RT-MAE 2001-15

ITEM RESPONSE THEORY FOR LONGITUDINAL DATA: ITEM PARAMETER ESTIMATION

by

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Classificação AMS: 62F10, 62H05, 62J02.
(AMS Classification)

- Julho de 2001 -

Item Response Theory for Longitudinal Data: Item Parameter Estimation

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Abstract

In this paper we deal with the problem of calibrating item parameters in longitudinal studies where a group of respondents is submitted to different tests along the time, having or not common items. Several covariance structures are explored to model the possible dependency among the abilities of the same respondent, measured at different instants. Maximum likelihood equations and some simulation results are presented. It is assumed that the ability distribution of the population, from where the respondents were randomly selected, is known.

Key words: Logistic model, covariance structures, multivariate latent distribution, repeated measure, binary response.

1 Introduction

Item Response Theory (IRT) has been widely applied in mental measurement. (See Lord (1980), Hambleton et al. (1991) and Andrade et al. (2000), among others for details on the fundamentals and applications of IRT). The major results presented in the literature are appropriate for the situations where the examinees belonging to one population or two or more populations are evaluated at only one instant in time. The reader is referred to Baker (1992) for details on the major estimation methods in IRT

In this paper we present a method to estimate the items parameters when the group of examinees, selected randomly from a population of examinees, is designed to be evaluated at T pre-specified instants. For instance, one group of students evaluated at the end of 5th to 8th grades. We consider that at pre-specified instant t, $t = 1, 2, \cdots, T$, the group of examinees is submitted to a Test t composed of n_t items corrected as *right/wrong*. The total number of items n will be less then or equal to $n_c = \sum_{t=1}^{T} n_t$, allowing for common items among the tests. It will be assumed that the ability distribution of the population, from where the examinees were randomly selected, is known.

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2 The Model

Let

$$P_{jit} = P(U_{jit} = 1|\theta_{jt}, \zeta_i)$$

be a twice-differentiable item response function (IRF) that describes the conditional probability of a correct response to item i, $i = 1, 2, \cdots, n$, of individual j, $j = 1, 2, \cdots, N$, in test t, $t = 1, 2, \cdots, T$, where U_{jit} represents the (binary) response, θ_{jt} the ability (latent trait) and ζ_i the vector of the item parameters. Examples of such function would be the logistic 1, 2 and 3 parameters models (see Hambleton et al. (1991) for details). Assuming the conditional independence of the responses to the items at test t, given θ_t, we have that

$$P(U_{j.t}|\theta_t, \zeta) = \prod_{i \in I_t} P(U_{jit}|\theta_t, \zeta_i),$$

where I_t is the set of the indexes of those items presented in Test t, $U_{j.t} = (U_{j1t}, U_{j2t}, \cdots, U_{jn_tt})'$ is the $(n_t \times 1)$ vector of responses of individual j at Test t and $\zeta = (\zeta_1', \zeta_2', \cdots, \zeta_n')'$. Notice that, for convenience and without loss of generality we dropped index j from the ability parameter because we are interested in the distribution of the ability at each instant t and not in any particular θ_{jt}. Furthermore, assuming the conditional longitudinal independence of the responses to the items along the T tests, given the abilities in the T tests, we will have

$$\begin{aligned} P(U_{j..}|\theta, \zeta) &= \prod_{t=1}^{T} P(U_{j.t}|\theta_t, \zeta) \qquad (1) \\ &= \prod_{t=1}^{T} \prod_{i \in I_t} P(U_{jit}|\theta_t, \zeta_i). \end{aligned}$$

with $U_{j..} = (U_{j.1}', U_{j.2}', \cdots, U_{j.N}')'$ being the $(n_c \times 1)$ vector of responses of individual j in all tests and $\theta = (\theta_1, \theta_2, \cdots, \theta_T)'$. Finally, assuming that θ follows a known multivariate continuous distribution with parameters η of finite elements, the unconditional, or marginal, probability of pattern $U_{j..}$ can be written as

$$P(U_{j..}|\zeta, \eta) = \int_{\mathbb{R}^T} P(U_{j..}|\theta, \zeta) g(\theta|\eta) d\theta. \qquad (2)$$

The above probability will depend on the items parameters and on θ only through its distribution.

3 The Likelihood Equations

Let index j represent one of the s different response patterns, $s \leq \min(N, 2^{n_c})$, instead of just individual j, $r_j > 0$ be the number of occurrences of the response pattern j, $j = 1, 2, \cdots, s$, and $\boldsymbol{R} = (r_1, r_2, \cdots, r_s)'$ be the $(s \times 1)$ vector of frequencies of the observed pattern $\boldsymbol{U}_{j..}$. From the independence among the responses of the different individuals, $\boldsymbol{R}$ follows a Multinomial distribution, that is

$$P(\boldsymbol{R}|\boldsymbol{\zeta}, \boldsymbol{\eta}) = \frac{N!}{\prod_{j=1}^{s} r_j!} \prod_{j=1}^{s} (P(\boldsymbol{U}_{j..}|\boldsymbol{\zeta}, \boldsymbol{\eta}))^{r_j}.$$

Therefore, the log-likelihood function is given by

$$\log L(\boldsymbol{\zeta}) = \log \left\{ \frac{N!}{\prod_{j=1}^{s} r_j!} \right\} + \sum_{j=1}^{s} r_j \log P(\boldsymbol{U}_{j..}|\boldsymbol{\zeta}, \boldsymbol{\eta})$$

and the set of estimating equations for the parameters of item i by

$$\frac{\partial \log L(\boldsymbol{\zeta})}{\partial \boldsymbol{\zeta}_i} = 0\,; \quad i = 1, 2, \cdots, n. \tag{3}$$

The evaluation of (3) involves some intermediate results that follow. First of all, notice that

$$\frac{\partial \log L(\boldsymbol{\zeta})}{\partial \boldsymbol{\zeta}_i} = \sum_{j=1}^{s} r_j \frac{1}{P(\boldsymbol{U}_{j..}|\boldsymbol{\zeta}, \boldsymbol{\eta})} \frac{\partial P(\boldsymbol{U}_{j..}|\boldsymbol{\zeta}, \boldsymbol{\eta})}{\partial \boldsymbol{\zeta}_i},$$

where

$$\begin{aligned}
\frac{\partial P(\boldsymbol{U}_{j..}|\boldsymbol{\zeta}, \boldsymbol{\eta})}{\partial \boldsymbol{\zeta}_i} &= \\
&= \frac{\partial}{\partial \boldsymbol{\zeta}_i} \left\{ \int_{\boldsymbol{R}^T} P(\boldsymbol{U}_{j..}|\boldsymbol{\theta}, \boldsymbol{\zeta}) g(\boldsymbol{\theta}|\boldsymbol{\eta}) d\boldsymbol{\theta} \right\} \\
&= \int_{\boldsymbol{R}^T} \left\{ \frac{\partial}{\partial \boldsymbol{\zeta}_i} P(\boldsymbol{U}_{j..}|\boldsymbol{\theta}, \boldsymbol{\zeta}) \right\} g(\boldsymbol{\theta}|\boldsymbol{\eta}) d\boldsymbol{\theta} \\
&= \int_{\boldsymbol{R}^T} \left\{ \sum_{t \in \boldsymbol{\tau}_i} \left(\frac{\partial P(U_{jit}|\theta_t, \boldsymbol{\zeta}_i)/\partial \boldsymbol{\zeta}_i}{P(U_{jit}|\theta_t, \boldsymbol{\zeta}_i)} \right) \right\} P(\boldsymbol{U}_{j..}|\boldsymbol{\theta}, \boldsymbol{\zeta}) g(\boldsymbol{\theta}|\boldsymbol{\eta}) d\boldsymbol{\theta},
\end{aligned} \tag{4}$$

and $\boldsymbol{\tau}_i$ is the set of the indexes of the tests that contains item i. The interchange of the differential and the integral signs was done based on the Dominated Convergence Theorem of Lebesgue (see Chow and Teicher, 1978). Writing

$$P(U_{jit}|\theta_t,\zeta) = P(U_{jit}=1|\theta_t,\zeta)^{U_{jit}} P(U_{jit}=0|\theta_t,\zeta)^{1-U_{jit}} = P_{it}^{U_{jit}} Q_{it}^{1-U_{jit}},$$

it follows that

$$\begin{aligned}\frac{\partial}{\partial\zeta_i} P(U_{jit}|\theta_t,\zeta_i) &= \frac{\partial}{\partial\zeta_i}\left(P_{it}^{U_{jit}} Q_{it}^{1-U_{jit}}\right)\\ &= \left[U_{jit} P_{it}^{U_{jit}-1} Q_{it}^{1-U_{jit}} - (1-U_{jit}) Q_{it}^{-U_{jit}} P_{it}^{U_{jit}}\right]\left(\frac{\partial P_{it}}{\partial\zeta_i}\right)\\ &= (-1)^{U_{jit}+1}\left(\frac{\partial P_{it}}{\partial\zeta_i}\right). \end{aligned} \tag{5}$$

The last equality follows from the fact that the term between brackets is 1 when U_{jit}=1 and -1 when U_{jit}=0. Finally, noticing that

$$\frac{(-1)^{U_{jit}+1} P_{it} Q_{it}}{P_{it}^{U_{jit}} Q_{it}^{1-U_{jit}}} = \begin{cases} Q_{it} & \text{if } U_{jit}=1\\ -P_{it} & \text{if } U_{jit}=0, \end{cases} \tag{6}$$

it follows from (4), (5) and (6) that

$$\frac{\partial P(U_{j..}|\zeta,\eta)}{\partial\zeta_i} = \int_{R^T}\left\{\sum_{t\in\tau_i}\frac{(U_{jit}-P_{it})}{P_{it}Q_{it}}\left(\frac{\partial P_{it}}{\partial\zeta_i}\right)\right\} P(U_{j..}|\theta,\zeta) g(\theta|\eta) d\theta.$$

From the above intermediate results the estimating function can be rewritten as

$$\frac{\partial \log L(\zeta)}{\partial\zeta_i} = \sum_{j=1}^{s} r_j \int_{R^T}\left\{\sum_{t\in\tau_i}\frac{(U_{jit}-P_{it})}{P_{it}Q_{it}}\left(\frac{\partial P_{it}}{\partial\zeta_i}\right)\right\} g_j^*(\theta) d\theta, \tag{7}$$

with

$$g_j^*(\theta) = \mathbb{P}(\theta|U_{j..},\zeta,\eta) = \frac{P(U_{j..}|\theta,\zeta) g(\theta|\eta)}{P(U_{j..}|\zeta,\eta)}.$$

Notice that the probability in the numerator is given by (1).

3.1 Application to Model LM3

As illustration, we will present the estimating equations for the parameters of the Logistic 3 Parameter Model (LM3), which is considered in most of the applications of the IRT and is given by

$$\begin{aligned} P_{it} &= P(U_{jit} = 1 | \theta_t, \zeta_i) \\ &= c_i + (1 - c_i)\{1 + e^{-Da_i(\theta_t - b_i)}\}^{-1} \\ &= c_i + (1 - c_i)P^*_{it} \end{aligned}$$

where a_i, b_i and c_i are the 3 parameters of item i, D is a constant equal to 1 and $P^*_{it} = \{1 + \exp[-Da_i(\theta_t - b_i)]\}^{-1}$. The value $D = 1.7$ is used when someone wants that the Logist Model gives results close to the Normal Model. See Hambleton et al. (1991) for details. Differentiating P_{it} with respect to each one of the 3 parameters we will have

$$\frac{\partial P_{it}}{\partial a_i} = D(1 - c_i)(\theta_t - b_i)P^*_{it}Q^*_{it} \tag{8}$$

$$\frac{\partial P_{it}}{\partial b_i} = -Da_i(1 - c_i)P^*_{it}Q^*_{it} \tag{9}$$

$$\frac{\partial P_{it}}{\partial c_i} = Q^*_{it} \tag{10}$$

Therefore, from (7), (8), (9), and (10) it follows that

$$\begin{aligned} \frac{\partial \log L(\zeta)}{\partial a_i} &= \sum_{j=1}^{s} r_j \int_{\mathbb{R}^T} \left\{ \sum_{t \in \mathcal{T}_i} (U_{jit} - P_{it}) \left(\frac{\partial P_{it}}{\partial a_i} \right) \frac{W_{it}}{P^*_{it}Q^*_{it}} \right\} g^*_j(\boldsymbol{\theta}) d\boldsymbol{\theta} \\ &= D(1 - c_i) \sum_{j=1}^{s} r_j \int_{\mathbb{R}^T} \left\{ \sum_{t \in \mathcal{T}_i} (U_{jit} - P_{it})(\theta_t - b_i)W_{it} \right\} g^*_j(\boldsymbol{\theta}) d\boldsymbol{\theta}, \end{aligned}$$

$$\begin{aligned} \frac{\partial \log L(\zeta)}{\partial b_i} &= \sum_{j=1}^{s} r_j \int_{\mathbb{R}^T} \left\{ \sum_{t \in \mathcal{T}_i} (U_{jit} - P_{it}) \left(\frac{\partial P_{it}}{\partial b_i} \right) \frac{W_{it}}{P^*_{it}Q^*_{it}} \right\} g^*_j(\boldsymbol{\theta}) d\boldsymbol{\theta} \\ &= -Da_i(1 - c_i) \sum_{j=1}^{s} r_j \int_{\mathbb{R}^T} \left\{ \sum_{t \in \mathcal{T}_i} (U_{jit} - P_{it})W_{it} \right\} g^*_j(\boldsymbol{\theta}) d\boldsymbol{\theta} \end{aligned}$$

and

$$\begin{aligned}\frac{\partial \log L(\zeta)}{\partial c_i} &= \sum_{j=1}^{s} r_j \int_{\mathbb{R}^T} \left\{ \sum_{t\in\tau_i} (U_{jit} - P_{it}) \left(\frac{\partial P_{it}}{\partial c_i} \right) \frac{W_{it}}{P_{it}^* Q_{it}^*} \right\} g_j^*(\theta) d\theta \\ &= \sum_{j=1}^{s} r_j \int_{\mathbb{R}^T} \left\{ \sum_{t\in\tau_i} (U_{jit} - P_{it}) \frac{W_{it}}{P_{it}^*} \right\} g_j^*(\theta) d\theta.\end{aligned}$$

with

$$W_{it} = \frac{P_{it}^* Q_{it}^*}{P_{it} Q_{it}}.$$

The estimating functions for the 3 parameters can then be written as

$$\begin{aligned} a_i: &\quad D(1-c_i) \sum_{j=1}^{s} r_j \int_{\mathbb{R}^T} \left\{ \sum_{t\in\tau_i} (U_{jit} - P_{it})(\theta_t - b_i) W_{it} \right\} g_j^*(\theta) d\theta = 0, \\ b_i: &\quad -D a_i (1-c_i) \sum_{j=1}^{s} r_j \int_{\mathbb{R}^T} \left\{ \sum_{t\in\tau_i} (U_{jit} - P_{it}) W_{it} \right\} g_j^*(\theta) d\theta = 0, \\ c_i: &\quad \sum_{j=1}^{s} r_j \int_{\mathbb{R}^T} \left\{ \sum_{t\in\tau_i} (U_{jit} - P_{it}) \frac{W_{it}}{P_{it}^*} \right\} g_j^*(\theta) d\theta = 0. \end{aligned}$$

There is no closed form solution for these equations and some iterative method, such as Newton-Raphson or Fisher Scoring algorithms, needs to be considered in order to solve them. We will consider in this work the EM algorithm as suggested in Bock and Aitkin (1981).

3.2 The EM algorithm

To shortly describe the EM algorithm to our problem, let us consider

$$f(\theta) = \sum_{j=1}^{s} r_j g_j^*(\theta) \quad \text{and} \quad r_{it}(\theta) = \sum_{j=1}^{s} r_j U_{jit} g_j^*(\theta),$$

respectively, the number of individuals that have vector of abilities θ and the number of such individuals that answered item i correctly in Test t. In the EM algorithm setup, $(U'_{...}, f', r')'$, with $U_{...} = (U'_{1..}, U'_{2..}, \cdots, U'_{N..})'$, $f = (f(\theta))$ and $r = (r_{it}(\theta))$, will represent the complete data, and $U_{...}$ the incomplete data. Notice that only $U_{...}$ will be observed. The estimating function (7) can be rewritten as

$$\frac{\partial \log L(\zeta)}{\partial \zeta_i} = \int_{R^T} \sum_{t \in \tau_i} \left\{ \sum_{j=1}^{s} r_j \left[(U_{jit} - P_{it}) g_j^*(\theta) \right] \left(\frac{\partial P_{it}}{\partial \zeta_i} \right) \frac{W_{it}}{P_{it}^* Q_{it}^*} \right\} d\theta$$
$$= \int_{R^T} \sum_{t \in \tau_i} \left\{ \left[r_{it}(\theta) - f(\theta) P_{it} \right] \left(\frac{\partial P_{it}}{\partial \zeta_i} \right) \frac{W_{it}}{P_{it}^* Q_{it}^*} \right\} d\theta.$$

In practice, the integrals are approximated through one of the many available methods in the literature. In IRT, it has been frequent the application of the method *Hermite-Gauss* (see Stroud and Secrest, 1966), usually referred as *gaussian quadrature method.* Assuming that for each one of the T dimensions we have q_t, $t = 1, \cdots, T$, quadrature points with $A_{l_t t}$, $l_t = 1, \cdots, q_t$, associated weights, in the multidimensional setup the total number of quadrature points will be $q = \prod_{t=1}^{T} q_t$, indicated by $\overline{\theta}_l = (\theta_{l_1 1}, \theta_{l_2 2}, \cdots, \theta_{l_T T})$, with $A_l = \prod_{t=1}^{T} A_{l_t t}$, $l = 1, \cdots, q$, associated weights.

The E and M steps can be written as

E-Step

To use the quadrature points $\overline{\theta}_l$, $l = 1, \cdots, q$, the weights A_l, $l = 1, \cdots, q$, and initial estimates of the parameters of the items, ζ_i, $i = 1, \cdots, n$, to generate $g_j^*(\overline{\theta}_l)$ and, later, $r_{lit} = r_{it}(\theta_l)$ and $f_l = f(\theta_l)$.

M-Step

With $\boldsymbol{r}$, $\boldsymbol{f}$ obtained in the E-Step, to solve the estimating equations for ζ_i, $i = 1, \cdots, n$, using the Newton-Raphson or Fisher Scoring algorithm.

These steps composes each iteration of the EM algorithm, which will be repeated until some stop criterion is reached.

4 Numerical Results

In this section we present one application of the proposed methodology in simulated data. The data where generated based on N = 1000 individuals submitted to T = 5 tests. The total simulation consisted of 500 replications. The known ability distribution and the composition of each one of the 5 tests are discussed below. All the calculations were done via a computer program developed by the authors using the computer language *Ox* (see Doornik, 2000).

4.1 Ability distribution

It was assumed that the vector of abilities $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_5)'$ followed a 5-variate normal distribution with parameters $\boldsymbol{\eta} = (\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu}$ is the (5×1) mean vector and $\boldsymbol{\Sigma}$ is the (5×5) covariance matrix.

Five different covariance structures where considered for illustration. Details on covariance structures can be found in Graybill (1969).

4.1.1 Diagonal covariance matrix

This would be the simplest structure for the covariance matrix. It implies that the abilities of any given examinee along the time are independent, a pattern not expected in our problem. The main reason to introduce this structure is to show how our proposition parallels the work by Bock and Zimowski (1997). From the independence, we can write the joint distribution of the abilities in the T instants of observation as the product of the marginal distributions, i.e.

$$g(\theta|\eta) = \prod_{t=1}^{T} g_t(\theta_t|\eta_t), \tag{11}$$

where $\eta = (\eta_1', \cdots, \eta_T')'$, with $\eta_t = (\mu_t, \sigma_t^2)'$, the mean and the variance of the distribution of the abilities at instant t, respectively. Applying (1) and (11) in (2), one can see that

$$\begin{aligned} P(U_{j..}|\zeta,\eta) &= \int_{\mathbb{R}^T} P(U_{j..}|\theta,\zeta) g(\theta|\eta) d\theta \\ &= \prod_{t=1}^{T} P(U_{j.t}|\zeta,\eta_t), \end{aligned}$$

and

$$g_j^*(\theta) = \prod_{t=1}^{T} g_{jt}^*(\theta_t),$$

with

$$g_{jt}^*(\theta_t) = \frac{P(U_{j.t}|\theta_t,\zeta) g_t(\theta_t|\eta_t)}{P(U_{j.t}|\zeta,\eta_t)}.$$

Therefore, expression (12) can be written as

$$\begin{aligned} \frac{\partial \log L(\zeta)}{\partial \zeta_i} &= \sum_{t\in\mathcal{T}_i} \sum_{j=1}^{s} r_j \int_{\mathbb{R}} \left\{ (U_{jit} - P_{it}) \left(\frac{\partial P_{it}}{\partial \zeta_i} \right) \frac{W_{it}}{P_{it}^* Q_{it}^*} \right\} g_{jt}^*(\theta) d\theta \\ &= \sum_{t\in\mathcal{T}_i} \sum_{j=1}^{s_t} r_{jt} \int_{\mathbb{R}} \left\{ (U_{jit} - P_{it}) \left(\frac{\partial P_{it}}{\partial \zeta_i} \right) \frac{W_{it}}{P_{it}^* Q_{it}^*} \right\} g_{jt}^*(\theta) d\theta, \end{aligned}$$

where r_{jt} represents the number of occurrences of pattern j and s_t the number of patterns with $r_{jt} > 0$ in Test t. This result is exactly the same as the one obtained by Bock and Zimowski (1997).

4.1.2 Uniform covariance matrix

This type of structure has been considered in many longitudinal studies (see Singer and Andrade, 2000 for details). It assumes that the variances are all equal and the covariances between pair $(\theta_t, \theta_{t'})$ are equal too. This covariance structure can be represented by

$$\Sigma = \sigma^2 \begin{pmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho & 1 & \rho & \cdots & \rho \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \rho & \cdots & 1 \end{pmatrix},$$

with $\rho \in (-1, 1)$.

4.1.3 Bands covariance matrix

This structure, also called Toeplitz structure, is different from the uniform structure in the sense that only the covariances between the abilities in two consecutive instants are not null. It is given by

$$\Sigma = \sigma^2 \begin{pmatrix} 1 & \rho & 0 & 0 & \cdots & 0 \\ \rho & 1 & \rho & 0 & \cdots & 0 \\ 0 & \rho & 1 & \rho & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

with $\rho \in (-\frac{1}{2}, \frac{1}{2})$.

4.1.4 AR(1) covariance matrix

This structure assumes that the correlations between the abilities decrease as long as the distances between the instants of observations increase. As in of above structures, the variances are assumed to be the same at all the instants. The form of this covariance matrix is

$$\Sigma = \sigma^2 \begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\ \rho & 1 & \rho & \cdots & \rho^{T-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & \cdots & 1 \end{pmatrix},$$

with $\rho \in (-1, 1)$.

4.1.5 Hankel's covariance matrix

Differently from the four structures presented, this structure allows for that the variance varies along the instants of observation. It assumes that all the covariances are the same but not the correlations. It is given by

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{12} & \cdots & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 & \sigma_{12} & \cdots & \sigma_{12} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{12} & \sigma_{12} & \sigma_{12} & \cdots & \sigma_T^2 \end{pmatrix},$$

with $|\sigma_{12}| \leq \min_{t,s} \sqrt{\sigma_t^2 \sigma_s^2}$.

4.1.6 Simulated distribution parameters

Table 1 contains the values for the ability distribution parameters used in the simulation process. These values where stated take into account the standard normal distribution . We selected two populations with variances below and two variances above the unit. The values for the means where selected to have populations close to each other and also apart from each other. It was also considered low and high correlations among the abilities, keeping in mind the restrictions on these values in order to guarantee positive definite covariance matrix.

Table 1: Values of the ability distribution parameters used in the simulations

Covariance structure	Location parameters					Dispersion parameters					
	μ_1	μ_2	μ_3	μ_4	μ_5	ρ or σ_{12}	σ^2 or σ_1^2	σ_2^2	σ_3^2	σ_4^2	σ_5^2
Diagonal	0	1	2	3	4		1	0.8	0.9	1.1	1.2
Uniform	0	1	2	3	4	0.25 , 0.75	1				
Bands	0	1	2	3	4	0.25 , 0.75	1				
AR(1)	0	1	2	3	4	0.25 , 0.75	1				
Hankel	0	1	2	3	4	0.25 , 0.75	1	0.8	0.9	1.1	1.2

4.2 The T tests

In order to generate the data it was assumed that at each one of the five instants of evaluation, the examinees where submitted to a test composed of 24 items with 6 common items between two consecutive instants. The items were of multiple choice with five categories of response. The total number of different items considered in the five tests was 96. Their parameters values and distribution along the tests are presented in Appendix A. The values for parameter a (discrimination) varied from 0.6 (low discrimination) to 1.4 (high discrimination) and the values for parameter b (difficulty) varied from -0.7 to 4.7. As the degree of difficulty of one item depends on the ability of the respondent, the items with higher (lower) values of b where allocated to examinees with higher (lower) abilities. This was done in order to avoid estimation problems. For the guessing parameter c it was considered only one value (0.20). It was considered the 3 parameter logistic model (LM3) with $D = 1.7$.

4.3 Results and comments

From the results of the 500 iterations of the simulation process, it was calculated the mean of the estimates, for each one of the 96 items and 3 parameters. The sum of the squares of the deviations of these means from the true values and their variances, were than evaluated and aggregated by each type of parameter (discrimination, difficulty and guessing). The results are presented in Tables 2 and 3.

In Table 2 we have a measurement of the bias and in Table 3 a measurement of the precision of the estimates. In both cases we got small values, indicating that the proposed methodology provides good estimates of the item parameters when the ability distribution is known. In general, the bias was a little higher for the discrimination and guessing parameters in the highly correlated situation.

Table 2: Sum of the squares of the deviations

Covariance structure	$\rho = 0.25$			$\rho = 0.75$ (or $\rho = 0.45$)		
	a	b	c	a	b	c
Diagonal*	0.1444	0.0761	0.0078			
Uniforme	0.3290	0.0561	0.0092	0.4483	0.0470	0.0121
Bandas**	0.3294	0.0561	0.0095	0.3429	0.0478	0.0089
AR(1)	0.3305	0.0583	0.0097	0.4216	0.0466	0.0106
Hankel	0.3542	0.0564	0.0096	0.5441	0.0520	0.0123

* This structure doesn't depend on ρ

** In this case it was considered $\rho = 0.45$ instead of $\rho = 0.75$

Table 3: Sum of variances

Covariance structure	$\rho = 0.25$			$\rho = 0.75$ (or $\rho = 0.45$)		
	a	b	c	a	b	c
Diagonal*	2.8221	4.0138	0.5348			
Uniforme	1.9250	3.2079	0.4012	1.8898	3.4182	0.4246
Bandas**	1.9095	3.1904	0.3977	1.9050	3.2658	0.4074
AR(1)	1.8995	3.1709	0.3934	1.9288	3.3900	0.4205
Hankel	1.9238	3.2366	0.4062	1.8633	3.4773	0.4285

* This structure doesn't depend on ρ

** In this case it was considered $\rho = 0.45$ instead of $\rho = 0.75$

Acknowledgment

This work was partially supported by grants from Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Coordenadoria para o Aperfeiçoamento de Pessoal de Nível Superior (CAPES), National Program of Excellency (PRONEX) contract No 76.97.1081.00, Thematic Project of Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) contract No 96/01741-7 and the Integrated Program for Teaching, Research and Extension (PROINT) of the UFPA, Number 0904PROI031 and 0904PROI032. It was partially developed when the first author was a doctoral student at the Department of Statistics of the University of São Paulo.

A Appendix

Table A: Simulated item parameters

Item	Test	a_i	b_i	c_i	Item	Test	a_i	b_i	c_i	Item	Test	a_i	b_i	c_i
1	1	0.6	-0.7	0.2	33	2	1.4	1.1	0.2	65	4	1.0	3.0	0.2
2	1	1.0	-0.7	0.2	34	2	0.6	1.3	0.2	66	4	1.4	3.0	0.2
3	1	1.4	-0.7	0.2	35	2	0.6	1.5	0.2	67	4	0.6	3.1	0.2
4	1	0.6	-0.5	0.2	36	2	0.6	1.7	0.2	68	4	1.0	3.1	0.2
5	1	1.0	-0.5	0.2	37	2,3	1.0	1.3	0.2	69	4	1.4	3.1	0.2
6	1	1.4	-0.5	0.2	38	2,3	1.0	1.5	0.2	70	4	0.6	3.3	0.2
7	1	0.6	-0.3	0.2	39	2,3	1.0	1.7	0.2	71	4	0.6	3.5	0.2
8	1	1.0	-0.3	0.2	40	2,3	1.4	1.3	0.2	72	4	0.6	3.7	0.2
9	1	1.4	-0.3	0.2	41	2,3	1.4	1.5	0.2	73	4,5	1.0	3.3	0.2
10	1	0.6	-0.1	0.2	42	2,3	1.4	1.7	0.2	74	4,5	1.0	3.5	0.2
11	1	1.0	-0.1	0.2	43	3	0.6	1.9	0.2	75	4,5	1.0	3.7	0.2
12	1	1.4	-0.1	0.2	44	3	1.0	1.9	0.2	76	4,5	1.4	3.3	0.2
13	1	0.6	0.1	0.2	45	3	1.4	1.9	0.2	77	4,5	1.4	3.5	0.2
14	1	1.0	0.1	0.2	46	3	0.6	2.0	0.2	78	4,5	1.4	3.7	0.2
15	1	1.4	0.1	0.2	47	3	1.0	2.0	0.2	79	5	0.6	3.9	0.2
16	1	0.6	0.3	0.2	48	3	1.4	2.0	0.2	80	5	1.0	3.9	0.2
17	1	0.6	0.5	0.2	49	3	0.6	2.1	0.2	81	5	1.4	3.9	0.2
18	1	0.6	0.7	0.2	50	3	1.0	2.1	0.2	82	5	0.6	4.0	0.2
19	1,2	1.0	0.3	0.2	51	3	1.4	2.1	0.2	83	5	1.0	4.0	0.2
20	1,2	1.0	0.5	0.2	52	3	0.6	2.3	0.2	84	5	1.4	4.0	0.2
21	1,2	1.0	0.7	0.2	53	3	0.6	2.5	0.2	85	5	0.6	4.1	0.2
22	1,2	1.4	0.3	0.2	54	3	0.6	2.7	0.2	86	5	1.0	4.1	0.2
23	1,2	1.4	0.5	0.2	55	3,4	1.0	2.3	0.2	87	5	1.4	4.1	0.2
24	1,2	1.4	0.7	0.2	56	3,4	1.0	2.5	0.2	88	5	0.6	4.3	0.2
25	2	0.6	0.9	0.2	57	3,4	1.0	2.7	0.2	89	5	0.6	4.5	0.2
26	2	1.0	0.9	0.2	58	3,4	1.4	2.3	0.2	90	5	0.6	4.7	0.2
27	2	1.4	0.9	0.2	59	3,4	1.4	2.5	0.2	91	5	1.0	4.3	0.2
28	2	0.6	1.0	0.2	60	3,4	1.4	2.7	0.2	92	5	1.0	4.5	0.2
29	2	1.0	1.0	0.2	61	4	0.6	2.9	0.2	93	5	1.0	4.7	0.2
30	2	1.4	1.0	0.2	62	4	1.0	2.9	0.2	94	5	1.4	4.3	0.2
31	2	0.6	1.1	0.2	63	4	1.4	2.9	0.2	95	5	1.4	4.5	0.2
32	2	1.0	1.1	0.2	64	4	0.6	3.0	0.2	96	5	1.4	4.7	0.2

References

[1] Andrade, D.F., Tavares, H.R., Valle, R.C. (2000). *Teoria da Resposta ao Item: Conceitos e Aplicações*. Associação Brasileira de Estatística: São Paulo.

[2] Baker, F.B. (1992). *Item response Theory - Parameter Estimation Techniques*. New York: marcel Dekker, Inc.

[3] Bock, R. D. and Aitkin, M. (1981). Marginal maximum likelihood estimation of item parameters: An application of a EM algorithm. *Psychometrika*, *46*, 433-459.

[4] Bock, R. D. and Lieberman, M. (1970). Fitting a response model for n dichotomously scored items. *Psychometrika*, *35*, 179-197.

[5] Bock, R.D. and Zimowski, M.F. (1997). Multiple group IRT. In Handbook of Modern Item response Theory. W.J. van der Linder and R.K. Hambleton Eds. New York: Spring-Verlag.

[6] Chow, Y.S. and Teicher, H. (1978). *Probability Theory: Independence, Interchangeability, Martingales*. New York: Springer-Verlag.

[7] Doornik, J.A. (2000). *Object-Oriented Matrix Programming using Ox 2.0*. London: Timberlake Consultants Ltd and Oxford: www.nuff.ox.ac.uk/Users/Doornik.

[8] Graybill, F. A. (1969). *Introduction to Matrices with Aplications in Statistics*. Belmont, CA: Wadsworth Publishing Company, Inc.

[9] Hambleton, R.K., Swaminathan, H. and Rogers, H.J. (1991). *Fundamentals of Item Response Theory*. Newburg Park: Sage Publications.

[10] Lord, F.M. (1980). *Applications of Item Response Theory to Practical Testing Problems*. Hillsdale: Lawrence Erlbaum Associates, Inc.

[11] Singer, J. M., Andrade, D. A. (2000). Analysis of longitudinal data. In P. K. Sen & C. R. Rao, Eds. *Handbook of Statistics, 18* (p. 115-160).

[12] Stroud, A. H., Secrest, D. (1966). *Gaussian Quadrature Formulas*. Englewood: Cliffs, New Jersey: Prentice-Hall.

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