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Minimum-density locating-dominating sets on infinite hexagonal grids with bounded height

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Abstract

A locating-dominating set of a graph G is a dominating set C of G such that, for each pair of distinct vertices u and v not in C, the neighborhood of u in C and the neighborhood of v in C are distinct. We study locating-dominating sets of minimum density on the infinite hexagonal grid H_k with finite height k. We show optimal solutions for H_k, $k \leq 5$, and when k is a multiple of 3. We also present an ILP formulation to find periodic locating-dominating sets for H_k, which may be solved in reasonable time, when k is not so large. With this approach, we found feasible solutions for $k = 7$ and $k = 8$, which are within at most 1.3% of the optimum. Combining these results, we obtain upper bounds for minimum-density locating-dominating sets on H_k, for all fixed $k \geq 10$, which are within 1% of the optimal solution. For the hexagonal grids, only results for the unrestricted case (unbounded height) have appeared in the literature. Results for H_k, $k \geq 2$, presented here have not appeared in the literature.

Keywords: Locating-dominating set; infinite hexagonal grid; finite height; minimum-density; integer linear program
2000 MSC: 05C69; 05C42; 05C85

1. Introduction

The concept of locating-dominating sets was introduced by Slater [13]. A *locating-dominating set* (for short, LDS) of a connected graph $G = (V, E)$ is a set $C \subseteq V$ such that each vertex v not in C has a neighbor in C (that is, C is a dominating set of G), and for each pair of distinct vertices u and v not in C, we have $N(u) \cap C \neq N(v) \cap C$, where $N(v)$ is the open neighborhood of vertex v.

A motivation for the study of locating-dominating sets is the detection of intruders in buildings. Consider that each vertex of a graph $G = (V, E)$ is a room in a building, and each edge represents neighboring rooms. Let C be an LDS of G. We can place a motion sensor in each room of C such that, if the sensor is non-operational, then there are no intruders in that room nor in any of its neighbors. If the sensor sends a 0 bit, then there is an intruder in that room and a 1 bit means there is an intruder in a neighboring room. If there is an intruder in a room that is not in C, since the

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neighbors in C of that room are unique, a central controller is able to locate the intruder using the information sent by the motion sensors.

Thus, it is of interest to find locating-dominating sets of minimum cardinality. However, finding such sets is an NP-hard problem, even for very specific graph classes such as bipartite [2], interval and intersection graphs [4].

While for finite graphs we are interested in the cardinality of an LDS, in the case of infinite graphs, we use the concept of *density* of an LDS to capture the idea of "ratio" of the elements in the LDS with respect to the whole graph. Let G be an infinite graph. If v is a vertex of G and $r \geq 1$ is a natural number, then the r-open neighborhood of v is defined as $N_r(v) := \{w \in V : 0 < \text{dist}(v, w) \leq r\}$, where $\text{dist}(v, w)$ is the distance between vertices v and w. The *density* of an LDS C in G, denoted $d(C, G)$, is defined as

$$d(C,G) := \inf\{d_w(C,G) : w \in V\}, \quad \text{where} \;\; d_w(C,G) := \limsup_{r\to\infty} |N_r(w) \cap C|/|N_r(w)|.$$

The *minimum density of an* LDS of a graph G, denoted $d^*(G)$, is defined as $d^*(G) := \inf\{d(C, G) : C \text{ is an LDS of } G\}$.

An updated and comprehensive source on locating-dominating sets for finite and infinite graphs, and related topics can be found in the bibliography maintained by Jean [8] (continuing the work done by A. Lobstein for many years). In the infinite case, mostly regular grids such as the square, triangular, hexagonal and king grids have been studied, all of which have vertex set $\mathbb{Z} \times \mathbb{Z}$. For these grids, locating-dominating sets with minimum density have already been found [14, 6, 7].

Let $\mathcal{G}_H$ denote the *infinite hexagonal grid*, represented in Figure 1. We define its vertex set as $V = \mathbb{Z} \times \mathbb{Z}$, and its edge set as $E = \{(u, v) : u = (i, j) \text{ and } u - v \in \{(0, \pm 1), ((-1)^{i+j+1}, 0)\}\}$.

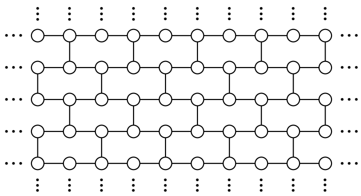

Fig. 1. The infinite hexagonal grid $\mathcal{G}_H$.

In this paper, we focus on subgraphs of the infinite hexagonal grid $\mathcal{G}_H$ which have a finite height (but infinite columns). Henceforth, $k \geq 2$ is always a natural number, and we denote by H_k *the subgraph of* $\mathcal{G}_H$ *with height* k (or equivalently, with k rows). More precisely, H_k is the subgraph of $\mathcal{G}_H$ induced by the vertices in the set $[k] \times \mathbb{Z}$, where $[k] := \{1, 2, \ldots, k\}$. In the forthcoming figures we show $H_2, H_3, \ldots, H_8$.

Bouznif et al. [1] determined locating-dominating sets for the square, triangular and king grids with k rows, for $k \leq 3$. These grids are defined similarly to H_k, considering their respective infinite grids.

We show how we obtained minimum-density locating-dominating sets on the hexagonal grids H_2, H_3, H_4 and H_5. We also present an integer linear program to find feasible LDS for H_k, for any fixed k. In practice, only when k is small, a solution may be found in a reasonable time. Finally, we show how to obtain an LDS for H_k, for each $k \geq 10$, by combining results we have found for $k \leq 9$. When k is a multiple of 3, we can assert that $1/3$ is the optimal density for an LDS on H_k, as this is the optimal density known for $\mathcal{G}_H$.

2. Minimum density locating-dominating sets in finite-height hexagonal grids

In this section, we describe an algorithm that finds a minimum-density LDS in the hexagonal grid H_k. This algorithm was first described in 2018 by Jiang [9] for the identifying code problem on infinite square grids with height k (for which he was able to find optimal solutions when $k \leq 5$). Computer-aided approach was also used by Bouznif et al. [1] for the identifying code and the LDS problems on square, triangular and king grids with height at most 3. They also showed results for the minimum-density locating total-domination problem (a slight variation of LDS problem), previously studied by Junnila [10], who showed an analytical proof for the case of square grid of height 3. Sampaio et al. [12] found optimal solutions for the identifying code problem on hexagonal grids with height at most 5. We note that an identifying code of a graph is also an LDS (the converse may not be true), and therefore, as expected, the densities of optimal locating-dominating sets that we exhibit here for H_k, $k \leq 5$, are smaller than those obtained by Sampaio et al. [12]. Our algorithm to find an optimal LDS is based on the work of Jiang [9] for the square grid. It is based on the idea of a configuration digraph, whose construction we describe next. To explain which is the problem we have to solve in this digraph, we define first some concepts.

Consider the hexagonal grid H_k, where $k \geq 2$. An *ℓ-bar* of H_k is a subgraph of it induced by the vertices in the set $[k] \times \{j_1, \dots, j_\ell\}$, where $j_1, \dots, j_\ell$ are ℓ consecutive columns of H_k.

Let R be an ℓ-bar of H_k, and let C be a subset of vertices of R. We say that C is a *barcode* of R if the following holds for the $(\ell-2)$-bar, say R', indexed by the columns 2 to $\ell-1$: for each vertex v in $V(R') \setminus C$, we have $N(v) \cap C \neq \emptyset$, and for each pair of distinct vertices u, v in $V(R') \setminus C$, we have $N(u) \cap C \neq N(v) \cap C$. That is, the vertices in the interior columns of the ℓ-bar satisfy the properties of an LDS.

The (weighted) *configuration digraph* (G_k, w) is constructed in the following way:

(i) Each vertex v_B of G_k represents a different barcode B of a 4-bar;

(ii) There is an arc $(v_B, v_{B'})$ in G_k if the last two columns of barcode B are identical to the first two columns of barcode B', and the 6-bar formed by the overlap of B and B' on the two identical columns is a barcode;

(iii) The weight w_e of an arc $e = (v_B, v_{B'})$ is the number of vertices in C in the last two columns of the barcode B'.

The *mean of a weighted cycle* in a weighted graph is defined as the sum of the weights of the arcs in the cycle divided by the number of arcs in it. (On the digraph G_k we consider only directed cycles.) Let C^* be a minimum mean cycle of the configuration digraph G_k, and let λ be the mean of C^*. Theorem 2.1 (for LDS) corresponds to an analogous version of the result presented by Jiang [9] (for identifying codes), but our proof is different to the one presented by Jiang. Owing to space limitation, we only comment on a sketch of this proof.

To refer to a feasible solution (that is, an LDS of a grid H_k) that is periodic, we use the term *pattern*. It is a solution on a subgrid with fixed number of columns that has the property that, when they are placed one after the other (horizontally), the result is an LDS for the corresponding grid with an infinite number of columns. A pattern in H_k corresponds to a solution given by a cycle in G_k. As the outdegree of every vertex in G_k is positive, and G_k is finite, then clearly, G_k has a cycle, and therefore a minimum mean cycle. We say that a pattern is optimal when its repetition gives rise to a minimum-density LDS.

Theorem 2.1. *Let H_k be the infinite hexagonal grid with $k \geq 2$ rows, and let (G_k, w) be the weighted configuration digraph as defined previously. A pattern P, defined by a set $C \subseteq V(H_k)$, is a periodic* LDS *of H_k with n columns if, and only if, G_k has a cycle $\widehat{C}$ with $n/2$ arcs and weight $w(\widehat{C})$ that is equal to the number of vertices of C in the pattern P, and the overlapping of consecutive barcodes associated with the vertices of $\widehat{C}$ makes the pattern P. Moreover, if λ is the mean of a minimum mean cycle of G_k, then the minimum density of an* LDS *in H_k is given by $d^*(H_k) = \lambda/2k$.*

(Sketch of a proof of Theorem 2.1:) Note that there is a straightforward correspondence between a periodic pattern P of an LDS in H_k and a cycle in the configuration digraph G_k. If we take a pattern whose corresponding cycle in G_k has mean λ, it is immediate that this pattern has density $\lambda/2k$. To show that $\lambda/2k$ is a lower bound for any LDS, assume that there exists a pattern with smaller density and notice that, in that case, there would be a cycle in G_k with mean smaller than λ, yielding a contradiction.

For a fixed k, to find a minimum density LDS in H_k, we implemented a program in C++ that constructs (G_k, w) and finds a minimum mean cycle in it. For that, we used the LEMON library for the graph data structure and a minimum mean cycle algorithm. For the latter, instead of using Karp's [11] or Hartmann and Orlin's [5] algorithms, we used the LEMON implementation of Howard's policy iteration algorithm, which is an algorithm for finding optimal policies in *Markov Decision Processes*, but can be adapted to find a minimum mean cycle in a digraph [3]. While the best known bounds for Howard's algorithm are exponential, it proved to be, in practice, faster than the other two (polynomial-time) algorithms.

Table 1 shows the size of the configuration digraph, as well as the runtime of our code for H_k, for $k \in \{2, 3, 4, 5\}$. The number of vertices (resp. arcs) in the configuration digraph G_k is at most 2^{4k} (resp. 2^{6k}). The code was compiled using the g++ compiler with the -O2 option. The code was executed on an HPC computer using 20 cores of an Intel(R) Xeon(R) CPU E7- 2870@ 2.40GHz processor, with 480 GB of RAM available.

The results found with our code are shown in Figures 2, 3 and 4. In these figures, the two consecutive red vertical dashed bars indicate the beginning and ending of the corresponding periodic pattern (all of them with period in the range from 4 to 16).

Table 1. Size of the configuration digraph G_k and time spent running the algorithm for $k \in \{2, 3, 4, 5\}$.

k	Number of vertices	Number of arcs	Digraph construction runtime	Howard's algorithm runtime	Total runtime
2	181	242	0.0016s	0.0001s	0.0018s
3	2581	104218	0.27s	0.01s	0.28s
4	36917	$\approx$ 5.4 million	41.38s	0.95s	42.33s
5	528005	$\approx$ 276 million	8711.49s	98.39s	8809.88s

Theorem 2.2. *The locating-dominating sets shown in Figures 2, 3 and 4 are of minimum density on the corresponding grids. The optimal densities are* $d^*(H_2) = 3/8 = 0.375$*,* $d^*(H_3) = 1/3$*,* $d^*(H_4) = 11/32 = 0.34375$ *and* $d^*(H_5) = 12/35 \approx 0.3428$*.*

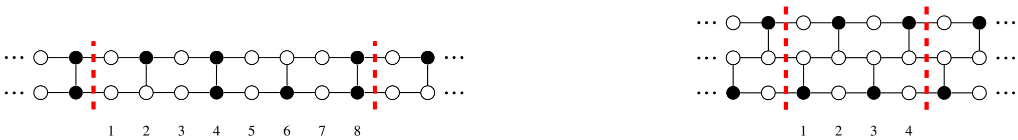

Fig. 2. Optimal periodic pattern of an LDS of the grids H_2 and H_3 with densities $3/8 = 0.375$ and $1/3$, respectively.

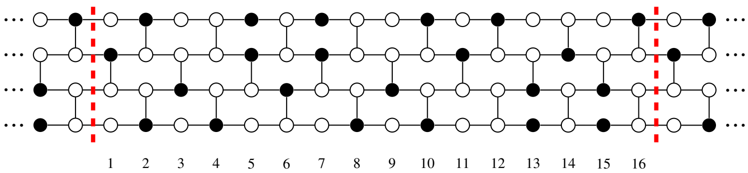

Fig. 3. Optimal periodic pattern of an LDS of the grid H_4 with density $11/32 = 0.34375$.

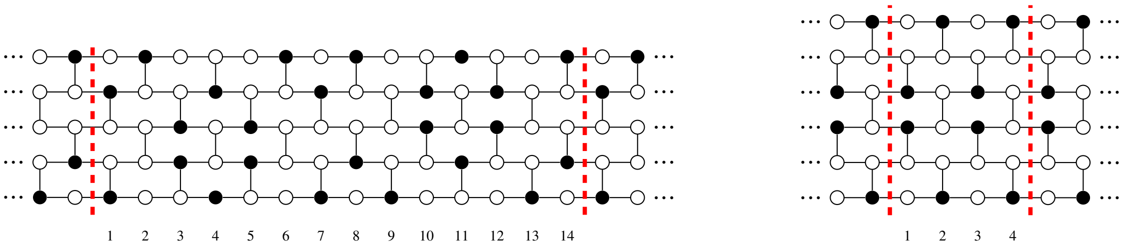

Fig. 4. Optimal periodic pattern of an LDS of the grids H_5 and H_6 with densities $12/35 \approx 0.3428$ and $1/3$, respectively.

3. An integer program to find feasible periodic locating-dominating sets on H_k

As the size of the configuration digraph grows exponentially with k, we were not able to find minimum-density locating-dominating sets for H_k when $k \geq 6$. (For $k = 6$, we have an optimal solution obtained from optimal solutions for $k = 3$.) Therefore, for grids with $k \geq 7$, instead of looking for optimal solutions, we decided to look for feasible solutions.

Motivated by the fact that even for large k, a pattern for H_k with not so large period may exist, we implemented the following strategy. We use a 0/1 integer linear program that looks for a minimum density pattern of a given length ℓ

(number of columns). Of course, such a pattern always exists, but it only gives an upper bound for $d^*(H_k)$. The idea is then to consider a larger value for ℓ, and see if there is a better pattern. When ℓ becomes large, the ILP may become very large, and we may not be able to find an optimal solution. But for $k = 7$ and $k = 8$ we were able to find patterns with period 22 (resp. 20) with density $26/77 \approx 0.3376$ (resp. $27/80 = 0.3375$), which we show in the next section that are within 1.3% of the optimum. Now we present the details of the 0/1 ILP that we have implemented.

Let B be an ℓ-bar of the grid H_k with $k \geq 2$. To find a periodic LDS with period ℓ we need to consider the $(\ell+4)$-bar B' formed by the ℓ-bar B appended with two columns at the beginning (left side) and two columns at the end. The reason for considering B' is that we must have the properties of an LDS holding on the transition from one instance of the pattern with ℓ columns to another. (Note that we do not have to consider more than 2 columns on each side.) Let B' have columns indexed by the set $\{-1, 0, \ldots, \ell+1, \ell+2\}$, where $1, \ldots, \ell$ are the columns of B.

The ILP looks for a set C of vertices that is an LDS of H_k with period ℓ (that is, within bar B). To find such a set C, we use decision variables x_v for each vertex v in B', so that $x_v = 1$ if $v \in C$ and $x_v = 0$, otherwise. For that, we impose conditions on $x = (x_v)_{v \in V(B')}$ to guarantee that the set $\{v \in V(B) : x_v = 1\}$ is an LDS of H_k and its cardinality is as small as possible.

The first observation we make is that, since columns -1 and 0 are the ending part of the previous (same) pattern, they must be equal to the columns $\ell-1$ and ℓ (ending part of bar B); so the values of the variables in these columns must be the same. Similarly, columns 1 and 2 are the beginning of the pattern, and must be equal to the columns $\ell+1$ and $\ell+2$. Thus, we have constraints $x_u - x_v = 0$ for each pair of vertices u and v such that $u = (i, p)$ and $v = (i, q)$, where $(p, q) \in \{(\ell+1, 1), (\ell+2, 2), (0, \ell), (-1, \ell-1)\}$ and $i \in [k]$.

Let $N[v] \coloneqq N(v) \cup \{v\}$ be the closed neighborhood of a vertex v. If F is a set of vertices, instead of writing $\sum_{v \in F} x_v$, we use the simplified notation $x(F)$. To ensure that each vertex v of B that is not in C has at least one neighbor in C, we write the constraint $x(N[v]) \geq 1$. Moreover, in the ℓ-bar B, each pair of distinct vertices u and v that are not in C must have distinct neighborhoods in C. To achieve this, we impose that the symmetric difference of their closed neighborhoods must be non-empty. Thus, we write the constraint $x(N[u] \triangle N[v]) \geq 1$ for each pair of distinct vertices u, v in $V(B)$ at distance 2. We can see that we only need to impose this constraint for pairs of vertices at distance 2. To make explicit the pairs of vertices that need to be compared, we replace this general constraint by constraints (11), (12) and (13).

Finally, we need to consider the pairs of vertices that are on the extremities of the pattern. Constraints (7), (8), (9) and (10) deal with this, as they consider only pairs of vertices that have some vertex in common in their open neighborhood. Putting together all restrictions, we have the following ILP (for a fixed height k and a fixed period ℓ).

$$
\begin{aligned}
&\text{minimize} \quad x(V(B)) && && (1)\\
&\text{subject to} \quad x(N[v]) \geq 1 && && \text{for } v \in V(B), && (2)\\
&x_u - x_v = 0, && u = (i, \ell+1),\ v = (i, 1) && \text{for } i \in [k], && (3)\\
&x_u - x_v = 0, && u = (i, \ell+2),\ v = (i, 2) && \text{for } i \in [k], && (4)\\
&x_u - x_v = 0, && u = (i, 0),\ v = (i, \ell) && \text{for } i \in [k], && (5)\\
&x_u - x_v = 0, && u = (i, -1),\ v = (i, \ell-1) && \text{for } i \in [k], && (6)\\
&x(N[u] \triangle N[v]) \geq 1, && u = (i, 0),\ v = (i, 2) && \text{for } i \in [k], && (7)\\
&x(N[u] \triangle N[v]) \geq 1, && u = (i, \ell+1),\ v = (i, \ell-1) && \text{for } i \in [k], && (8)\\
&x(N[u] \triangle N[v]) \geq 1, && u = (i, 0),\ v = (i+1, 1) && \text{for } i \in [k-1], && (9)\\
&x(N[u] \triangle N[v]) \geq 1, && u = (i, 1),\ v = (i+1, 0) && \text{for } i \in [k-1], && (10)\\
&x(N[u] \triangle N[v]) \geq 1, && u = (i, j),\ v = (i, j+2) && \text{for } i \in [k] \text{ and } j \in [\ell-2], && (11)\\
&x(N[u] \triangle N[v]) \geq 1, && u = (i, j),\ v = (i+1, j-1) && \text{for } i \in [k-1] \text{ and } j \in \{2, \ldots, \ell\}, && (12)\\
&x(N[u] \triangle N[v]) \geq 1, && u = (i, j),\ v = (i+1, j+1) && \text{for } i \in [k-1] \text{ and } j \in [\ell-1], && (13)\\
&x_v \in \{0, 1\}, && && \text{for } v \in V(B'). && (14)
\end{aligned}
$$

This integer linear program finds a minimum-density periodic locating-dominating set with a fixed period $\ell \geq 4$ and even. In order to find a period ℓ that yields a pattern with low density, we solve the integer program for bars of length $\ell = 4, 6, \ldots, n$, where n is an even natural number, and select the LDS with the smallest density found.

We implemented this integer program in C++ using the *Gurobi* optimizer for $k = 7$ considering up to $n = 40$ columns and for $k = 8$ considering up to 26 columns. Figures 5 and 6 show the best densities found for the grids H_7 and H_8. We were able to find minimum-density locating-dominating sets for the grids H_6 and H_9, which will be discussed in Section 4.

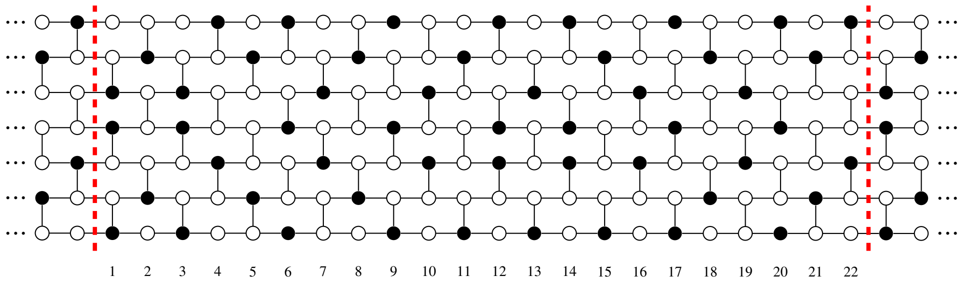

Fig. 5. Periodic pattern of an LDS C of the grid H_7 with density $d(C, G) = 26/77 \approx 0.3376$.

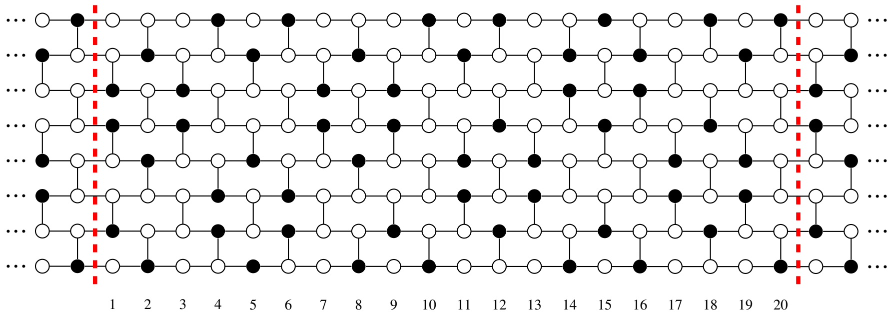

Fig. 6. Periodic pattern of an LDS C of the grid H_8 with density $d(C, G) = 27/80 = 0.3375$.

4. Explicit construction of feasible locating-dominating sets on H_k for any $k \geq 10$

In Section 2, we showed that the minimum density of a locating-dominating set in the grid H_3 is $d^*(H_3) = 1/3$. The pattern of an LDS for H_6, shown in Figure 4, also has density $1/3$ and consists of two stacked copies of the pattern for H_3, seen in Figure 2, with one of the copies shifted by one column. From this observation, we notice that, to construct locating-dominating sets with density $1/3$ for each H_{3m}, with $m \geq 1$, we can stack m copies of the pattern for H_3 on top of each other (shifting them accordingly). Honkala and Laihonen [7] proved the following result:

Theorem 4.1. *The minimum density of a locating-dominating set on the infinite hexagonal grid $\mathcal{G}_H$ is $d^*(\mathcal{G}_H) = 1/3$.*

Thus, from Theorem 4.1, we have that $1/3$ is the minimum density of an LDS for the grids H_{3m}, with $m \geq 1$, since, otherwise, we would obtain an LDS for $\mathcal{G}_H$ with density smaller than $1/3$, which is a contradiction. In fact, Lemma 4.1, together with Theorem 4.1, implies that $1/3$ is a lower bound for the density of any LDS C in any grid H_k, with $k \geq 2$.

Let P_i and P_j be patterns of locating-dominating sets C_i and C_j for the grids H_i and H_j, with periods p and q, respectively. Let t be the lowest common multiple of p and q. Let P_i^* and P_j^* be patterns obtained by repeating P_i and P_j horizontally t/p and t/q times, respectively. Note that both P_i^* and P_j^* have the same period t. Moreover, their densities remain the same, i.e., $d(P_i^*, H_i) = d(C_i, H_i)$ and $d(P_j^*, H_j) = d(C_j, H_j)$.

Let P^* be a pattern formed by taking the pattern P_i^* on the first i rows of H_{i+j} and the pattern P_j^* on the next j rows of H_{i+j}. Depending on the parity of j, we may need to shift the pattern P_j^* (by one column) so that the last line of P_i^* and the first line of P_j^* become consistent with respect to H_{i+j} (see Figure 7).

Lemma 4.1. *Let P_i and P_j be patterns of locating-dominating sets C_i and C_j for the grids H_i and H_j, respectively. Let P^* be the pattern formed by stacking P^*_j on top of P^*_i, where P^*_i and P^*_j are patterns with the same period, constructed by the repetition of P_i and P_j, respectively. Then, P^* is a pattern of an* LDS *for the grid H_{i+j} with density*

$$d(P^*, H_{i+j}) = \frac{i \cdot d(C_i, H_i) + j \cdot d(C_j, H_j)}{i + j}.$$

The previous lemma gives a way to obtain LDS for new grids, using the fact that we know solutions for grids H_k, for $k \in \{2, 3, \ldots, 8\}$. We notice that among all these grids, the ones with smallest densities are the grids H_3 and its multiples, H_7 and H_8. Therefore, the best way to stack grids to construct an LDS for any hexagonal grid with height at least 10 that is not a multiple of 3 is to stack one of the patterns for H_7 or H_8, depending on the height of the grid, with the necessary number of patterns for H_3, so that we can get closer to the lower bound of $1/3$.

Consider grids of the form H_{3m+1}, with $m \geq 3$. Since $3m + 1 = 3(m - 2) + 7$, we can stack $m - 2$ patterns for the grid H_3 (shown in Figure 2) and first obtain a pattern for $H_{3(m-2)}$, then stack the obtained pattern on top of a pattern for the grid H_7 (Figure 5). In this case, from Lemma 4.1, we get a pattern for H_{3m+1}, with density $(m+4/11)/(3m+1)$. Similarly, for grids of the form H_{3m+2} with $m \geq 3$, we can stack $m - 2$ patterns for the grid H_3 and then stack the obtained pattern on top of a pattern for the grid H_8 (Figure 6). This way, we obtain a pattern for H_{3m+2} that has density $(m + 7/10)/(3m + 2)$. In Figure 7, we show a pattern for the grid H_{11}. Since $11 = 3 + 8$, we stack one pattern that consists of 5 (horizontal) repetitions of the pattern for H_3 (that has period 4) on top of one pattern for H_8 (that has period 20). We obtain an LDS C for the grid H_{11} with density $d(C, H_{11}) = 37/110 \approx 0.3363$ and period 20.

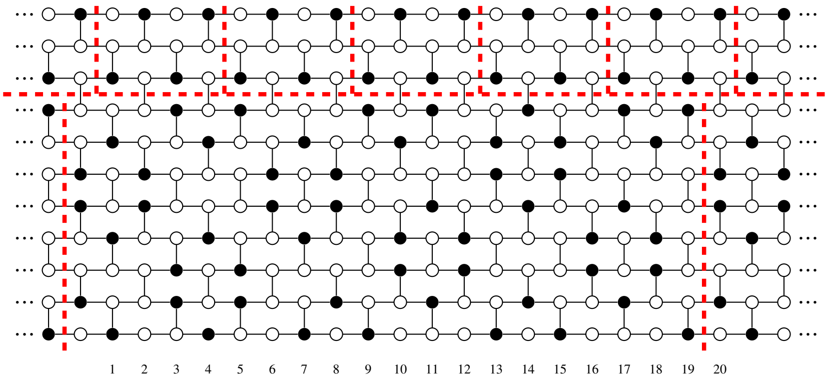

Fig. 7. Periodic pattern of an LDS C of the grid H_{11} with density $d(C, G) = 37/110 \approx 0.3363$.

Theorem 4.2 summarizes the results obtained in this section.

Theorem 4.2. *Let H_k be the infinite hexagonal grid with k rows. For each natural number $m \geq 1$, if $k = 3m$, the minimum density of a locating-dominating set for the grid H_{3m} is $d^*(H_{3m}) = 1/3$. For each $m \geq 3$, if $k = 3m + 1$, then $d^*(H_{3m+1}) \leq (m + 4/11)/(3m + 1)$, and if $k = 3m + 2$, then $d^*(H_{3m+2}) \leq (m + 7/10)/(3m + 2)$. Moreover, for $m \geq 3$, we have an explicit construction to obtain these solutions that are optimal or within less than 1% of the optimal one.*

5. Concluding Remarks

We note that the solutions we have exhibited for $k = 3m + 1$ or $k = 3m + 2$, $m \geq 3$, tend to $1/3$ when m tends to infinity. Thus, except for H_7 and H_8, whose densities are within 1.3% of the optimal, the solutions we have shown are optimal or within less than 1% from the optimal one.

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