \text{Mod}$ is the category of $\Bbbk$-vector spaces. The category of finitely generated $\mathcal{C}$-modules we denote by $\mathcal{C} - \text{mod}$. If $\operatorname{Ob}\mathcal{C}$ is finite, then the categories $\mathcal{C} - \text{Mod}$ and $A(\mathcal{C}) - \text{Mod}$ are equivalent.

2.2. **Integral extensions.** Details of the facts listed in this section can be found in [Mat], [AM].

Let $A$ be an integral domain, $K$ its field of fractions and $\tilde{A}$ the integral closure of $A$ in $K$. The ring $A$ is called *normal* if $A = \tilde{A}$.

**Proposition 2.1.** *Let $A$ be a normal noetherian ring, $K \subset L$ a finite Galois extension, $\bar{A}$ is the integral closure of $A$ in $L$. Then $\bar{A}$ is a finite $A$-module.*

**Corollary 2.1.**
- *If $\tilde{A}$ is noetherian then $\bar{A}$ is finite over $\tilde{A}$.*
- *If $A$ is a finitely generated $\Bbbk$-algebra then $\bar{A}$ is finite over $A$. In particular, $\tilde{A}$ is finite over $A$.*

Denote by $\operatorname{Specm} A$ ($\operatorname{Spec} A$) the space of maximal (prime) ideals in $A$, endowed with Zarisky topology. Let $\imath : A \hookrightarrow B$ be an integral extension. Then it induces a surjective map $\operatorname{Specm} B \to \operatorname{Specm} A$ ($\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$). In particular, for any character $\chi : A \to \Bbbk$ there exists a character $\tilde{\chi} : B \to \Bbbk$ such that $\tilde{\chi}|_A = \chi$. If, in addition, $B$ is finite over $A$, i.e. finitely generated as an $A$-module, then the number of different characters of $B$ which correspond to the same character of $A$, is finite. Hence we have in particular

**Corollary 2.2.** *If $A$ is a finitely generated $\Bbbk$-algebra then for any character $\chi : A \to \Bbbk$ there exists finitely many characters $\tilde{\chi} : \bar{A} \to \Bbbk$ such that $\tilde{\chi}|_A = \chi$.*

2.3. **Skew (semi)group rings.** Let $R$ be a ring, $\mathcal{M}$ a semigroup and $f : \mathcal{M} \longrightarrow \operatorname{Aut}(R)$ a homomorphism. Then $\mathcal{M}$ acts naturally on $R$: $r^g = f(g)(r)$ for $g \in \mathcal{M}, r \in R$.

The *skew semigroup ring*, $R * \mathcal{M}$, associated with the left action of $\mathcal{M}$ on $R$, is a free left $R$-module, $\bigoplus_{m\in\mathcal{M}} Rm$, with a basis $\mathcal{M}$ and with the multiplication defined as follows

$$(r_1 m_1)\cdot(r_2 m_2) = (r_1 r_2^{m_1})(m_1 m_2), \quad m_1, m_2 \in \mathcal{M},\ r_1, r_2 \in R.$$

If the action of $\mathcal{M}$ is trivial on $R$ then $R * \mathcal{M}$ coincides with the semigroup ring $R[\mathcal{M}]$. If $x \in R*\mathcal{M}$ and $m \in \mathcal{M}$ then denote by $x_m$ the element in $R$ such that $x = \sum_{m\in\mathcal{M}} x_m m$.

Assume, a finite group $G$ acts by automorphisms on $R$ and by conjugations on $\mathcal{M}$. Then $G$ acts on $R * \mathcal{M}$ and $R * \mathcal{M}^G$ will denote the invariants under this action.

Denote

$$\operatorname{supp} x = \{m \in \mathcal{M} | x_m \neq 0\}$$

the *support* of $x$. Hence $x \in R * \mathcal{M}^G$ if and only if $x_{m^g} = x_m^g$ for $m \in \mathcal{M}, g \in G$. If $x \in R * \mathcal{M}^G$ then $\operatorname{supp} x$ is a finite $G$-invariant subset in $\mathcal{M}$.

For $\varphi \in \operatorname{Aut} R$ and $a \in R$ set $H_\varphi = \{h \in G | \varphi^h = \varphi\}$ and

(1)
$$[a\varphi] := \sum_{g\in G/H_\varphi} a^g \varphi^g \in R * \mathcal{M}^G.$$

Then

(2)
$$R * \mathcal{M}^G = \bigoplus_{\varphi\in G\backslash\mathcal{M}} (R*\mathcal{M})_\varphi^G, \quad \text{where}$$
$$(R*\mathcal{M})_\varphi^G = \{[a\varphi] \,|\, a \in R^{H_\varphi}\}.$$

Clearly,

(3)
$$\gamma \cdot [a\varphi] = [(a\gamma)\varphi],\ [a\varphi]\cdot\gamma = [(a\gamma^\varphi)\varphi],\ \gamma \in R^G,$$

$$[a\varphi] = \sum_{g\in G/H_\varphi} a^g\varphi^g = \sum_{g\in G/H_\varphi} \varphi^g(g\varphi^{-1}g^{-1}ga) = \sum_{g\in G/H_\varphi} \varphi^g(a^{\varphi^{-1}})^g = [\varphi a^{\varphi^{-1}}].$$

For $a, b \in R^{H_\varphi}$, $\gamma \in R^G$ denote

(4)
$$[a\varphi b] = \sum_{g\in G/H_\varphi} a^g\varphi^g b^g,$$
$$\gamma[a\varphi b] = [(\gamma a)\varphi b] = [a\varphi(b\gamma^{\varphi^{-1}})],\ [a\varphi b]\gamma = [(\gamma^\varphi a)\varphi b] = [a\varphi(b\gamma)].$$

2.4. **Galois algebras.** We will assume that $\Gamma$ is an integral domain, $K$ is the field of fractions of $\Gamma$, $K \subset L$ is a finite Galois extension with the Galois group $G$, $\imath : K \to L$ is a natural embedding, $\bar{\Gamma}$ is the integral closure of $\Gamma$ in $L$.

**Definition 2.** (1) *Monoid* $\mathcal{M} \subset \operatorname{Aut} L$ *is called* separating *(with respect to $K$) if for any $m_1, m_2 \in \mathcal{M}$ from*

$$m_1|_K = m_2|_K$$

*follows $m_1 = m_2$.*

(2) *An automorphism* $\varphi : L \longrightarrow L$ *is called* separating *(with respect to* $K$*) if the monoid generated by* $\{\varphi^g \mid g \in G\}$ *in* $\operatorname{Aut} L$ *is separating.*

Note that if $\mathcal{M}$ is separating then $\mathcal{M} \cap G = \{e\}$. The converse holds if $\mathcal{M}$ is a group.

**Remark 2.1.** *The following conditions are equivalent*

(1) *Monoid* $\mathcal{M}$ *is separating with respect to* $K$.
(2) *For any* $m \in \mathcal{M}, m \neq e$ *there exists* $\gamma \in K$ *such that* $\gamma^m \neq \gamma$.
(3) *If* $Gm_1G = Gm_2G$ *for some* $m_1, m_2 \in \mathcal{M}$, *then there exists* $g \in G$ *such that* $m_1 = m_2^g$.

We will assume that $\mathcal{M} \subset \operatorname{Aut} L$ is a separating monoid on which $G$ acts by conjugations. Let $U$ be a Galois algebra with respect to $\Gamma$.

**Lemma 2.1.** [FO]
*Let* $u \in U$ *be nonzero element,* $T = \operatorname{supp} u$, $u = \sum_{m \in T} [a_m m]$. *Then*

$$K(\Gamma u \Gamma) = (\Gamma u \Gamma) K = KuK = \bigoplus_{m \in T} V(a_m m).$$

In particular it shows that for every $m \in \mathcal{M}$ the algebra $U$ contains the elements $[b_1 m], \dots, [b_k m]$ where $b_1, \dots, b_k$ is a $K$-basis in $L^{H_m}$.

Let $e \in \mathcal{M}$ be the unit element, $Le \subset L * \mathcal{M}$ and $U_e = U \cap Le$.

**Theorem 2.1.** [FO] *Let* $U$ *be a Galois subalgebra in* $L * \mathcal{M}$. *Then*

(1) $U_e \subset K$.
(2) $U \cap K$ *is a maximal commutative* $\Bbbk$*-subalgebra in* $U$.
(3) *The center* $Z(U)$ *of algebra* $U$ *equals* $U \cap K^{\mathcal{M}}$.

## 3. Integral Galois algebras

3.1. **Characterization of integral Galois algebras.** Let $M$ be a right $\Gamma$-submodule in a Galois algebra $U$. Set

$$\mathbb{D}_r(M) = \{u \in U \mid \text{there exists } \gamma \in \Gamma, \gamma \neq 0 \text{ such that } u \cdot \gamma \in M\}.$$

This is clearly a right $\Gamma$-module, which we call the *module of denominators* of $M$.

If $M_1, M_2$ are $\Gamma$-submodules in a $U$, then the notation $M_1 \dotplus M_2$ means $M_1 + M_2$ and $M_1 \cap M_2 = 0$.

**Lemma 3.1.** *For right* $\Gamma$*-submodules of* $U$ *holds the following.*

(1) $M \subset \mathbb{D}_r(M)$, $\mathbb{D}_r(\mathbb{D}_r(M)) = \mathbb{D}_r(M)$.
(2) $\mathbb{D}_r(M) = MK \cap U$.
(3) *If* $N \subset M$ *then* $\mathbb{D}_r(N) \subset \mathbb{D}_r(M)$.
(4) *If* $N \cap M = 0$ *then* $\mathbb{D}_r(N \dotplus M) = \mathbb{D}_r(N) \dotplus \mathbb{D}_r(M)$.
(5) $\mathbb{D}_r(\Gamma) = U_e$.

*Proof.* Statements (1) and (3) are obvious. Statements (2) and (4) follow from the fact that $U$ is torsion free over $\Gamma$. Theorem 2.1 (1) claims that $U_e \subset K$, implying (5). □

We have the following characterization of right integral Galois algebras. The case of left integral algebras is considered analogously.

**Lemma 3.2.** *A Galois algebra $U$ with respect to a noetherian $\Gamma$ is* right integral *if and only if for every finitely generated right $\Gamma$-module $M \subset U$, the right $\Gamma$-module $\mathbb{D}_r(M)$ is finitely generated.*

*Proof.* Assume $U$ is right integral. Then $MK$ is a finite dimensional right $K$-vector space, hence $D_r(M) = MK \cap M$ is finitely generated right $\Gamma$-module. Conversely, let $W \subset L * \mathcal{M}^G$ be a finite dimensional right $K$-vector space. Choose a basic $w_1, \ldots, w_n \in W$. Then for each $i = 1, \ldots, n$ there exists $\gamma_i \in \Gamma$, such that $w_i\gamma_i \in U$. Hence, for finitely generated $\Gamma$-module $M = w_1\gamma_1\Gamma + \cdots + w_n\gamma_n\Gamma$ holds $MK = W$. By conditions $D_r(M) = MK \cap M$ is finitely generated over $\Gamma$. Therefore $U$ is right integral. □

**Corollary 3.1.** *If $U$ is right (left) integral then $\Gamma \subset U_e$ is an integral extension. In particular $U_e$ is a normal ring.*

*Proof.* Lemma 3.1, (5) shows that $U_e = U \cap Le \subset K$ is finitely generated right (left) $\Gamma$-module. Moreover, it is finitely generated as left and right $\Gamma$-module simultaneously. Clearly, the statement now follows from Corollary 2.1. □

The notion of integrality of $U$ has the following immediate impact on the representation theory of $U$.

**Lemma 3.3.** *Let $M$ be a $\Gamma$-module. Then*

(1) *If $N \subset M$ is a right $\Gamma$-submodule, $\mathbb{D}_r(M) = M$ and $\mathbb{D}_r(N) = N$, then there exists a right submodule $N' \subset M$, such that $M = N \dot{+} N'$. For such submodule holds $\mathbb{D}_r(N') = N'$.*
(2) *Let $U$ be a right integral Galois algebra with respect to $\Gamma$, $\mathsf{m} \in \operatorname{Specm}\Gamma$. Then $U\mathsf{m} \neq U$, or equivalently $U \otimes_\Gamma \Gamma/\mathsf{m} \neq 0$.*

*Proof.* Choose a maximal right $\Gamma$-submodule $N' \subset M$, such that $N \cap N' = 0$. It exists by the Zorn lemma. Then for every nonzero $m \in M$ holds $N \cap (N' + m\Gamma) \neq 0$, or equivalently, for some nonzero $\gamma \in \Gamma$ holds $m\gamma \in N \dot{+} N'$. Hence $M \subset \mathbb{D}_r(N \dot{+} N') = \mathbb{D}_r(N) + \mathbb{D}_r(N') \subset \mathbb{D}_r(M) = M$. It proves (1).

To show (2) assume the opposite. Then $1 \in U\mathsf{m}$, i.e $1 = \sum_{i=1}^{n} u_i\mu_i$, $u_i \in U, \mu_i \in \mathsf{m}$. Consider the module of denominators $M = \mathbb{D}_r(\sum_{i=1}^{n} u_i\Gamma)$. Then $u_i \in M$ for all $i = 1, \ldots, n$ and $1 \in M\mathsf{m}$. Note that $M$ contains a $\Gamma$-submodule $U_e = \mathbb{D}_r(\Gamma)$. Applying (1), we obtain that $M \simeq U_e \dot{+} N$ for some right $\Gamma$-submodule $N \subset M$. Note that $\Gamma \subset U_e$ is an integral extension of finite rank, and hence $U_e\mathsf{m} \neq U_e$. In particular $1 \notin U_e\mathsf{m}$. But then $1 \notin M\mathsf{m} = U_e\mathsf{m} \dot{+} N\mathsf{m}$, which is a contradiction. □

## 3.2. Examples of integral Galois algebras.

**Example 3.1.** *Following Section 7.1 in* [FO], *commutative Galois algebras with respect to* $\Gamma$ *are just finitely generated over* $\Gamma$ *subrings in* $K$. *Such Galois algebra is integral only if the extension* $\Gamma \subset U$ *is integral. Indeed, assume that* $U$ *is integral. Let* $u \in U$ *be a non-integral element. Then* $\Gamma[u]$ *is not a finitely generated* $\Gamma$*-module. On the other hand, let*

$$a_0u^n + a_1u^{n-1} + \ldots + a_n = 0, a_0, \ldots, a_n \in \Gamma, a_0 \neq 0.$$

*Set* $M = \Gamma u^{n-1} + \Gamma u^{n-2} + \ldots + \Gamma$. *Then* $\mathbb{D}_r(M) = \Gamma[u]$, *since* $a_0^{k-n+1}u^k \in M$, *for all* $k \geq n$. *Since* $M$ *is finitely generated we obtain a contradiction with the integrality of* $U$.

*Suppose now that the extension* $\Gamma \subset U$ *is integral and* $\Gamma$ *is noetherian then immediately* $U$ *is integral over* $\Gamma$, *since any* $\Gamma$*-submodule in* $\bar{\Gamma}$ *is finitely generated.*

Next we establish the following convenient sufficient condition of the integrality.

**Proposition 3.1.** *Let* $U$ *be a Galois algebra with respect to* $\Gamma$. *If* $U$ *is free as a right (left)* $\Gamma$*-module and* $\Gamma$ *is a noetherian algebra then* $U$ *is right (left) integral.*

*Proof.* Indeed, every finitely generated right $\Gamma$-submodule $M \subset U$ belongs to $F$, where $U = F \oplus F'$, $F, F'$ are free right modules and $F$ is of finite rank. Then $\mathbb{D}_r(M) \subset F$. Moreover, it is finitely generated, since $\Gamma$ is noetherian. □

**Example 3.2.** *Recall that* $U(\mathrm{gl}_n)$ *is a Galois algebra with respect to its Gelfand-Tsetlin subalgebra,* [FO], *Corollary 7.2. Hence* $U(\mathrm{gl}_n)$ *is integral due to Proposition 3.1 and* [Ov].

**Example 3.3.** *If* $U = \mathrm{Y}_p(\mathrm{gl}_2)$ *is a restricted Yangian of level* $p$ *for* $\mathrm{gl}_2$ *(*[FMO]*) then* $U$ *is a Galois algebra with respect to the Gelfand-Tsetlin subalgebra* $\Gamma$ *(cf.*[FO], *Section 7.3.2). Moreover,* $U$ *is free over* $\Gamma$ *by* [FMO], *Theorem 3.4. Applying Proposition 3.1 we conclude that* $U$ *is integral.*

**Example 3.4.** *If* $U = D(\sigma, a)$ *is a generalized Weyl Algebra (*[FO]*), then due to Proposition 3.1* $U$ *is an integral Galois algebra.*

**Example 3.5.** *Let* $\mathcal{M}$ *be a separating subgroup in* $\mathrm{Aut}\, L$, $\mathcal{M} \cdot \bar{\Gamma} = \bar{\Gamma}$, $\varphi_1, \ldots, \varphi_n \in \mathcal{M}$ *a set of generators of* $\mathcal{M}$ *as a semigroup,* $a_1, \ldots, a_n \in \bar{\Gamma}$. *If* $\Gamma$ *is normal then the subalgebra* $U$ *in* $L * \mathcal{M}^G$ *generated by* $\Gamma$ *and* $[\varphi_1], \ldots, [\varphi_n]$ *is an integral Galois algebra with respect to* $\Gamma$. *Indeed, since* $\mathcal{M} \cdot \bar{\Gamma} = \bar{\Gamma}$, *for any* $u \in U$,

$$u = \sum_{m \in \mathcal{M}} [a_m m],$$

*all* $a_m$ *are in* $\bar{\Gamma}$. *In particular, if* $u \in U_e$ *then* $u = [a_e e]$, *where* $a_e \in K \cap \bar{\Gamma}$. *Since* $\Gamma$ *is normal then* $a_e \in \Gamma$ *and* $U_e = \Gamma$. *Applying Theorem 3.2,* (2) *we obtain the integrality of* $U$.

3.3. **Harish-Chandra subalgebras.** A $\Gamma$-bimodule $V$ we call *quasi-central* if for any $v \in V$, the $\Gamma$-bimodule $\Gamma v \Gamma$ is finitely generated both as a left and as a right $\Gamma$-module. In particular, commutative subalgebra $\Gamma \subset U$ is called a *Harish-Chandra subalgebra* in $U$ if $U$ is a quasi-central $\Gamma$-bimodule [DFO].

We have the following characterization of Harish-Chandra subalgebras in Galois algebras.

**Proposition 3.2.** *Assume that $\Gamma$ is finitely generated as an algebra over $\Bbbk$. Then $\Gamma \subset U \subset L * \mathcal{M}$ is Harish-Chandra if and only if $m \cdot \bar{\Gamma} = \bar{\Gamma}$ for every $m \in \mathcal{M}$.*

*Proof.* Note that $\bar{\Gamma}$ if finitely generated as $\Gamma$-module. Suppose first $m \cdot \bar{\Gamma} = \bar{\Gamma}$ for every $m \in \mathcal{M}$. Note that $m^{-1} \cdot \bar{\Gamma} = \bar{\Gamma}$. It is enough to prove that $\Gamma[a\varphi]\Gamma$ is quasi-central for any standard generator $[a\varphi]$ of $U$. Then

$$\Gamma[a\varphi]\Gamma = [\Gamma a\varphi\Gamma] = [\Gamma \cdot \varphi(\Gamma) a\varphi] = [a\varphi\Gamma \cdot \varphi^{-1}(\Gamma)] \tag{5}$$

is finitely generated over $\Gamma$ from the left, since $\varphi(\Gamma) \subset \bar{\Gamma}$, and it is finitely generated from the right, since $\varphi^{-1}(\Gamma) \subset \bar{\Gamma}$.

Conversely, assume $\Gamma[a\varphi]\Gamma$ is finitely generated right $\Gamma$-module for any generator $[a\varphi]$. By (5) it means that $\Gamma \cdot \varphi^{-1}(\Gamma)$ is finite over $\Gamma$, i.e. $\varphi^{-1}(\Gamma) \subset \bar{\Gamma}$. Analogously, $\varphi(\Gamma) \subset \bar{\Gamma}$. □

The following example shows that the condition $\mathcal{M} \cdot \Gamma \subset \bar{\Gamma}$ does not imply the condition $\mathcal{M}^{-1} \cdot \Gamma \subset \bar{\Gamma}$.

**Example 3.6.** *Let $\Gamma = \Bbbk[x, y]$, $K = \Bbbk(x, y)$, $L = \Bbbk(\sqrt{x}, \sqrt{y})$. Then $\bar{\Gamma} = \Gamma[\sqrt{x}, \sqrt{y}]$. Consider an element $\varphi \in \operatorname{Aut} L$ such that $x \mapsto x$, $y \mapsto xy$, $\sqrt{x} \mapsto \sqrt{x}$ and $\frac{\sqrt{y}}{\sqrt{x}} \mapsto \sqrt{y}$. We see that $\varphi(\bar{\Gamma}) \subset \bar{\Gamma}$ but $\varphi^{-1}(\bar{\Gamma})$ does not belong to $\bar{\Gamma}$.*

**Example 3.7.** *Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra, $H$ a Cartan subalgebra of $\mathfrak{g}$, $U(\mathfrak{g})$ and $U(H)$ are universal enveloping algebras of $\mathfrak{g}$ and $H$ respectively. Then $U(H)$ is a Harish-Chandra subalgebra in $U(\mathfrak{g})$. But $U(H)$ is not maximal commutative subalgebra of $U(\mathfrak{g})$. Hence $U(\mathfrak{g})$ is not Galois algebra with respect to $U(H)$ by Theorem 2.1,* (2).

**Proposition 3.3.** *If $U$ is a right (left) integral Galois algebra with respect to noetherian $\Gamma$ then for any $m \in \mathcal{M}$ holds $m^{-1}(\Gamma) \subset \bar{\Gamma}$ ($m(\Gamma) \subset \bar{\Gamma}$).*

*Proof.* Let $U$ be right integral and $[a\varphi]$ a standard generator of $U$. It is enough to check that $\varphi^{-1}(\Gamma) \subset \bar{\Gamma}$. Assume $\gamma \in \Gamma$ is such that $x = \varphi^{-1}(\gamma) \notin \bar{\Gamma}$. In particular, it implies that the right $\Gamma$-submodule of $U$,

$$M = \sum_{i=0}^{\infty} \gamma^i [a\varphi]\Gamma = \sum_{i=0}^{\infty} [a\varphi x^i \Gamma],$$

is not finitely generated. On the other hand, $x$ is an algebraic over $\Gamma$. Let $\gamma_0 x^n + \gamma_1 x^{n-1} + \cdots + \gamma_n = 0$, $\gamma_i \in \Gamma$, $\gamma_0 \neq 0$. Consider the following finitely generated right $\Gamma$-module

$$N = \sum_{i=0}^{n-1} \gamma^i [a\varphi]\Gamma = \sum_{i=0}^{n-1} [a\varphi x^i \Gamma].$$

Since $U$ is right integral then $\mathbb{D}_r(N) = M$ is finitely generated, which is a contradiction. Hence $\varphi^{-1}(\Gamma) \subset \bar{\Gamma}$. The case of left Galois algebras can be considered similarly. □

¿From Proposition 3.3 and Proposition 3.2 we immediately obtain

**Corollary 3.2.** *Let $\Gamma$ be a noetherian $\Bbbk$-algebra without zero divisors and $U$ an integral Galois algebra with respect to $\Gamma$. Then $\Gamma$ is a Harish-Chandra subalgebra in $U$.*

**Remark 3.1.** *Note that the converse statement in Proposition 3.3 is not true in general. We will show it for right integral Galois algebras. Consider the case when* $\Gamma$ *is integrally closed in* $K$ *and there is an automorphism* $\varphi : K \longrightarrow K$ *of infinite order, such that* $\varphi(\Gamma)$ *is a proper subset in* $\Gamma$. *In this case set* $V = K_{\varphi}$. *Then* $L = K$, $\mathcal{M} = \{\varphi^n | n \geq 0\}$ *and* $L * \mathcal{M}$ *is isomorphic to the skew polynomial algebra* $K[x;\varphi]$ *([MCR]). Its subalgebra* $U$ *generated by* $\Gamma$ *and* $x$ *is a Galois algebra. Let* $U_n \subset U$ *be the* $\Gamma$*-subbimodule of monomials of degree* $n \geq 0$ *and* $\Gamma_m \subset K$, $m \geq 0$, *the subalgebra generated by all* $\varphi^i(\Gamma)$, *where* $i = -m, \ldots, 0$.

*Then we have*

$$U_n = \Gamma x^n \Gamma = \Gamma x^n = x^n \Gamma_n, \ n \geq 0. \tag{6}$$

*Since* $\varphi(\Gamma)$ *is a proper subset in* $\Gamma$, *then for some* $\gamma \in \Gamma$, $a = \varphi^{-1}(\gamma) \notin \Gamma$. *Hence, for any* $n > 0$, $\Gamma_n$ *contains a non-integral over* $\Gamma$ *element* $a^n = \varphi^{-n}(\gamma)$.

*Consider a right* $\Gamma$*-module* $x\Gamma$ *generated by* $x$. *Then* $\mathbb{D}_r(x\Gamma)$ *contains* $x\Gamma[a]$, *which is not finitely generated, since the extension* $\Gamma \subset \Gamma[a]$ *is not integral. Hence* $\mathbb{D}_r(x\Gamma)$ *is not finitely generated and thus* $U$ *is not right integral. On the other hand, clearly,* $U$ *is left integral.*

**Example 3.8.** *As an example of the situation in Remark 3.1 one can consider* $\Gamma = \Bbbk[x_1, x_2], K = \Bbbk(x_1, x_2)$, *and an automorphism* $\varphi \in \operatorname{Aut} K$ *such that* $\varphi(x_1) = x_1$ *and* $\varphi(x_2) = x_1 x_2$.

3.4. **Properties of integral Galois algebras.** Let $U$ be a Galois algebra with respect to $\Gamma$.

Let $S \subset \mathcal{M}$ be a finite $G$-invariant subset. Denote $U(S) = \{u \in U \,|\, \operatorname{supp} u \subset S\}$. Obviously, it is a $\Gamma$-subbimodule in $U$ and $\mathbb{D}_r(U(S)) = \mathbb{D}_l(U(S)) = U(S)$, since the multiplication on $0 \neq \gamma \in \Gamma$ does not change the support.

For every $f \in \Gamma$ consider $f_S^r \subset \Gamma \otimes_{\Bbbk} K$ (respectively $f_S^l \subset K \otimes_{\Bbbk} \Gamma$) as follows

$$f_S^r = \prod_{s \in S} (f \otimes 1 - 1 \otimes f^{s^{-1}}) = \sum_{i=0}^{|S|} f^{|S|-i} \otimes T_i, \ (T_0 = 1). \tag{7}$$

(respectively $f_S^l = \prod_{s \in S} (f^s \otimes 1 - 1 \otimes f)$).

The following lemma describes the properties of $f_S^r$.

**Lemma 3.4.** [FO] *Let* $S \subset \mathcal{M}$ *be a* $G$*-invariant subset and* $m^{-1}(\Gamma) \subset \bar{\Gamma}$ *for all* $m \in \mathcal{M}$. *For any subset* $X \subset \mathcal{M}$ *set* $f_X = f_X^r$.

(1) *Let* $u \in U$. *Then* $u \in U(S)$ *if and only if* $f_S \cdot u = 0$ *for every* $f \in \Gamma$.

(2) *Let* $u \in U$ *and* $T = \operatorname{supp} u \setminus S$. *Then* $f_T \cdot u \in U(S)$.

(3) *Let* $S \subset T$ *be* $G$*-invariant subsets in* $\mathcal{M}$, $f \in \Gamma$, $f_{T \setminus S} = \sum_{i=1}^{n} f_i \otimes g_i \in \Gamma \otimes_{\Bbbk} \Gamma$, $a \in L$, $m \in \mathcal{M}$. *Then* $f_{T \setminus S} \cdot [am] = [(\sum_{i=1}^{n} f_i g_i^m a) m]$.

(4) *If* $f \in \Gamma$, $S = \{e\}$ *and* $f_{T \setminus S} = \sum_{i=1}^{n} f_i \otimes g_i \in \Gamma \otimes_{\Bbbk} \Gamma$, *then* $f_{T \setminus S} \cdot u = (\sum_{i=1}^{n} f_i g_i) u$.

(5) *Let $S$ be a $G$-orbit. The $\Gamma$-bimodule homomorphism* $P^T_S : U(T) \longrightarrow U(S)$, $u \mapsto f_{T\setminus S} \cdot u$ *is either zero or* $\operatorname{Ker} P^T_S = U(T \setminus S)$ *(both cases are possible, cf. (1)).*

(6) *Let* $S = S_1 \sqcup \cdots \sqcup S_n$ *be the decomposition of $S$ in $G$-orbits and* $P^S_{S_i} : U(S) \longrightarrow U(S_i)$, $i = 1, \ldots, n$ *are defined in (5) nonzero homomorphisms. Then the homomorphism*

$$P^S : U(S) \longrightarrow \bigoplus_{i=1}^{n} U(S_i),\ P^S = (P^S_{S_1}, \ldots, P^S_{S_n}), \tag{8}$$

*is a monomorphism.*

The case of $f^l_S$ is treated analogously, substituting $m^{-1}(\Gamma) \subset \bar{\Gamma}$ by $m(\Gamma) \subset \bar{\Gamma}$ in the conditions of lemma. In particular all statements are valid in the case when $\Gamma \subset U$ is a Harish-Chandra subalgebra.

**Corollary 3.3.** *Assume $U$ is a right (respectively left) integral Galois algebra with respect to $\Gamma$, $\{\mathcal{O}_i\}_{i\in\mathbb{N}}$ is an ordering of the orbits of $\mathcal{M}$ with respect to the $G$-action. Then there exists a right (respectively left) $\Gamma$-module decomposition,*

$$U = \bigoplus_{i=0}^{\infty} U_i, \tag{9}$$

$$\text{such that } \bigoplus_{i=0}^{n} U_i = U\Big(\bigsqcup_{i=0}^{n} \mathcal{O}_i\Big) \text{ for any } n \geq 0.$$

*Besides*

$$\bigoplus_{i=0}^{n} U_i K = \bigoplus_{i=0}^{n} (L * \mathcal{M})^G_{\varphi_i} \text{ (respectively } \bigoplus_{i=0}^{n} K U_i = \bigoplus_{i=0}^{n} (L * \mathcal{M})^G_{\varphi_i}),$$

*where* $\varphi_i \in \mathcal{O}_i$, $i = 0, \ldots, n$.

*Proof.* Following Lemma 3.3, (1) one can choose $U_i$ as a a right (respectively left) complement to the submodule $U(\bigsqcup_{i=0}^{n-1} \mathcal{O}_i)$ in $U(\bigsqcup_{i=0}^{n} \mathcal{O}_i)$. The necessary decomposition is obtained by induction on $n$. The second statement follows from (2). □

Our goal now is to prove the following

**Theorem 3.1.** *Let $U$ be a Galois algebra with respect to a noetherian Harish-Chandra subalgebra $\Gamma$. Then the following statements are equivalent.*

(1) *$U$ is right (respectively left) integral.*
(2) *$U(S)$ is finitely generated right (respectively left) $\Gamma$-module for any finite $G$-invariant $S \subset \mathcal{M}$.*
(3) *$U(G \cdot m)$ is finitely generated right (respectively left) $\Gamma$-module for any $m \in \mathcal{M}$.*

*Proof.* Assume $U$ is right integral. Consider a $G$-invariant finite subset $S \subset \mathcal{M}$. Since the dimension $\dim^r_K U(S)K$ is finite (cf. [FO]), there exist $u_1, \ldots, u_k \in U(S)$, which form a basis of $U(S)K$ as a right $K$-space. Then

$$\mathbb{D}_r\big(\sum_{i=1}^{k} u_i\Gamma\big) = \big(\sum_{i=1}^{k} u_i\Gamma\big)K \cap U = U(S)K \cap U = \mathbb{D}_r(U(S)) = U(S).$$

Therefore, $U(S) = \mathbb{D}_r\big(\sum_{i=1}^{k} u_i\Gamma\big)$, which proves (2). Obviously, (2) implies (3).

Assume (3) holds. We will show that $U$ is right integral. Let $M$ be a finitely generated right $\Gamma$-submodule in $U$. Then $M \subset U(S)$ for some finite $G$-invariant subset $S \subset \mathcal{M}$, and $\mathbb{D}_r(M) \subset \mathbb{D}_r(U(S))$. Since $\mathbb{D}_r(U(S)) = U(S)$, it remains to prove that $U(S)$ is finitely generated as a right $\Gamma$-module by Lemma 3.2. Let $S = S_1 \sqcup \cdots \sqcup S_n$ be the decomposition of $S$ into $G$-orbits. Then following Lemma 3.4, (6), we can identify $U(S)$ with its image under the monomorphism $P^S$. Since $U(S_i)$ is a finitely generated right $\Gamma$-module for every $i = 1, \dots, n$, we conclude that $U(S)$ is finitely generated right $\Gamma$-module, which completes the proof. □

**Theorem 3.2.** *Assume that $U \subset L * \mathcal{M}^G$ is a Galois algebra with respect to a noetherian $\Gamma$ and $\mathcal{M}$ is group.*

(1) *If $U_e$ is integral extension of $\Gamma$ and $m^{-1}(\Gamma) \subset \bar{\Gamma}$ (respectively $m(\Gamma) \subset \bar{\Gamma}$), then $U$ is right (respectively left) integral.*
(2) *If $U_e$ is integral extension of $\Gamma$ and $\Gamma$ is a Harish-Chandra subalgebra in $U$, then $U$ is integral.*

*Proof.* We will prove (1). Assume that $U_e$ is an integral extension of $\Gamma$, $m^{-1}(\Gamma) \subset \bar{\Gamma}$, but $U$ is not right integral. Following Theorem 3.1, (3) there exists $m \in \mathcal{M}$, such that $\mathbb{D}_r(M)$ is not finitely generated, where $M = U(G \cdot m)$. Consider in $\mathbb{D}_r(M)$ a strictly ascending chain of right $\Gamma$-modules

$$[mI_1] \subset [mI_2] \subset \cdots \subset \mathbb{D}_r(M), \tag{10}$$

where $I_k, k \geq 1$ are right $\Gamma$-submodules in $L$.

Since $\mathcal{M}$ is a group, then following Lemma 2.1, there exists $[bm^{-1}] \in U$. The multiplication by $[bm^{-1}]$ is injective on $[mL]$, since $([bm^{-1}][ma])_e = \dfrac{|G|}{|H_m|}ba$. Hence, multiplying (10) by $[bm^{-1}]$ from the left we obtain the strictly ascending chain of right $\Gamma$-modules

$$[bm^{-1}][mI_1] \subset [bm^{-1}][mI_2] \subset \cdots \subset [bm^{-1}]\mathbb{D}_r(M) = \mathbb{D}_r([bm^{-1}]M), \tag{11}$$

Let $S = \mathcal{O}_{m^{-1}}\mathcal{O}_m$. Since $m^{-1}(\Gamma) \subset \bar{\Gamma}$ there exists $F = \sum_{i=1}^{n} f_i \otimes g_i \in \Gamma \otimes_{\Bbbk} \Gamma$ (by Lemma 3.4, (3)), which defines a nonzero morphism $P_e^S : U(S) \longrightarrow U(\{e\}) = U_e$. Applying $P_e^S$ to the sequence (11), we obtain an infinite strictly ascending chain of right $\Gamma$-submodules in $\mathbb{D}_r(U_e)$,

$$b\gamma I_1 \subset b\gamma I_2 \subset \cdots \subset U_e,$$

where $\gamma = \sum_{i=1}^{n} f_i g_i$, which is a contradiction. Statement (2) follows immeaditely from Proposition 3.2. □

**Corollary 3.4.** *Let $U \subset L * \mathcal{M}$ be a Galois algebra over noetherian $\Gamma$. Assume that $\mathcal{M}$ is a group and $\Gamma$ is a normal $\Bbbk$-algebra. Then the following statements are equivalent*

(1) *$U$ is integral over $\Gamma$.*
(2) *$\Gamma$ is a Harish-Chandra subalgebra and, if for $u \in U$ there exists a nonzero $\gamma \in \Gamma$ such that $\gamma u \in \Gamma$ or $u\gamma \in \Gamma$, then $u \in \Gamma$.*

*Proof.* Assume (1). Then $\Gamma$ is a Harish-Chandra subalgebra by Corollary 3.2. Suppose that $u\gamma \in \Gamma$ for some $u \in U$ and $\gamma \in \Gamma$. Then (2) follows from Corollary 3.1, since $u \in \mathbb{D}_r(\Gamma) = U_e = \Gamma$. To prove the opposite implication consider $u \in U_e$. Since $U_e \subset K$ (Theorem 2.1, (1)), there exists $\gamma \in \Gamma$, such that $\gamma u \in \Gamma$. Thus, $u \in \Gamma$. Theorem 3.2, (2) completes the proof. □

The last corollary can be viewed as a non-commutative analogue of the following statement, which is probably well known. For the convenience of the reader we include the proof.

**Proposition 3.4.** *Let $i : A \subset B$ be an embedding of integral domains over $\Bbbk$, such that $A$ is non-singular. If the induced morphism of varieties $i^*$ : $\operatorname{Specm} B \to \operatorname{Specm} A$ is surjective then for any $b \in B$ such that $ab \in A$, for some nonzero $a \in A$, follows $b \in A$.*

*Proof.* We can assume that $i$ induces an epimorphism of the $\operatorname{Spec} B$ onto $\operatorname{Spec} A$ and will use the following property of non-singular rings: for every $\mathsf{m} \in \operatorname{Specm} A$ the localization $A_{\mathsf{m}}$ is a unique factorization domain. Assume $ab = a' \in A$ and fix $\mathsf{m} \in \operatorname{Specm} A$. Consider this equality in the ring $B_{\mathsf{m}}$. We can assume that $a$ and $a'$ are coprime in $A_{\mathsf{m}}$. If $a$ is invertible in $A_{\mathsf{m}}$ then $b \in A_{\mathsf{m}}$. In the opposite case there exists $P \in \operatorname{Spec} A$ such that $a \in P$ and $a' \notin P$, which shows that $P$ does not lift to the point of $\operatorname{Spec} B$. Since $b \in A_{\mathsf{m}}$ for every $\mathsf{m} \in \operatorname{Spec} A$, it implies $b \in A$. □

In particular, Proposition 3.4 holds in the case of an integral extension $A \subset B$ with nonsingular $A$.

## 4. Harish-Chandra categories

4.1. **Harish-Chandra modules.** Denote by $\operatorname{Specm}\Gamma$ the set of maximal ideals of $\Gamma$. A module $M \in U-\operatorname{mod}$ is called *Harish-Chandra module (with respect to $\Gamma$)*, provided that $M|_\Gamma$ is a direct sum of a finite dimensional $\Gamma$-modules $\bigoplus_{\mathsf{m} \in \operatorname{Specm}\Gamma} M(\mathsf{m})$, where $\mathsf{m}^k M(\mathsf{m}) = 0$ for some $k = k(\mathsf{m}) \geq 0$.

When for all $\mathbf{m} \in \operatorname{Specm}\Gamma$ and all $x \in M(\mathbf{m})$ holds $\mathbf{m}x = 0$ such Harish-Chandra module $M$ is called *weight module (with respect to $\Gamma$)*.

All Harish-Chandra modules form a full abelian subcategory $\mathbb{H}(U, \Gamma)$ in $U-\operatorname{mod}$. A full subcategory of $\mathbb{H}(U, \Gamma)$ consisting of weight modules we denote by $\mathbb{H}W(U, \Gamma)$ . The support of a Harish-Chandra module $M$ is a set $\operatorname{supp} M \subset \operatorname{Specm}\Gamma$ consisting of such $\mathbf{m}$ that $M(\mathbf{m}) \neq 0$. For $D \subset \operatorname{Specm}\Gamma$ denote by $\mathbb{H}(U,\Gamma,D)$ the full subcategory in $\mathbb{H}(U,\Gamma)$ formed by $M$ such that $\operatorname{supp} M \subset D$. For a given $\mathbf{m} \in \operatorname{Specm}\Gamma$ let $\chi_{\mathbf{m}} : \Gamma \to \Gamma/\mathbf{m}$ be a character of $\Gamma$. If there exists an irreducible Harish-Chandra module $M$ with $M(\mathbf{m}) \neq 0$ then we say that $\chi_{\mathbf{m}}$ extends to $M$. Since for any character $\chi : \Gamma \to \Bbbk$, $\operatorname{Ker}\chi \in \operatorname{Specm}\Gamma$, we will identify the set of all characters of $\Gamma$ with $\operatorname{Specm}\Gamma$.

Suppose that $\Gamma$ is a Harish-Chandra subalgebra in the algebra $U$. For $a \in U$ let

(12)
$$X_a = \{(\mathbf{m},\mathbf{n}) \in \operatorname{Specm}\Gamma \times \operatorname{Specm}\Gamma \mid \Gamma/\mathbf{n} \text{ is a subquotient of } \Gamma a\Gamma/\Gamma a\mathbf{m} \iff (\Gamma/\mathbf{n}) \otimes_\Gamma \Gamma a \Gamma \otimes_\Gamma (\Gamma/\mathbf{m}) \neq 0\}$$

Denote by $\Delta$ the minimal equivalence on $\operatorname{Specm}\Gamma$ containing all $X_a$, $a \in U$ and by $\Delta(U,\Gamma)$ the set of the $\Delta-$equivalence classes on $\operatorname{Specm}\Gamma$. Then for any $a \in U$ and $\mathbf{m} \in \operatorname{Specm}\Gamma$ holds

(13)
$$aM(\mathbf{m}) \subset \sum_{(\mathbf{m},\mathbf{n}) \in X_a} M(\mathbf{n}), \quad \mathbb{H}(U,\Gamma) = \bigoplus_{D \in \Delta(U,\Gamma)} \mathbb{H}(U,\Gamma,D).$$

In particular if $X$ is a finite-dimensional $\Gamma-$module then the module $U \otimes_\Gamma X$ is a Harish-Chandra module.

4.2. **Correspondences associated with a bimodule.** The situation described above allows the following generalization to the case of prime ideals. Let $\mathsf{p} \in \operatorname{Spec}\Gamma$, $S = S_\mathsf{p} = \Gamma \setminus \mathsf{p}$, $\varphi \in \operatorname{Aut}\Gamma$, $T = T_{\Gamma,\varphi} = \prod_{g \in G/H_\varphi} \varphi^g(S)$.

Assume $\Gamma$ is noetherian. Let $M$ be a finitely generated left (resp. right) module over $\Gamma$. For $m \in M$ denote $\operatorname{Ann}_\Gamma(m)$ the ideal of $\gamma \in \Gamma$ such that $\gamma m = 0$ (resp. $m\gamma = 0$). By $\operatorname{Ass}(M)$ ($= \operatorname{Ass}_\Gamma(M) \subset \operatorname{Spec}\Gamma$) we denote the set of prime ideals $\mathsf{p}$ in $\Gamma$ associated with $M$, i.e. there exists $m \in M$, such that $\operatorname{Ann}_\Gamma(m) = \mathsf{p}$. In particular any maximal annihilator is in $\operatorname{Spec}\Gamma$.

Let $\Gamma$ be a commutative ring, $V$ a quasi-central $\Gamma$-bimodule. Denote by $\mathfrak{X}_V \subset \operatorname{Spec}\Gamma \times \operatorname{Spec}\Gamma$ the associated with $V$ relation

(14)
$$\mathfrak{X}_V = \{(\mathsf{p},\mathsf{q}) \mid \mathsf{p} \in \operatorname{Spec}\Gamma,\ \mathsf{q} \in \operatorname{Ass}(V \otimes_\Gamma \Gamma/\mathsf{p})\}.$$

**Remark 4.1.** *Note that $\mathfrak{X}_V$ can be dually defined as*

(15)
$$\mathfrak{X}_V = \{(\mathsf{p},\mathsf{q}) \mid \mathsf{q} \in \operatorname{Spec}\Gamma,\ \mathsf{p} \in \operatorname{Ass}(\Gamma/\mathsf{q} \otimes_\Gamma V)\}.$$

For $\mathsf{p} \in \operatorname{Spec}\Gamma$ denote by $\Gamma_\mathsf{p}$ the localization of $\Gamma$ by the multiplicative set $S_\mathsf{p} = \Gamma \setminus \mathsf{p}$. Abusing notation we will denote again by $\mathsf{p}$ the corresponding ideal in $\Gamma_\mathsf{p}$. Denote by $K_\mathsf{q}$ the fraction field $\Gamma_\mathsf{p}/\mathsf{p}$.

Define the category $\mathcal{A} = \mathcal{A}(U,\Gamma)$ as follows

(16)
$$\operatorname{Ob}\mathcal{A} = \operatorname{Spec}\Gamma; \ \mathcal{A}(\mathsf{p},\mathsf{q}) = \varprojlim_{l,m} \mathcal{A}_{l,m}(\mathsf{p},\mathsf{q}), \text{ where}$$
$$\mathcal{A}_{l,m}(\mathsf{p},\mathsf{q}) = \Gamma_\mathsf{q}/\mathsf{q}^m \otimes_\Gamma U \otimes_\Gamma \Gamma_\mathsf{p}/\mathsf{p}^l.$$

Any element $x \in \Gamma_{\mathsf{q}} \otimes_\Gamma U \otimes_\Gamma \Gamma_{\mathsf{p}}$ can be presented in the form $x = \gamma_1 \otimes u_1 \otimes 1$ and in the form $x = 1 \otimes u_2 \otimes \gamma_2$, for some $\gamma_1, \gamma_2 \in \Gamma$, $u_1, u_2 \in U$. We prove the first statement, the second is analogous. For $s_{\mathsf{q}}^{-1} \otimes u \otimes s_{\mathsf{q}}$ there exists $u' \in U, s \in \Gamma$, such that $s_q u' = us$. Then $s_{\mathsf{q}}^{-1} \otimes u \otimes s_{\mathsf{q}}^{-1} = s_{\mathsf{q}}^{-1} \otimes s_{\mathsf{q}} u' \otimes s^{-1} s_{\mathsf{q}}^{-1} = 1 \otimes u' \otimes s^{-1} s_{\mathsf{q}}^{-1}$.

**Lemma 4.1.** *Assume, that if for nonzero* $u \in U$ *holds* $(\mathsf{p}, \mathsf{q}) \notin \mathfrak{X}_V$, *where Set*

$$V = \Gamma \otimes_\Gamma u \otimes_\Gamma \Gamma \simeq \Gamma u \Gamma,$$

*then* $\mathcal{A}(\mathsf{p}, \mathsf{q}) = 0$.

*Proof.* Consider $u \in U$ such that its class in $\mathcal{A}_{l,m}(\mathsf{p}, \mathsf{q})$ is nonzero. There is enough to prove, that if $(\mathsf{p}, \mathsf{q}) \notin \mathfrak{X}_V$, then $W = \Gamma_{\mathsf{q}}/\mathsf{q}^m \otimes_\Gamma u \otimes_\Gamma \Gamma_{\mathsf{p}}/\mathsf{p}^l = 0$. Remark, that $\Gamma_{\mathsf{q}}/\mathsf{q}^m \otimes_{\Bbbk} \Gamma_{\mathsf{p}}/\mathsf{p}^L$-module $W$ allows a filtration, induced by multiplication on the ideal $\mathbf{m} = \mathsf{q}/\mathsf{q}^m \otimes_{\Bbbk} \Gamma_{\mathsf{p}}/\mathsf{p}^L + \Gamma_{\mathsf{q}}/\mathsf{q}^m \otimes_{\Bbbk} \mathsf{p}/\mathsf{p}^L$

$$W \supset \mathbf{m} \cdot W \supset \mathbf{m}^2 \cdot W \supset \cdots \supset W \cdot \mathbf{m}^{m+n-1} \supset W \cdot \mathbf{m}^{m+n} = 0.$$

The factors of this filtration are isomorphic to the factors of the $\Gamma_{\mathsf{q}}/\mathsf{q}^m \otimes_{\Bbbk} \Gamma_{\mathsf{p}}/\mathsf{p}^L$-module $K_{\mathsf{q}} \otimes_\Gamma V \otimes_\Gamma K_{\mathsf{p}}$. Hence it is enough to prove, that $K_{\mathsf{q}} \otimes_\Gamma V \otimes_\Gamma K_{\mathsf{p}} = 0$. Consider any $s \in \mathsf{q}$, then there exist $s_1 \Gamma, u_1 \in V$, such that $su_1 = us_1$. Hence in $K_{\mathsf{q}} \otimes_\Gamma V \otimes_\Gamma K_{\mathsf{p}}$ holds $\bar{u} \cdot s_1$, where $\bar{u}$ is the class of $u$. If $s_1 \notin \mathsf{p}$, then $s_1$ acts bijectively, hence $\bar{u} = 0$. So $s_1 \in \mathsf{p}$ and $\mathsf{p} \in \operatorname{Ass}(W/\mathsf{q}W)$. □

The composition of morphisms is defined as follows. Let $a \in \mathcal{A}(\mathsf{p}, \mathsf{q}), b \in \mathcal{A}(\mathsf{q}, \mathsf{r})$. Choose for any $l, m, n \in \mathbb{N}$ their representatives $1 \otimes a_{l,m} \otimes s_{\mathsf{p}}^{-1} \in \mathcal{A}_{l,m}(\mathsf{p}, \mathsf{q})$ and $s_{\mathsf{r}}^{-1} \otimes a_{m,n} \otimes 1 \in \mathcal{A}_{m,n}(\mathsf{q}, \mathsf{r})$. Set $(ba)_{l,n} = \varprojlim_m s_{\mathsf{r}}^{-1} \otimes b_{m,n} a_{l,m} \otimes s_{\mathsf{p}}^{-1}$. We prove that the limit exists, i.e. there exists $M = M(a, b, p, q)$ such that for $m > M$ the element $s_{\mathsf{r}}^{-1} \otimes b_{m,n} a_{l,m} \otimes s_{\mathsf{p}}^{-1} \in \mathcal{A}_{l,m}(\mathsf{p}, \mathsf{r})$ does not depend on $m$ and on the choice of $a_{m,n}$ and $b_{l,m}$. This follows from the fact that there exists $M$, such that for every $m > M$ holds

$$1 \otimes U \mathsf{q}^m U \otimes 1 \subset \mathsf{r}^n \otimes U + U \mathsf{p}^l.$$

We define the functor $F : U - \mathrm{mod} \longrightarrow \mathcal{A} - \mathrm{mod}$ as follows:

$$F(M)(\mathsf{p}) = \varprojlim_n \Gamma_{\mathsf{p}}/\mathsf{p}^n \otimes_\Gamma M, \; M \in U - \mathrm{mod}, \mathsf{p} \in \operatorname{Spec} \Gamma,$$

$$\text{for } f \in \operatorname{Hom}_U(M, N) \; F(f) = \varprojlim_n \mathbb{1}_{\Gamma_{\mathsf{p}}/\mathsf{p}^n} \otimes_\Gamma f : F(M) \longrightarrow F(N). \tag{17}$$

It is easy to check, that $F$ is a functor.

4.3. **Case of the maximal spectrum.** Let $\Gamma$ be an integral domain and a $\Bbbk$-algebra and $U$ a Galois algebra with respect to $\Gamma$. We assume that $\Gamma$ is a Harish-Chandra subalgebra in $U$.

Define a category $\mathcal{A} = \mathcal{A}_{U,\Gamma}$ with objects $\operatorname{Ob} \mathcal{A} = \Gamma$ and the space of morphisms from $\mathbf{m}$ to $\mathbf{n}$ being

$$\mathcal{A}(\mathrm{m},\mathrm{n}) = \lim_{\leftarrow n,m} U/(\mathrm{n}^n U + U\mathrm{m}^m). \tag{18}$$

Then we have $\mathcal{A} = \bigoplus_{D\in\Delta(U,\Gamma)} \mathcal{A}_D$, where $\mathcal{A}_D$ is the restriction of $\mathcal{A}$ on $D$. For $\mathsf{m} \in \operatorname{Specm}\Gamma$ denote by $D(\mathsf{m})$ denote the class of $\Delta$-equivalence, containing $\mathsf{m}$. The category $\mathcal{A}$ is endowed with the topology of the inverse limit and the category of $\Bbbk$-vector spaces ($\Bbbk$ – mod) with the discrete topology. Consider the category $\mathcal{A} - \mathrm{mod}_d$ of continuous functors $M : \mathcal{A}\longrightarrow\Bbbk - \mathrm{mod}$ ( discrete modules in [DFO], 1.5). For any discrete $\mathcal{A}$–module $N$ define a Harish-Chandra $U$–module $\mathbb{F}(N) = \oplus_{\mathrm{m}\in\operatorname{Specm}\Gamma} N(\mathrm{m})$ and for $x \in N(\mathrm{m})$ and $a \in U$ define

$$ax = \sum_{\mathrm{n}\in\operatorname{Specm}\Gamma} a_{\mathrm{n}}x$$

where $a_{\mathrm{n}}$ is the image of $a$ in $\mathcal{A}(\mathrm{m},\mathrm{n})$. If $f : M\longrightarrow N$ is a morphism in $\mathcal{A} - \mathrm{mod}_d$ then define $\mathbb{F}(f) = \oplus_{\mathrm{m}\in\operatorname{Specm}\Gamma} f(\mathrm{m})$. Hence we have a functor $\mathbb{F} : \mathcal{A} - \mathrm{mod}_d \longrightarrow \mathbb{H}(U,\Gamma)$.

**Theorem 4.1.** *([DFO], Theorem 17) The functor $\mathbb{F}$ is an equivalence.*

We will identify a discrete $\mathcal{A}$-module $N$ with the corresponding Harish-Chandra module $\mathbb{F}(N)$. Let $\Gamma_{\mathrm{m}} = \lim_{\leftarrow m} \Gamma/\mathrm{m}^m$ be the completion of $\Gamma$ by $\mathrm{m} \in \operatorname{Specm}\Gamma$. Then the space $\mathcal{A}(\mathrm{m},\mathrm{n})$ has a structure of $\Gamma_{\mathrm{n}} - \Gamma_{\mathrm{m}}$-bimodule

For $\mathrm{m} \in \operatorname{Specm}\Gamma$ denote by $\hat{\mathrm{m}}$ a completion of $\mathrm{m}$. Consider a two-sided ideal $I \subset \mathcal{A}$ generated by $\hat{\mathrm{m}}$ for all $\mathrm{m} \in \operatorname{Specm}\Gamma$ and set $\mathcal{A}(W) = \mathcal{A}/I$. Then Proposition 4.1 implies the following statement.

**Corollary 4.1.** *The categories $\mathbb{H}W(U,\Gamma)$ and $\mathcal{A}(W) - \mathrm{mod}_d$ are equivalent.*

The subalgebra $\Gamma$ is called *big* in $\mathrm{m} \in \operatorname{Specm}\Gamma$ if $\mathcal{A}(\mathrm{m},\mathrm{m})$ is finitely generated as $\Gamma_{\mathrm{m}}$–module.

The importance of the concept of a big subalgebra is described in the following statement.

**Lemma 4.2.** *([DFO], Corollary 19) If $\Gamma$ is big in $\mathrm{m} \in \operatorname{Specm}\Gamma$ then there exists finitely many non-isomorphic irreducible Harish-Chandra $U$–modules $M$ such that $M(\mathrm{m}) \neq 0$. For any such module $M$, $\dim M(\mathrm{m}) < \infty$.*

Note that the regular $\mathcal{A}$-module does not belong to the category $\mathcal{A} - \mathrm{mod}_d$. This leads to the following generalization of the category $\mathbb{H}(U,\Gamma)$. A $U$-module is called *topologized Harish-Chandra module* (with respect to $\Gamma$), if $M|_\Gamma$ is a direct sum of $\Gamma$-modules $\bigoplus_{\mathsf{m}\in\operatorname{Specm}\Gamma} M(\mathsf{m})$, such that $M(\mathsf{m})$ is a complete separated (i.e. Hausdorff) in $\mathsf{m}$-adic topology. A morphism $f : M \to N$ of two such modules is a homomorphism such that $f = \oplus_{\mathsf{m}\in\operatorname{Specm}\Gamma} f_{\mathsf{m}}$, where $f_{\mathsf{m}} : M(\mathsf{m}) \to N(\mathsf{m})$ is continuous in $\mathsf{m}$-adic topology. The category of such modules $T\mathbb{H}(U,\Gamma)$ contains $\mathbb{H}(U,\Gamma)$ as a full subcategory. When necessary we will work within of the category $T\mathbb{H}(U,\Gamma)$.

A functor $F : \mathcal{A} \to \Bbbk - \mathrm{Mod}$ is called topologized if $F(\mathsf{m})$ is complete and separated in $\mathsf{m}$-adic topology for any $\mathsf{m} \in \operatorname{Specm}\Gamma$. Let $T\mathcal{A} - \mathrm{Mod}$ be the category of topologized functors. Then

$$T\mathbb{H}(U,\Gamma) \simeq T\mathcal{A} - \mathrm{Mod}\,.$$

We will show next that an integral Galois algebra acts faithfully in the category of Harish-Chandra modules. First we need the following lemma.

**Lemma 4.3.** *Let $\Gamma$ be noetherian and $M$ a finitely generated right $\Gamma$-module. Then the set of $\mathsf{m} \in \operatorname{Specm}\Gamma$ such that $\operatorname{Tor}_1^\Gamma(M,\Gamma/\mathsf{m}) = 0$ contains an open dense subset in $\operatorname{Specm}\Gamma$.*

*Proof.* Let

$$R^\bullet \;:\; \ldots \xrightarrow{d^2} \Gamma^{n_2} \xrightarrow{d^1} \Gamma^{n_1} \xrightarrow{d^0} \Gamma^{n_0} \longrightarrow 0 \ldots \tag{19}$$

Let (19) be a free resolution of $M$. It induces the resolution $R^\bullet \otimes_\Gamma K$ of $M \otimes_\Gamma K$. Denote $r = \dim_K \operatorname{Im}(d^1 \otimes \mathbb{1}_K) = \operatorname{Ker}(d^0 \otimes \mathbb{1}_K)$. Denote by $D_i$ the matrix of $d^i$ and for $\mathsf{m} \in \operatorname{Specm}\Gamma$ by $D_i(\mathsf{m})$ the specialization of $D_i$ in $\mathsf{m}$, $i = 1, 2$. Then always

$$\operatorname{rank} D_2(\mathsf{m}) \leq r$$

and the set

$$V = \{\mathsf{m} \in \operatorname{Specm}\Gamma \mid \operatorname{rank} D_2(\mathsf{m}) = r\}$$

is open dense. Analogously,

$$\operatorname{rank} D_1(\mathsf{m}) \leq n_1 - r$$

and the set

$$V' = \{\mathsf{m} \in \operatorname{Specm}\Gamma \mid \operatorname{rank} D_1(\mathsf{m}) = n_1 - r\}$$

is open dense in $\operatorname{Specm}\Gamma$. Hence for any $\mathsf{m} \in V \cap V'$ the first cohomology of the complex $R^\bullet \otimes_\Gamma \Gamma/\mathsf{m}$ equals 0. □

**Proposition 4.1.** *Let $U$ be an integral Galois algebra with respect to a noetherian algebra $\Gamma$. Then for every $u \in U, u \neq 0$ the set $\Omega_u$ of $\mathsf{m} \in \operatorname{Specm}\Gamma$ for which there exists $\mathsf{n} \in \operatorname{Specm}\Gamma$, such that the image of $u$ in $\mathcal{A}(\mathsf{m},\mathsf{n})$ is nonzero, contains a massive subset.*

*Proof.* We prove a stronger statement: there exists a massive set $X_u \subset \operatorname{Specm}\Gamma$ such that for every $\mathsf{m} \in X_u$ the image $\bar{u}$ of $u$ in $U/U\mathsf{m}$ is nonzero. Fix $\mathsf{m} \in \operatorname{Specm}\Gamma$ and let $N = u\Gamma \simeq \Gamma$. Then $\bar{u} = 0$ if and only if

$$u = \sum_{i=1}^{n} u_i m_i, \;\; u_i \in U, m_i \in \mathsf{m}, \;\; i = 1, \ldots, n.$$

Assume this the case. Let $S = \bigcup_{i=1}^{n} \operatorname{supp} u_i$ and $M = U(S)$. Then the exact sequence of right $\Gamma$-modules

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0 \tag{20}$$

becoms non-exact after tensoring with $\Gamma/\mathrm{m}$, i.e. $\mathrm{Tor}_1^\Gamma(M/N,\Gamma/\mathrm{m}) \neq 0$. But the set $U(u,S)$ of $\mathrm{m} \in \mathrm{Specm}\,\Gamma$, such that $\mathrm{Tor}_1^\Gamma(M/N,\Gamma/\mathrm{m}) = 0$ contains an open in $\mathrm{Specm}\,\Gamma$ subset (Lemma 4.3). Hence for the massive set

$$X_u = \bigcap_{S \subset \mathcal{M}} U(u,S)$$

the element $u \in U$ acts non-trivially on $U/U\mathrm{m}$. Now the statement follows from Theorem 4.1. □

## 5. Representations of Galois algebras

Let $U$ be a Galois algebra with respect to $\Gamma$ in $L * \mathcal{M}$.

5.1. **Extension of characters.** We would like to know for which $\mathrm{m} \in \mathrm{Specm}\,\Gamma$, the character $\chi = \chi_{\mathrm{m}}$ extends to an irreducible Harish-Chandra $U$-module. Denote by $U_\chi$ the left module $U/(U \,\mathrm{Ker}\,\chi)$. We will call $U_\chi$ *the universal module, generated by a $\chi$-eigenvector.* As we saw above $U_\chi \in \mathbb{H}(U,\Gamma)$ if $\Gamma$ is a Harish-Chandra subalgebra. The problem is that in general we can not guarantee this. Moreover, $U_\chi$ could be zero.

It is more convenient to work with the following extension of $\Gamma$. Denote by $\mathbb{L}$ the subalgebra in $L$ generated by all $\gamma^m$, where $m$ runs $\mathcal{M}$ and $\gamma$ runs $\Gamma$. If $\mathbb{L} \subset \bar{\Gamma}$, i.e. every $m \in \mathcal{M}$ is integral, then any character on $\Gamma$ can be extended to a character on $\mathbb{L}$. In this case the extension $\Gamma \subset \mathbb{L}$ is analogous to the extension $Sym[x_1,\ldots,x_n] \subset \Bbbk[x_1,\ldots,x_n]$. Denote by $\bar{\mathbb{L}}$ the integral closure of $\mathbb{L}$ in $L$. Then $L$ is a field of fractions of $\bar{\mathbb{L}}$. If $\Gamma \subset \mathbb{L}$ is an integral extension then $\bar{\Gamma} = \bar{\mathbb{L}}$ and any character of $\Gamma$ can be extended to a character on $\bar{\mathbb{L}}$.

Let $\mathcal{L} = \mathrm{Specm}\,\bar{\mathbb{L}}$. The elements $\ell$ of $\mathcal{L}$ will be called *tableau.* The canonical embedding $\Gamma \hookrightarrow \bar{\mathbb{L}}$ induces the projection $\pi : \mathcal{L} \longrightarrow \mathrm{Specm}\,\Gamma$. The Galois group $G$ acts on $\mathcal{L}$ and the orbits of this action are in the canonical bijection with $\mathrm{Specm}\,\Gamma$, i.e. for $\mathrm{m} \in \mathrm{Specm}\,\Gamma$ the group $G$ acts transitively on $\pi^{-1}(\mathrm{m})$ (cf. Proposition 2.1, chapter VII, [La] ). If $\mathrm{m} \in \mathrm{Specm}\,\Gamma$ and $|G \cdot \mathrm{m}| = |G|$ then the tableau $\ell \in \mathcal{L}$, such that $\pi(\ell) = \mathrm{m}$, will be called *regular*; otherwise the tableau $\ell$ is called *non-regular.* If $\mathrm{m} \in \mathrm{Specm}\,\Gamma$ then we will denote by $\ell_{\mathrm{m}}$ an element of $\mathcal{L}$ such that $\pi(\ell_{\mathrm{m}}) = \mathrm{m}$. We will say that $\ell_{\mathrm{m}}$ lies over $\mathrm{m}$.

Since $\Gamma$ is a subalgebra in $\bar{\mathbb{L}}$, we can for $\gamma \in \Gamma$ and $\ell \in \mathcal{L}$ write $\gamma(\ell)$ instead of $\gamma(\pi(\ell))$. If $\varphi \in \mathrm{Aut}\,\bar{\mathbb{L}}$ and $\gamma \in \Gamma$, then there holds $\gamma^\varphi(\ell) = \gamma(\varphi^{-1} \cdot \ell)$.

We will use the following localizations of $\bar{\mathbb{L}}$. Let $A^{\mathcal{M}}$ be the set of all $a^m$, where $m \in \mathcal{M}$ and $[a\varphi]$ run all standard generators of $U$. Denote by $\Lambda_1$ an algebra generated over $\bar{\mathbb{L}}$ by $A^{\mathcal{M}}$, and let $\Lambda_2$ be an algebra generated over $\Lambda_1$ by all $a^{-1}$, $a \in A^{\mathcal{M}}$. Denote $\mathcal{L}_i = \mathrm{Specm}\,\Lambda_i$, $\Omega_i = \pi(\mathcal{L}_i)$, $i = 1,2$. Then we have the following standard embeddings:

$$\mathcal{L}_2 \subset \mathcal{L}_1 \subset \mathcal{L}, \quad \Omega_2 \subset \Omega_1 \subset \mathrm{Specm}\,\Gamma.$$

By $\mathcal{L}_r \subset \mathcal{L}$ denote the set of $\ell = \ell_{\mathrm{m}}$, such that $\mathcal{M}$ acts on $\ell$ without stabilizer and $\mathcal{M}\cdot\ell \cap G\cdot\ell$ consists just of $\ell$. In other words, for $\mathrm{m} = \pi(\ell_{\mathrm{m}}) \in \mathrm{Specm}\,\Gamma$ holds $S(\mathrm{m},\mathrm{m}) = \{e\}$ (see 6.1) Set $\Omega_r = \pi(\mathcal{L}_r)$.

The following useful fact is obvious.

**Lemma 5.1.** *If $\ell_1, \ell_2 \in \mathcal{L}$ belong to different orbits of $G$, then there exists $\gamma \in \Gamma$, such that $\gamma(\ell_1) \neq \gamma(\ell_2)$, in other words $\Gamma$ distinguishes the orbits of $G$.*

Let $\chi$ be a character of $\bar{\mathbb{L}}$, and hence of $\Gamma$, $\mathsf{m} = \operatorname{Ker}\chi$. It defines a representation $M_\chi$ of $\bar{\mathbb{L}} * \mathcal{M} \subset L * \mathcal{M}$ as follows:

$$M_\chi = (\bar{\mathbb{L}} * \mathcal{M}) \otimes_{\bar{\mathbb{L}}} (\bar{\mathbb{L}}/\mathsf{m}) \simeq \bigoplus_{\mathsf{n} \in \mathcal{O}_\mathsf{m}} \bar{\mathbb{L}}/\mathsf{n}, \tag{21}$$

where $\mathcal{O}_\mathsf{m}$ is the orbit of $\mathsf{m}$ in $\operatorname{Specm} \bar{\mathbb{L}}$ with respect to the action of $\mathcal{M}$. For any Harish-Chandra module $M$ generated by a $\chi$-eigenvector $v \in M$, $\operatorname{supp} M \subset \mathcal{O}_\mathsf{m}$. In particular, if $U_\chi \neq 0$ then $U_\chi$ is a Harish-Chandra module and $\operatorname{supp} U_\chi \subset \mathcal{O}_\mathsf{m}$. In this case $U_\chi$ is isomorphic to $M_\chi$ as $\bar{\mathbb{L}} * \mathcal{M}$-module.

**Example 5.1.** *In the case of Generalized Weyl algebras, $\mathbb{L} = \Gamma$, and the the structure of $U$-module on $M_\chi$ can be defined for any character $\chi$ of $\Gamma$, identifying $M_\chi$ with the universal module $U_\chi$ (cf. [BBF]).*

Consider a skew semigroup algebra $\Lambda_1 * \mathcal{M} \subset L * \mathcal{M}$. Clearly $\Lambda_1 * \mathcal{M}$ contains the Galois algebra $U$. If $\tilde{\chi} : \Lambda_1 \to \mathbb{k}$ is a character then we can construct the universal $\Lambda_1 * \mathcal{M}$-module

$$M(\tilde{\chi}) = (\Lambda_1 * \mathcal{M}) \otimes_{\Lambda_1} \Lambda_1 / \operatorname{Ker} \tilde{\chi}.$$

Denote by $M(U, \tilde{\chi})$ the restriction of $M(\tilde{\chi})$ on $U$. The properties of $U$-module $M(U, \tilde{\chi})$ are collected in the following statement. If $M = \oplus_{m \in \mathcal{M}} M^m$ is a $\mathcal{M}$-graded module then its support $\Omega = \{m \in \mathcal{M} | M^m \neq 0\}$ is called oriented connected if for any $\varphi, \psi \in \Omega$ there exists $u \in U$ such that $0 \neq u M^\varphi \subset M^\psi$.

**Theorem 5.1.** *Let $\tilde{\chi} : \Lambda_1 \to \mathbb{k}$ be a character, $\chi = \tilde{\chi}|_\Gamma$.*

1. *$M(U, \tilde{\chi})$ is a Harish-Chandra (with respect to $\Gamma$) $U$-module.*
2. *$U_\chi \neq 0$ and $U_\chi \subset M(U, \tilde{\chi})$.*
3. *Module $U_\chi$ has a unique $\mathcal{M}$-graded maximal submodule and unique graded irreducible quotient.*
4. *The module $U_\chi$ is graded irreducible if and only if its support $\operatorname{supp}_\mathcal{M} U_\chi$ as a $\mathcal{M}$-graded module, is oriented connected.*

*Proof.* The module $M(U, \tilde{\chi})$ is a Harish-Chandra module by construction. It has a $U$-submodule isomorphic to $U_\chi$, which is obviously nonzero. Also, $U_\chi$ is $\mathcal{M}$-graded module with 1-dimensional components. Note that this gradation may not coincide with the gradation by $\mathcal{M} \cdot \chi$ as a Harish-Chandra module. This happens when some $\varphi \in \mathcal{M}$ act periodically on $\chi$. As a result the components in $\mathcal{M} \cdot \chi$-gradation can be more than 1-dimensional. Since all components in $\mathcal{M}$-gradation are 1-dimensional, $U_\chi$ has a unique $\mathcal{M}$-graded maximal submodule which does not intersect the $\chi$-component. The basis elements of $U_\chi$ are labelled by the elements of $\mathcal{M}$ and thus $\operatorname{supp}_\mathcal{M} U_\chi = \mathcal{M}$. Clearly, $U_\chi$ is generated by any $\mathcal{M}$-graded component if and only if its support is oriented connected. □

**Corollary 5.1.** *A character $\chi : \Gamma \to \mathbb{k}$ can be extended to an irreducible Harish-Chandra module if $\chi = \tilde{\chi}|_\Gamma$ for some character $\tilde{\chi} : \Lambda_1 \to \mathbb{k}$.*

Recall, that a non-empty set $X \subset \operatorname{Specm}\Gamma$ is called *massive*, provided that $X$ is a complement of countable many subvarieties of $X$ of nonzero codimension. If the field $\Bbbk$ is uncountable then a massive set is dense (in Zariski topology) in $\operatorname{Specm}\Gamma$. We will show that there exists a massive subset of characters in $\operatorname{Specm}\Gamma$ which can be extended to Harish-Chandra $U$-modules. We will use the following standard fact.

**Lemma 5.2.** *Let $\pi : \operatorname{Specm}\bar{\mathbb{L}} \hookrightarrow \operatorname{Specm}\Gamma$ be the canonical projection. If $X \subset \operatorname{Specm}\bar{\mathbb{L}}$ is a massive subset then $\pi(X)$ is massive in $\operatorname{Specm}\Gamma$.*

*Proof.* Since $X$ is massive in $\operatorname{Specm}\bar{\mathbb{L}}$ then $X = \cap_{i\in\mathbb{Z}} U_i$, where $U_i$ is open in $\operatorname{Specm}\bar{\mathbb{L}}$ for any $i \in \mathbb{Z}$. Moreover, $\pi(U_i)$ contains an open set $U_i' \subset \operatorname{Specm}\Gamma$ for every $i$, and hence

$$\pi(X) \supset \bigcap_{i\in\mathbb{Z}} U_i',$$

i.e. $\pi(X)$ is massive in $\operatorname{Specm}\Gamma$. □

**Corollary 5.2.** *Suppose that $\mathcal{M}\cdot\Gamma \subset \bar{\Gamma}$.*

(1) *Given $\chi : \Gamma \to \Bbbk$ there exists finitely many (possible none) $\tilde{\chi} : \bar{\Gamma}[A^{\mathcal{M}}] \to \Bbbk$ such that $\chi = \tilde{\chi}|_{\Gamma}$.*
(2) *There exists a massive set $X \subset \operatorname{Specm}\Gamma$ such that any $\chi \in X$ can be extended to a character $\tilde{\chi} \in \operatorname{Specm}\bar{\Gamma}[A^{\mathcal{M}}]$ such that $\chi = \tilde{\chi}|_{\Gamma}$.*

*Proof.* We have that $\mathbb{L} \subset \bar{\Gamma}$ and $\bar{\mathbb{L}} = \bar{\Gamma}$. Hence any character of $\Gamma$ has finitely many extensions to the characters of $\bar{\Gamma}$, and for any character of $\bar{\Gamma}$ there either exists a unique extension to a character of $\bar{\Gamma}[A^{\mathcal{M}}]$ or none. This implies (1). The statement (2) follows from the fact that $L$ is the field of fractions of $\bar{\Gamma}$. □

Note, that if $\mathcal{M}\cdot\Gamma \subset \bar{\Gamma}$ then any $m \in \mathcal{M}$ defines an automorphism of $\bar{\mathbb{L}} = \bar{\Gamma}$ and, hence induces a continuous automorphism of $\mathcal{L}$. In particular, this holds when $\Gamma$ is a Harish-Chandra subalgebra.

**Lemma 5.3.** *Suppose that the field $\Bbbk$ is uncountable. Then the sets $\mathcal{L}_i \subset \mathcal{L}$ and $\Omega_i \subset \operatorname{Specm}\Gamma$, $i = 1, 2$, are massive. Moreover, if $\mathcal{M}\cdot\Gamma \subset \bar{\Gamma}$ then $\mathcal{L}_r$ and $\Omega_r$ are massive.*

*Proof.* Note that $\Lambda_1$, $\Lambda_2$ are countably generated over $\bar{\mathbb{L}}$. Due to Lemma 5.2 it is enough to show that the corresponding subsets are massive in $\operatorname{Specm}\bar{\mathbb{L}}$.

The sets $\mathcal{L}_1$ and $\mathcal{L}_2$ can be characterized in the following way. Let $Z_1, \ldots, Z_N$ be the canonical generators of $U$ ($Z_i = [a_i\varphi_i]$, $i = 1, \ldots, N$). Let $I = (i_1, \ldots, i_k) \in \{1, \ldots, N\}^k$, $k \geq 0$, $w_I = Z_{i_1} \ldots Z_{i_k} = \sum_{h\in\mathcal{M}} [x_h^I h]$, $x_h^I \in L$. If $x_h^I \neq 0$ then it defines a rational function on $\operatorname{Specm}\bar{\mathbb{L}}$. Let $D(x_h^I)$ be the domain of regularity of $x_h^I$, and $O_0(I) = \cap_{h\in\mathcal{M}} D(x_h^I)$. Set

$$O_1(I) = \{p \in O_0(I) \mid x_h^I(p) \neq 0 \text{ for any } h \in \operatorname{supp} w_I\},$$

where $\operatorname{supp} w_I$ consists of those $h \in \mathcal{M}$ for which $x_h^I \neq 0$. Note that both $O_0(I)$ and $O_1(I)$ are nonempty open sets in $\operatorname{Specm}\bar{\mathbb{L}}$.

Denote by $\mathcal{J}$ the space of all sequences $I = (i_1, \ldots, i_k)$ for all $k \geq 0$. Then $\mathcal{L}_1 = \bigcap_{I \in \mathcal{J}} O_0(I)$ and $\mathcal{L}_2 = \bigcap_{I \in \mathcal{J}} O_1(I)$. Hence $\mathcal{L}_i \subset \operatorname{Specm} \bar{\mathbb{L}}$ is massive for $i = 1, 2$.

For any $m \in \mathcal{M}, m \neq e$, set

$$\mathcal{X}_m = \{\ell \in \mathcal{L} | m \cdot \ell \in G \cdot \ell\}.$$

Then $\mathcal{X}_m$ is a proper closed subset in $\mathcal{L}$. It is obviously closed since $G$ is finite. If $m \in \mathcal{M}$ and $g \in G$ then denote by $\mathcal{L}(m, g)$ the set of those $\ell$ for which $m \cdot \ell = g \cdot \ell$. Hence, $\mathcal{X}_m = \cup_{g \in G} \mathcal{L}(m, g)$. Assume that $\mathcal{L} = \mathcal{X}_m$ for some $m \in \mathcal{M}$. Since the variety $\operatorname{Specm} \bar{\mathbb{L}}$ is irreducible, we conclude that $\mathcal{L}(m, g) = \mathcal{L}$ for some $g \in G$, and hence $m = g$. But this is impossible, since $\mathcal{M}$ is separating. Thus $\bigcup_{m \in \mathcal{M}, m \neq e} \mathcal{X}_m$ is the complement of $\Omega_r$ in $\mathcal{L}$ and $\mathcal{L}_r$ is massive. The sets $\Omega_i$, $i = 1, 2, r$, are massive by Lemma 5.2. □

We have the following stronger version of Theorem 5.1.

**Theorem 5.2.** *Suppose that the field $\Bbbk$ is uncountable.*

(1) *There exists a massive subset $X_1 \subset \operatorname{Specm} \Gamma$, such that for every $\chi \in X_1$, $U_\chi$ is nonzero Harish-Chandra module and* $\operatorname{supp} U_\chi \subset \mathcal{O}_{\mathsf{m}}$, *where $\chi = \chi_{\mathsf{m}}$.*
(2) *If $\mathcal{M}$ is a group, then there exists a massive set $X_2 \subset X_1$, such that for any $\chi \in X_2$ the module $U_\chi$ is a unique $\mathcal{O}_{\mathsf{m}}$-graded irreducible $U$-module generated by a $\chi$-eigenvector and* $\operatorname{supp} U_\chi = \mathcal{O}_{\mathsf{m}}$.
(3) *If $\mathcal{M}$ is a group and $\mathcal{M} \cdot \Gamma \subset \bar{\Gamma}$, then there exists a massive set $X_r \subset X_2$, such that for any $\chi \in X_r$ the module $U_\chi$ is irreducible $U$-module with all 1-dimensional components. In this case there is a canonical isomorphism of $\Bbbk$-vector spaces $\Bbbk\mathcal{M} \simeq U_\chi$.*

*Proof.* Let $X_1 = \Omega_1$. Then for every $\chi \in X_1$, $U_\chi \neq 0$ by Theorem 5.1. Hence $U_\chi$ is a Harish-Chandra module. Moreover, since $U_\chi$ is $\operatorname{Specm} \bar{\mathbb{L}}$-graded, it has an irreducible quotient with a nonzero $\chi$-eigenvector. This implies (1).

Assume now that $\mathcal{M}$ is a group and set $X_2 = \Omega_2$. Let $\chi \in X_2$ and $Z = \sum_{h \in \mathcal{M}} [x_h h]$ is a generator of $U$. By assumption, for every $\mathsf{n} \in \mathcal{M} \cdot \mathsf{m}$ holds $x_h(\mathsf{n}) \neq 0$, hence every component of $U_\chi$, $\chi = \chi_{\mathsf{m}}$, generates the whole $U_\chi$. Therefore, $U_\chi$ is irreducible as $\mathcal{O}_{\mathsf{m}}$-graded $U$-module. Moreover, since $U_\chi$ is the universal module generated by a $\chi$-eigenvector, it is a unique such graded irreducible module, implying statement (2). Note that if $\mathcal{M}$ acts on $\mathsf{m}$ with a nontrivial stabilizer then $U_\chi$ is not irreducible.

Suppose now that $\mathcal{M} \cdot \Gamma \subset \bar{\Gamma}$. Then $\Omega_r$ is massive by Lemma 5.3. Consider a subset $X_r = X_2 \cap \Omega_r$. Since $\Gamma$ distinguishes the components by Lemma 5.1, it implies the irreducibility of $U_\chi$ for any $\chi \in X_r$. The basis elements of $U_\chi$ in this case are labelled by the elements of $\Bbbk\mathcal{M}$ which completes the proof of (3). □

## 6. Representations of integral Galois algebras

6.1. **Extension of characters for integral algebras.** We are in the position now to prove **Theorem A** stated in the Introduction.

Let $U$ be a right integral Galois algebra with respect to $\Gamma$. Consider an arbitrary character $\chi : \Gamma \to \Bbbk$ and let $\mathsf{m} = \operatorname{Ker}\chi \in \operatorname{Specm}\Gamma$. Then by Lemma 3.3, (2) the module $U/U\mathsf{m}$ is nonzero. Denote by $v$ the image of 1 in $U/U\mathsf{m}$. Then $\mathsf{m}v = 0$ which defines a gradation on $U/U\mathsf{m}$ by $\operatorname{Specm}\Gamma$. Any non-zero graded simple quotient of $U/U\mathsf{m}$ satisfies the theorem. Therefore, there exists a simple $U$-module $M$ extending the character $\chi$ and proving Theorem A.

The following corollary generalizes Theorem A for $\operatorname{Spec}\Gamma$.

**Corollary 6.1.** *If $U$ is right integral then for any $\mathsf{p} \in \operatorname{Spec}\Gamma$ there exists a $U$-module $N$, such that $\mathsf{p} \in \operatorname{Ass}_\Gamma(N)$.*

*Proof.* If $N = U\otimes_\Gamma K_\mathsf{p} \neq 0$ and $\bar{1} = 1\otimes 1 \in N$ then $\Gamma\cdot\bar{1} \simeq \Gamma/\mathsf{p}$. Note that $N \simeq (U/U\mathsf{p})[S_\mathsf{p}^{-1}]$. Hence $\bar{1} = 0$ in $N$ means that $s \in U\mathsf{p}$ for some $s \in S_\mathsf{p}$. Write

$$s = \sum_{i=1}^{n} u_i p_i,\ \ u_i \in U, p_i \in \mathsf{p}. \tag{22}$$

Then there exists $\mathsf{m} \in \operatorname{Specm}\Gamma$ such that $p_i(\mathsf{m}) = 0$, $i = 1,\dots,n$, $s(\mathsf{m}) \neq 0$. Consider the character $\chi : \Gamma \longrightarrow \Gamma/\mathsf{m}$ and a simple $U$-module $M$ with a nonzero element $v$ such that $\mathsf{m}v = 0$. Applying the equality (22) to $v$ we obtain a contradiction. □

Of course the property of a Galois algebra to be right integral is not a necessary condition to guarantee an extension of an arbitrary character of $\Gamma$ to a $U$-module. On the other hand we have the following

**Lemma 6.1.** *Let $U \subset L * \mathcal{M}$ be a Galois algebra with respect to a noetherian $\Gamma$. If every character $\chi : \Gamma \longrightarrow \Bbbk$ extends to a representation of $U$ then $U_e \subset \bar{\Gamma} \cap K$. If in addition $\mathcal{M}$ is a group and $\Gamma$ is a Harish-Chandra subalgebra then $U$ is integral.*

*Proof.* If $\chi$ extends to a representation of $U$, then it extends to a representation of $U_e \subset K$ in particular. It implies that $U_e$ belongs to the integral closure of $\Gamma$ in $K$. The second statement follows immediately from Theorem 3.2. □

The following corollary gives a module-theoretic characterization of integral Galois algebra.

**Corollary 6.2.** *Let $U$ be a Galois algebra with respect to a noetherian algebra $\Gamma$, $\mathcal{M}$ a group and $m^{-1}(\Gamma) \subset \bar{\Gamma}$ for any $m \in \mathcal{M}$. Then every character $\chi : \Gamma \to \Bbbk$ lifts to a simple left (right) $U$-module if and only if $U$ is right (left) integral.*

6.2. **Harish-Chandra modules for integral Galois algebras.** We assume that $\Gamma$ is normal and that it is finitely generated as an algebra over $\Bbbk$. In particular, $\Gamma = \tilde{\Gamma} = U_e$ and $\bar{\Gamma}$ is finite over $\Gamma$ by Corollary 2.1.

Let $\ell_{\mathsf{m}}$ and $\ell_{\mathsf{n}}$ be some maximal ideals of $\bar{\Gamma}$, lying over $\mathsf{m}$ and $\mathsf{n}$ correspondingly. Note that given $\mathsf{m} \in \operatorname{Specm}\Gamma$ the number of different $\ell_{\mathsf{m}}$ is finite due to Corollary 2.2. Monoid $\mathcal{M}$ acts on both $\operatorname{Specm}\Gamma$ and $\operatorname{Specm}\bar{\Gamma}$. Denote by $S(\mathsf{m},\mathsf{n})$ the following $G$-invariant subset in $\mathcal{M}$

$$S(\mathsf{m},\mathsf{n}) = \{m \in \mathcal{M} \mid \ell_{\mathsf{n}} \in GmG \cdot \ell_{\mathsf{m}}\} = \{m \in \mathcal{M} \mid \ell_{\mathsf{n}}\ell_{\mathsf{m}}^{-1} \in GmG\}. \tag{23}$$

Note that the set $S(\mathsf{m},\mathsf{n})$ can be empty and it does not depend on the choice of $\ell_{\mathsf{m}}$ and $\ell_{\mathsf{n}}$. Really, if $\ell'_{\mathsf{m}}, \ell'_{\mathsf{n}}$, lying over $\mathsf{m}$ and $\mathsf{n}$ correspondingly, then ([Mat], Theorem 9.3, III)) there exist $g', g''$, such that $\ell'_{\mathsf{m}} = g'\ell_{\mathsf{m}}$, $\ell'_{\mathsf{n}} = g''\ell_{\mathsf{n}}$. Hence $\ell'_{\mathsf{m}}(\ell'_{\mathsf{n}})^{-1}$ belongs to $GmG$.

Given $\mathsf{m} \in \operatorname{Specm}\Gamma$ denote by $\operatorname{St}_{\mathcal{M}}(\mathsf{m})$ the stabilizer of $\mathsf{m}$ (as a set) in $\mathcal{M}$.

**Lemma 6.2.** *Let $\mathsf{m} \in \operatorname{Specm}\Gamma$. The set $S(\mathsf{m},\mathsf{m})$ is finite if and only if $\operatorname{St}_{\mathcal{M}}(\mathsf{m})$ is finite. Moreover, if $\mathcal{M}$ is a group and $\operatorname{St}_{\mathcal{M}}(\mathsf{m})$ is finite, then for any $\mathsf{n} \in \operatorname{Specm}\Gamma$ both $S(\mathsf{m},\mathsf{n})$ and $S(\mathsf{n},\mathsf{m})$ are finite.*

*Proof.* Since the Galois group $G$ is finite the proof follows immediately from Corollary 2.2. □

For $\mathsf{m} \in \operatorname{Specm}\Gamma$ denote by $\hat{\Gamma}_{\mathsf{m}}$ the completion of $\Gamma$ by $\mathsf{m}$.

**Proposition 6.1.** *Let $U$ be an integral Galois algebra with respect to $\Gamma$. Then for any $\mathsf{m},\mathsf{n} \in \operatorname{Specm}\Gamma$, such that $S(\mathsf{m},\mathsf{n})$ is finite, the $\hat{\Gamma}_{\mathsf{n}} - \hat{\Gamma}_{\mathsf{m}}$-bimodule*

$$\mathcal{A}(\mathsf{m},\mathsf{n}) = \varprojlim_{n,m} U/(\mathsf{n}^n U + U\mathsf{m}^m) \tag{24}$$

*is finitely generated.*

The proof of the proposition is based on the following lemma.

**Lemma 6.3.** *Let $\Gamma$ be a Harish-Chandra subalgebra in $U$, $\mathsf{m},\mathsf{n} \in \operatorname{Specm}\Gamma$, $S = S(\mathsf{m},\mathsf{n})$, $m, n \geq 0$. Then*

$$U = U(S) + \mathsf{n}^n U + U\mathsf{m}^m.$$

*Proof.* Fix $u \in U$ and denote $T = \operatorname{supp} u \setminus S$. If $T = \varnothing$ then $u \in U(S)$. Let $T \neq \varnothing$. We show by induction in $k$, that there exists $u_k \in U(S)$, such that

$$u \in u_k + \sum_{i=0}^{k} \mathsf{n}^{k-i} u \mathsf{m}^i, \ u_k \in U(S) \text{ ( hence } u_{k+1} - u_k \in \sum_{i=0}^{k} \mathsf{n}^{k-i} u \mathsf{m}^i). \tag{25}$$

Since $\ell_{\mathsf{m}}^t$ and $\ell_{\mathsf{n}}$ belong to different $G$-orbits if $t \notin S$, then by Lemma 5.1 there exists $f \in \Gamma$ such that $f(\ell_{\mathsf{n}}) \neq f(\ell_{\mathsf{m}}^t)$ for every $t \in T$. Without loss of generality we can assume that $f_T(\mathsf{n},\mathsf{m}) = \prod_{t\in T}(f(\ell_{\mathsf{n}}) - f^{t^{-1}}(\ell_{\mathsf{m}})) = 1$, which implies $f_T \in 1 + \mathsf{n} \otimes \Gamma + \Gamma \otimes \mathsf{m}$. Set $u_1 = f_T \cdot u$. Then $u_1$ belongs to $u + \mathsf{n}u\Gamma + \Gamma u\mathsf{m}$ and, hence, $u \in u_1 + \mathsf{n}u\Gamma + \Gamma u\mathsf{m}$. Moreover, by Lemma 3.4, (2) $u_1 \in U(S)$.

We prove the induction step $k \Rightarrow k+1$. Writing in (25) the expression for $u$ in the right hand side we obtain

$$u \in u_k + \sum_{i=0}^{k} \mathsf{n}^{k-i}(u_k + \sum_{j=0}^{k} \mathsf{n}^{k-j} u \mathsf{m}^j)\mathsf{m}^i \subset u_k + \sum_{i=0}^{k} \mathsf{n}^{k-i} u_k \mathsf{m}^i + \sum_{i=0}^{k+1} \mathsf{n}^{k+1-i} u \mathsf{m}^i,$$

that finishes the proof of the induction step, since $u_k + \sum_{i=0}^{k} \mathsf{n}^{k-i} u_k \mathsf{m}^i \subset U(S)$. □

In the assumptions of Proposition 6.1 we have

$$\begin{aligned} \mathcal{A}(\mathsf{m},\mathsf{n}) &= \lim_{\leftarrow n,m} U/(\mathsf{n}^n U + U\mathsf{m}^m) \simeq \\ &\lim_{\leftarrow n,m} U(S)/(\mathsf{n}^n U + U\mathsf{m}^m) \cap U(S). \end{aligned} \tag{26}$$

Since $U(S)$ is a noetherian $\Gamma$-bimodule by Theorem 3.1, the generators of $U(S)$ as a $\Gamma$-bimodule generate any $\lim\limits_{\leftarrow n,m} U(S)/(\mathsf{n}^n U + U\mathsf{m}^m) \cap U(S)$ as a $\Gamma$-bimodule, and hence generate $\mathcal{A}(\mathsf{m},\mathsf{n})$ as $\hat{\Gamma}_{\mathsf{n}} - \hat{\Gamma}_{\mathsf{m}}$-bimodule. This completes the proof of Proposition 6.1.

**Corollary 6.3.** *Let $\Gamma$ be a normal finitely generated $\Bbbk$-algebra, $U$ integral $\Gamma$-algebra. If for some $\mathsf{m} \in D$ the group $\mathrm{St}_{\mathcal{M}}(\mathsf{m})$ is finite then for every $M \in \mathbb{H}(U,\Gamma,D)$ the space $M(\mathsf{m})$ is finite dimensional. Moreover, if in addition $\mathcal{M}$ is a group then for any $\mathsf{n} \in D$ the space $M(\mathsf{n})$ is finite dimensional.*

*Proof.* By Lemma 6.2 it is enough to prove that if $x \in M(\mathsf{m})$ and $S(\mathsf{m},\mathsf{n})$ is finite, then $\mathcal{A}_D(\mathsf{m},\mathsf{n}) \cdot x$ is finite dimensional. But this follows immediately from Proposition 6.1. □

6.3. **Proof of Theorem B.** We will show that under some conditions for integral Galois algebras there exists (up to isomorphism) finitely many simple Harish-Chandra modules extending a given character of $\Gamma$ (hence we will prove **Theorem B**).

Let $U$ be an integral Galois algebra $U$ with respect to $\Gamma$, $\mathsf{m} \in \operatorname{Specm}\Gamma$. Assume that $\Gamma$ is finitely generated over $\Bbbk$ and $\mathrm{St}_{\mathcal{M}}(\mathsf{m})$ is finite. Then $S(\mathsf{m},\mathsf{m})$ is finite by Lemma 6.2. Consider $\chi : \Gamma \longrightarrow \Bbbk$ such that $\mathsf{m} = \operatorname{Ker}\chi$. If $\Gamma$ is not normal then $\tilde{\Gamma}$ is a finite $\Gamma$-module and $\chi$ admits finitely many extensions to $\tilde{\Gamma}$, by Corollary 2.2. Hence, it is enough to prove the statement in the case $\tilde{\Gamma} = \Gamma$. But then Proposition 6.1 implies that $\Gamma$ is big in $\mathsf{m}$. By Lemma 4.2 there exists only finitely many non-isomorphic extensions of $\chi$ to simple $U$-modules, which completes the proof of Theorem B.

6.4. **Harish-Chandra categories for integral Galois algebras.** In this subsection we study in details the category of Harish-Chandra modules over integral $U$. We assume that $\Gamma$ is finitely generated normal $\Bbbk$-algebra.

Assume that $\Omega_2$ and $\Omega_r$ are as in 5.1.

**Theorem 6.1.** (1) *If $\mathsf{m},\mathsf{n} \in \operatorname{Specm}\Gamma$ and $S(\mathsf{m},\mathsf{n}) = \varnothing$, then $\mathcal{A}(\mathsf{m},\mathsf{n}) = 0$.*

(2) *Let $a \in U$, $a = \sum_{i=1}^{n} [a_i m_i]$, $m_i \in \mathcal{M}$, $a_i \in L^*$ and*

$$\mathcal{X}(a) = \{(m_i \cdot \ell^g, \ell) \,|\, \ell \in \mathcal{L}, i = 1, \ldots, n, g \in G\}.$$

*Then* $X_a \subset (\pi \times \pi)(\mathcal{X}(a))$.

(3) *If* $\mathsf{m} \in \Omega_r$, *then* $\mathcal{A}(\mathsf{m}, \mathsf{m})$ *is a homomorphic image of* $\hat{\Gamma}_{\mathsf{m}}$. *In particular, there exists a unique up to isomorphism irreducible* $U$*-module* $M$, *extending the character* $\chi : \Gamma \longrightarrow \Gamma/\mathsf{m}$.

(4) *Let* $\mathsf{m} \in \Omega_r$, $D = D(\mathsf{m})$, $M_{\mathsf{m}} = \mathcal{A}_D/\mathcal{A}_D\hat{\mathsf{m}}$, *where* $\hat{\mathsf{m}} \subset \hat{\Gamma}_{\mathsf{m}}$ *is the completed ideal. Then* $U/U\mathsf{m}$ *is canonically isomorphic to* $\mathbb{F}(M_{\mathsf{m}})$.

(5) *Let* $\mathcal{M}$ *be a group,* $\mathsf{m} \in \Omega_r \cap \Omega_2$. *Then for every* $\mathsf{n} \in D(\mathsf{m})$,

$$\mathcal{A}(\mathsf{n}, \mathsf{n}) \simeq \hat{\Gamma}_{\mathsf{n}},$$

*and all objects of* $\mathcal{A}_D$ *are isomorphic.*

*Proof.* The statement (1) follows from Lemma 6.3 and (18).

To prove (2) note, that the $\Gamma$-bimodule $\Gamma a \Gamma$ is a factor of $\bigoplus_{i=1}^{n} \Gamma[am_i]\Gamma$, so it is enough to prove, that if for $\mathsf{m}, \mathsf{n} \in \operatorname{Specm}\Gamma$, $\Gamma/\mathsf{n}$ is a subfactor of $\Gamma[am]\Gamma/\mathsf{m}$, then $(\mathsf{m}, \mathsf{m}) \in (\pi \times \pi)(\mathcal{X}(a))$. Since $\operatorname{Ext}^1$-spaces between non-isomorphic simples in commutative case are zero, we can consider just factors instead of subfactors.

Consider the canonical embedding of $\Gamma$-subbimodules in $L * \mathcal{M}$

$$i : \Gamma[am]\Gamma \hookrightarrow \bar{\Gamma} \otimes_\Gamma \Gamma[am]\Gamma \otimes_\Gamma \bar{\Gamma} \simeq \bar{\Gamma}[am]\bar{\Gamma}.$$

Then, since $\bar{\Gamma}$ is a faithfully flat (i.e. $\bar{\Gamma}$ is flat and $\bar{\Gamma} \otimes_\Gamma M \neq 0$ for any nonzero $\Gamma$-module $M$, [Mat], Theorem 7.2) $\Gamma$-module, every $\chi : \Gamma[am]\Gamma \longrightarrow \Bbbk$ lifts to

$$\bar{\chi} : \bar{\Gamma} \otimes_\Gamma \Gamma[am]\Gamma \otimes_\Gamma \bar{\Gamma} \longrightarrow \Bbbk.$$

Hence, every simple factor of $\Gamma[am]\Gamma$ as $\Gamma$-bimodule factorizes through $i$. Consider a homomorphism of $\Gamma$-bimodules

$$\bar{\Gamma} \otimes_{\Bbbk} \bar{\Gamma} \xrightarrow{\pi} \bar{\Gamma}[am]\bar{\Gamma} \xrightarrow{\bar{\chi}} \Bbbk.$$

Then the composition is just the pair $(m \cdot \ell^g, \ell)$, where $\ell \in \mathcal{L}$ and $g \in G$. It proves (2).

To prove the statement (3) we note that, by Lemma 6.3 and by (18), $\mathcal{A}(\mathsf{m}, \mathsf{m})$ is generated as $\hat{\Gamma}_{\mathsf{m}}$-bimodule by the central element $\bar{e}$, which is the class of $e \in U$. On other hand, there exists the canonical complete algebra homomorphism $i : \hat{\Gamma}_{\mathsf{m}} \longrightarrow \mathcal{A}(\mathsf{m}, \mathsf{m})$, $i(1) = \bar{e}$, which is clearly surjective.

Obviously, $\mathcal{A}_D-\mathrm{mod}_d$ is equivalent to the full subcategory $\mathbb{F}(\mathcal{A}_D-\mathrm{mod}_d) \subset U-\mathrm{mod}$. For $\mathsf{m} \in \operatorname{Specm}\Gamma$ consider the functor $W_{\mathsf{m}} : U-\mathrm{mod} \longrightarrow \Bbbk-\mathrm{Mod}$, which sends $M \in U-\mathrm{mod}$ to

$$W_{\mathsf{m}}(M) = \{m \in M \mid \mathsf{m} \cdot m = 0\}.$$

This is obviously a representable functor, namely $W_{\mathsf{m}} \simeq \operatorname{Hom}_U(U/U\mathsf{m}, -)$. On the other hand, for an indecomposable $N$, which does not belong to the image of $\mathbb{F}$, holds

$$\operatorname{Hom}_U(\mathbb{F}(M_{\mathsf{m}}), N) = 0,$$

and for $N = \mathbb{F}(N')$ we have

$$\mathrm{Hom}_U(\mathbb{F}(M_{\mathsf{m}}),\mathbb{F}(N')) \simeq \mathrm{Hom}_{\mathcal{A}}(M_{\mathsf{m}},N') \simeq W_{\mathsf{m}}(\mathbb{F}(N')) = W_{\mathsf{m}}(N),$$

where all isomorphisms are functorial, i.e, $U/U\mathsf{m} \simeq \mathbb{F}(M_{\mathsf{m}})$, implying (4).

Let us prove (5). As proved, $\mathcal{A}(\mathsf{m},\mathsf{m})$ is a homomorphic image of $\hat{\Gamma}_{\mathsf{m}}$. Analogously to (21) consider in $\mathbb{F}(\mathcal{A}_{D(\mathsf{m})} - \mathrm{mod}_d)$ a $U$-module:

$$M_{\chi,n} = (\bar{\mathbb{L}} * \mathcal{M}) \otimes_{\bar{\mathbb{L}}} (\bar{\mathbb{L}}/\mathsf{m}^n) \simeq \bigoplus_{\mathsf{n}\in\mathcal{O}_{\mathsf{m}}} \bar{\mathbb{L}}/\mathsf{n}^n.$$

Any nonzero element from $\Gamma_{\mathsf{m}}$ acts nontrivially on $M_{\chi,n}$ for any $n$. Thus $\mathcal{A}(\mathsf{m},\mathsf{m}) \simeq \Gamma_{\mathsf{m}}$ by the Krull intersection theorem ([Mat], Theorem 8.10, (II)).

To prove that all objects are isomorphic, it is enough to show, that for every standard generator $[a_+\varphi]$ of $U$ the objects $\mathsf{m}$ and $\mathsf{n} = \pi(\varphi^g \cdot \ell_{\mathsf{m}})$ are isomorphic. Since $\mathcal{M}$ is a group there exists a generator $[a_-\varphi^{-1}]$ of $U$ and we can consider the element $x = [a_-\varphi^{-1}][a_+\varphi]$, which has the coefficient $a_-a_+^{\varphi^{-1}}$ by $e$. Then, as in Lemma 3.4, there exists a function $f_+ \in \Gamma$, such that $f_+(\ell_{\mathsf{n}}^{\varphi}) = 1$ and $f_+(\mathsf{n}') = 0$ on all other $\mathsf{n}' = \pi(\varphi^g \cdot \ell_{\mathsf{m}})$, $g \in G$. Since $a_-$ and $a_+^{\varphi^{-1}}$ are nonzero in $\mathsf{m}$ and $\mathsf{n}$ (due to the fact, that $\mathsf{m},\mathsf{m} \in \Omega_2$) we conclude, that in $\mathcal{A}$ exist $g : \mathsf{m} \longrightarrow \mathsf{n}$ and $f : \mathsf{n} \longrightarrow \mathsf{m}$, such that $fg = 1_{\mathsf{m}}$. Analogously one constructs $f', g'$ such that $g'f' = 1_{\mathsf{n}}$. □

**Corollary 6.4.** *Let $\mathcal{M}$ be a group, $U$ integral $\Gamma$-algebra, $D = D(\mathsf{m}) \subset \mathrm{Specm}\,\Gamma$ a $\Delta(U,\Gamma)$-equivalence class of a maximal ideal $\mathsf{m} \in \Omega_r \cap \Omega_2$. Then the category $\mathbb{H}(U,\Gamma,D)$ is equivalent to the category $\hat{\Gamma}_{\mathsf{m}} - \mathrm{mod}$.*

*Proof.* Since all the objects in $\mathcal{A}_D$ are isomorphic by Theorem 6.1, (5) the categories $\mathcal{A}_D - \mathrm{mod}$ and $\mathcal{A}(\mathsf{m},\mathsf{m}) - \mathrm{mod}$ are equivalent. Note that the functors of restriction $res : \mathbb{H}(U,\Gamma,D) \to \mathcal{A}(\mathsf{m},\mathsf{m}) - \mathrm{mod}$ and of induction $ind : \mathcal{A}(\mathsf{m},\mathsf{m}) \to H(U,\Gamma,D)$ are quasi-inverse. □

**Remark 6.1.** *Recall that if $\mathsf{m}$ is non-singular point of $\mathrm{Specm}\,\Gamma$, then $\hat{\Gamma}_{\mathsf{m}}$ is isomorphic to the algebra of formal power series in $\mathrm{GKdim}\,\Gamma$ variables.*

Theorem 5.2, Theorem 6.1 and Corollary 6.4 immediately imply **Theorem C**.

6.5. **Proof of Theorem D.** Theorem D stated in Introduction is an analogue of the Harish-Chandra theorem for the universal enveloping algebras ([D]). In particular it shows that the subcategories in $U - \mathrm{mod}$, described in Corollary 6.4, contain enough modules.

Suppose that conditions of Theorem D are satisfied. Consider the set $\Omega_u$ constructed in Proposition 4.1 and set

$$\Omega'_u = \Omega_u \cap \Omega_2 \cap \Omega_r.$$

Then for any $\mathsf{m} \in \Omega'_u$ the element $u$ acts nontrivially on module $U/U\mathsf{m}$ which is simple by Theorem 6.1. Now Proposition 4.1 completes the proof of Theorem D.

6.6. **Tableau modules.** Consider an arbitrary generator $[ma]$ of $L * \mathcal{M}^G$, $m \in \mathcal{M}$, $a \in L$, and an arbitrary tableau $[\ell] \in \mathcal{L}$. Suppose that the rational function on $\mathbb{L}$, $a^g$, is defined on $\ell$ for all $g \in G/H_m$. Then the action of $[ma]$ is defined on $[\ell]$ by

$$[ma] \cdot [\ell] = \sum_{g \in G/H_m} a^g(\ell)[\ell^g].$$

One can check that

$$[m'a']([ma][\ell]) = ([m'a'][ma])[\ell],$$

if both sides are defined. Hence, we can define a *partial* action of $L * \mathcal{M}^G$ on $\operatorname{Specm} \mathbb{L}$ in the spirit of [Ex].

Let $U \subset L * \mathcal{M}$ be a Galois algebra. Consider a massive subset $X(U) \subset \operatorname{Specm} \mathbb{L}$ consisting of those $\ell \in \operatorname{Specm} \mathbb{L}$ for which $a^g(\ell)$ is defined for all $g \in G/H_m$, $m \in \mathcal{M}$ and $[ma] \in U$. Then $U_\ell \neq 0$ for all $\ell \in X(U)$.

Fix $\ell \in X(U)$ and consider the orbit $\mathcal{O}_\ell = \mathcal{M} \cdot \ell$. Then $U$ acts on $\mathcal{O}_\ell$ and this action defines a Harish-Chandra $U$-module $M[\ell]$ whose support is $\mathcal{O}_\ell$. Clearly, $M[\ell]$ is a weight Harish-Chandra module.

A tableau $\ell \in X(U)$ will be called $\mathcal{M}$- *regular* if $\pi(\ell) \neq \pi(\ell^m)$ for all $m \in \mathcal{M}$. If $\ell$ is $\mathcal{M}$-regular then all weight spaces of $M[\ell]$ are 1-dimensional and it has a basis consisting of tableaux $[\ell']$, $\ell' \in \mathcal{O}_\ell$.

We will denote by $GT(U)$ a full subcategory in $U - \operatorname{mod}$ consisting of modules with a basis labelled by the subsets of the orbits of $\mathcal{M}$-regular tableau in $X(U)$. The category $GT(U)$ will be called the *Gelfand-Tsetlin category.* The action of the generators of the Galois algebra $U$ on basis elements of any $V \in GT(U)$ is analogous to the classical Gelfand-Tsetlin formulas for finite-dimensional representations of $\mathrm{gl}_n$.

Theorem D immediately implies

**Corollary 6.5.** *Let $\mathcal{M}$ be a group, $u$ a nonzero element of an integral Galois $\Gamma$-algebra $U$ with a noetherian normal $\Gamma$. If $u \in U$ acts trivially on all tableaux modules then $u = 0$.*

## 7. Acknowledgment

The first author is supported in part by the CNPq grant (processo 307812/2004-9) and by the Fapesp grant (processo 2005/60337-2). The second author is grateful to Fapesp for the financial support (processo 2004/02850-2) and to the University of São Paulo for the hospitality during his visits.

## References

[AM] Atiyah M.F., Macdonald I.G., Introduction to commutative algebra, Addison-Wesley, 1969.

[Ba] Bavula V., Generalized Weyl algebras and their representations, Algebra i Analiz 4 (1992), 75–97. (English translation: St. Petersburg Math. J. 4 (1993), 71–92 .)

[BavO] Bavula V., Oystaeyen F., Simple Modules of the Witten-Woronowicz algebra, Journal of Algebra **271** (2004), 827–845.

[BBF] Bekkert V., Benkart G., Futorny V., Weight Modules for Weyl Algebras, Contemporary Mathematics **343** (2004), 17–42.

[D] Dixmier J., *Algèbres Enveloppantes.* Paris: Gauthier-Villars, 1974.

[DFO] Drozd Yu., Ovsienko S., Futorny V., *Harish - Chandra subalgebras and Gelfand Zetlin modules*, in: "Finite dimensional algebras and related topics", NATO ASI Ser. C., Math. and Phys. Sci., **424**, (1994), 79-93.

[Ex] Exel R., *Partial actions of groups and actions of inverse semigroups*, Proc. Amer. Math. Soc., 126 (1998), 3481-3494, [arXiv:funct-an/9511003], MR 99b:46102.

[FMO] Futorny V., Molev A., Ovsienko S., *Harish-Chandra modules for Yangians*, Representation Theory, AMS, 9 (2005), 426-454.

[FO] Futorny V., Ovsienko S., *Galois algebras I: Structure Theory*, preprint, USP, 2006.

[KS] Klimyk A. U., Schmüdgen K. *Quantum groups and their representations*, Text and monographs in physics, Springer-Verlag, Berlin, 1997.

[La] Lang S. *Algebra*, Third edition, 1994.

[MCR] McConnell J. C., Robson J.C., Noncommutative noetherian rings, Chichester, Wiley, 1987.

[Mat] Matsumura H., Commutative ring theory, Cambridge Studies in Advanced Mathematics, 8, Cambridge University Press, 1997.

[Ov] Ovsienko S.*Strongly nilpotent matrices and Gelfand - Zetlin modules*, Linear Algebra and Its Applications, 365 (2003), 349-367.

[Zh] Zhelobenko D.P. *Compact Lie groups and their representations*, Nauka, Moscow, 1970 (Translations of mathematical monographs, **40**, AMS, Providence, Rhode Island, 1973).



Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281, São Paulo, CEP 05315-970, Brasil

*E-mail address*: futorny@ime.usp.br

Faculty of Mechanics and Mathematics, Kiev Taras Shevchenko University, Vladimirskaya 64, 00133, Kiev, Ukraine

*E-mail address*: ovsienko@zeos.net

## TRABALHOS DO DEPARTAMENTO DE MATEMÁTICA

## TÍTULOS PUBLICADOS

| | |
|---|---|
| 2005-01 | PEREIRA, A. L. and PEREIRA, M. C. Generic simplicity for the solutions of a nonlinear plate equation. 22p. |
| 2005-02 | ARAGONA, J. Colombeau generalized functions on quasi-regular sets. 25p. |
| 2005-03 | PEREIRA. A. L. Global attractor and nonhomogeneous equilibria for a non local evolution equation in an unbounded domain. 26p. |
| 2005-04 | SHESTAKOV, I. and ZHUKAVETS, N. Malcev Poisson superalgebra of free Malcev superalgebra on one odd generator. 19p. |
| 2005-05 | SHESTAKOV, I. and ZHUKAVETS, N. Speciality of Malcev superalgebras on one odd generator. 15p. |
| 2005-06 | BILLIG, Y., FUTORNY, V. and MOLEV, A. Verma modules for Yangians. 18p. |
| 2005-07 | GORODSKI, C. Jacobi fields on normal homogeneous Riemannian manifolds. 6p. |
| 2005-08 | GRISHKOV, A. Notes on simple Lie algebras over a field of characteristic 2. 7p. |
| 2005-09 | MIRANDA, J. C. S. and MORETTIN, P. A. Estimation of the density of point processes on $\mathbb{R}^m$ via wavelets. 15p. |
| 2005-10 | KASHUBA, I., OVSIENKO, S. and SHESTAKOV, I. Jordan algebras versus associative algebras. 35p. |
| 2005-11 | IWAKI, E., JESPERS, E. and JURIAANS, S. O. The hypercentre of the unit group of an integral group ring. 11p. |
| 2005-12 | FUTORNY, V. and OVSIENKO, S. Theory of Galois algebras. 54p. |
| 2005-13 | MIRANDA, J. C. S. Characterization of the intensity for a class of point processes. 16p. |
| 2005-14 | MIRANDA, J. C. S. Neglecting higher order infinitesimals and the study of some measures. 16p. |
| 2005-15 | MIRANDA, J. C. S. Some wide classes of point processes interrelationships and three working properties. 15p. |
| 2005-16 | MIRANDA, J. C. S. Sure inference analysis. 27p. |
| 2005-17 | GRISHKOV, A. N. and ZAVARNITSINE, A. V. Sylow's theorem for Moufang loops I. 14p. |
| 2005-18 | LOURENÇO, M. L. and TOCHA, N. N. Zeros of complex homogeneous polynomials. 7p. |

2005-19 ALEXANDRINO, M. M. Proofs of conjectures about singular riemannian foliations. 19p.

2005-20 MELO, S. T. and SILVA, C. C. K-Theory of pseudodifferential operators with semi-periodic symbols. 13p

2005-21 COX, B. L. and FUTORNY, V. Structure of intermediate Wakimoto modules. 20p.

2005-22 BAHTURIN, Y. A., SHESTAKOV, I. P. and ZAICEV, M. V. Group Gradings on Jordan Algebras $M_n^{(+)}$.30p.

2005-23 BILIOTTI, L. and GORODSKI, C. Polar actions on compact irreducible symmetric spaces are taut. 7p.

2005-24 LOURENÇO, M. L. and PELLEGRINI, L. Interpolation by analytic functions on preduals of Lorentz sequence spaces. 9p.

2005-25 LOURENÇO, M. L. and ACOSTA, M. D. Šilov boundary for holomorphic functions on some classical Banach spaces.13p.

2006-01 MIRANDA, J. C. S. Adaptive maximum probability estimation of multidimensional Poisson processes intensity function. 9p.

2006-02 MIRANDA, J. C. S. Some inequalities and asymptotics for a weighted alternate binomial sum. 7p.

2006-03 BELITSKII, G., BERSHADSKY, M., SERGEICHUK, V. and ZHARKO, N. Classification of 2 x 2 x 2 matrices; criterion of positivity. 16p.

2006-04 FUTORNY, V. and SERGEICHUK, V. Classification of sesquilinear forms whit the first argument on a subspace or a factor space. 25p.

2006-05 DOKUCHAEV, M. A., KIRICHENKO, V. V., NOVIKOV, B.V. and PETRAVCHUK, A. P. On incidence modulo ideal rings. 41p.

2006-06 DOKUCHAEV, M., DEL RÍO, Á. and SIMÓN, J. J. Globalizations of partial actions on non unital rings. 10p.

2006-07 DOKUCHAEV, M., FERRERO, M. and PAQUES, A. Partial actions and Galois theory. 20p.

2006-08 KIRICHENKO, V. V. On semiperfect rings of injective dimension one. 25p.

2006-09 SHESTAKOV, I. and ZHUKAVETS. N. The free alternative superalgebra on one old generator. 30p.

2006-10 MIRANDA, J. C. S. Probability density functions of the wavelet coefficients of a wavelet multidimensional poisson intensity estimator. 7p.

2006-11 ALEXANDRINO, M. M and GORODSKI, C. Singular Riemannian foliations with sections, transnormal maps and basic forms. 17p.

2006-12 FERNANDEZ, J. C. G. On right-nilalgebras of index 4. 21p.

2006-13 FUTORNY, V. and OVSIENKO, S. Galois Algebras I: Structure Theory. 52p.

2006-14 FUTORNY, V. and OVSIENKO, S. Galois Algebras II: Representation Theory. 36p.

Nota: Os títulos publicados nos Relatórios Técnicos dos anos de 1980 a 2005 estão à disposição no Departamento de Matemática do IME-USP.
Cidade Universitária "Armando de Salles Oliveira"
Rua do Matão, 1010 - Cidade Universitária
Caixa Postal 66281 - CEP 05315-970