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MULTIPLICITIES OF ZERO-SCHEMES IN QUASIHOMOGENEOUS CORANK-1 SINGULARITIES

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RESUMO

Dado um germe de aplicação $F : \mathbb{C}^n \to \mathbb{C}^n$, de corank 1 na origem. Consideramos uma perturbação estável F_ε de F e investigamos a multiplicidade dos esquemas zero-dimensionais que ocorrem no discriminante $\triangle(F_\varepsilon)$. Quando F é quase-homogêneo obtemos uma fórmula exprimindo as multiplicidades em termos dos pesos e do grau.

Multiplicities of zero-schemes in quasihomogeneous corank-1 singularities

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Abstract

How many cusps does a swallowtail have,
After it becomes a stable map,
Oh and how many swallowtails does a butterfly have,
After ...
(with apologies to B. Dylan)

Introduction

Consider the map

$$\begin{array}{rcl} F: \mathbf{C}^2 & \to & \mathbf{C}^2 \\ (x,y) & \mapsto & (x, y^4 + xy), \end{array}$$

(which is a section of the swallowtail singularity) and its perturbation

$$F_\varepsilon(x,y) = (x, y^4 + xy + \varepsilon y^2).$$

The singular set of F is given by $4y^3 + x = 0$, and the discriminant $\Delta(F)$ of F (the image of its singular set) is a curve with a singular point at the origin. The singular set of F_ε is also a smooth curve, but its image $\Delta(F_\varepsilon)$ is a curve with 2 cusps (A_2-points) and a double point (an $A_{(1,1)}$-point) — see Figure 1.

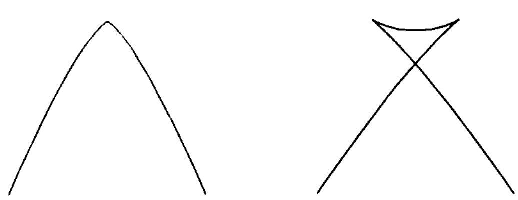

Figure 1: Discriminants of F and F_ε — the swallowtail

It turns out (and is well-known) that the number of cusps and double points is independent of the perturbation, provided the perturbation is a stable map. T. Fukuda and G. Ishikawa [2] show that the number of cusps is given by the dimension of a local algebra associated to F, and independently J. Rieger [8] gives formulae for both the number of cusps and the number of double points in the case that F is of corank 1 — see also [9]. T. Gaffney and D. Mond [4] give formulae for both the number of cusps and the number of double points for a general finitely-determined map-germ $\mathbf{C}^2 \to \mathbf{C}^2$.

In this paper, we consider the analogous problem for map-germs $F : \mathbf{C}^n \to \mathbf{C}^n$; that is, given such a map-germ, consider a perturbation which is stable, and ask how many occurrences of each isolated feature in $\Delta(F_\varepsilon)$ there are. The features are the *zero-schemes* of the title, and the numbers are the *multiplicities*. We are able to give answers in the case that F *is of corank 1*. In particular, if F is weighted homogeneous, then we give a closed formula for these numbers in terms of the weights and degrees of F. However, unlike Fukuda, Ishikawa and Rieger, we do not consider the case of real map-germs $\mathbf{R}^2 \to \mathbf{R}^2$.

A 3-dimensional example analogous to the swallowtail one above can be obtained by taking a section of the butterfly:

$$\begin{array}{rccc} F: & \mathbf{C}^3 & \to & \mathbf{C}^3 \\ & (x_1, x_2, y) & \mapsto & (x_1, x_2, y^5 + x_1 y^2 + x_2 y). \end{array}$$

Here the singular set is a smooth surface in $\mathbf{C}^3$, whose image $\Delta(F)$ is a surface with a cuspidal edge and a more degenerate point at the origin — see Figure 2. A stable perturbation (or stabilization) F_ε can be given by

$$F_\varepsilon(x_1, x_2, y) = (x_1, x_2, y^5 + x_1 y^2 + x_2 y + \varepsilon y^3).$$

The interesting isolated features (zero-schemes) of $\Delta(F_\varepsilon)$ are the 2 swallowtail points (A_3-points), and the 2 points where a cuspidal edge passes through a smooth sheet ($A_{(2,1)}$-points). There could in principle be a further isolated feature, namely a triple point of $\Delta(F_\varepsilon)$ where three smooth sheets intersect ($A_{(1,1,1)}$-points), but in fact it doesn't occur in this example. The purpose of this paper is to be able to predict these numbers from the form of F, without studying F_ε explicitly. For example, if y^5 were replaced by y^6 in the butterfly example above, then according to Theorem 1, any stabilization would have 1 $A_{(1,1,1)}$-point, 6 $A_{(2,1)}$-points and 3 A_3-points.

In general, let $F : (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ be a map-germ with a degenerate (non-stable) singularity, and let $\tilde{F}$ be a 1-parameter *stabilization* of F. We assume that F is of corank 1 (that is, dF_0 has rank $n-1$). If F is finitely-determined, then the singularity of F (at 0) splits up into a number of non-degenerate zero-dimensional (stable) singularities of $\tilde{F}$, which we now describe.

A stable map-germ $G : (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ has an A_k singularity ($k \leq n$) if it is left-right equivalent to the germ,

$$(x_1, \ldots, x_{n-1}, y) \mapsto (x_1, \ldots, x_{n-1}, y^{k+1} + x_1 y^{k-1} + \cdots x_{k-1} y).$$

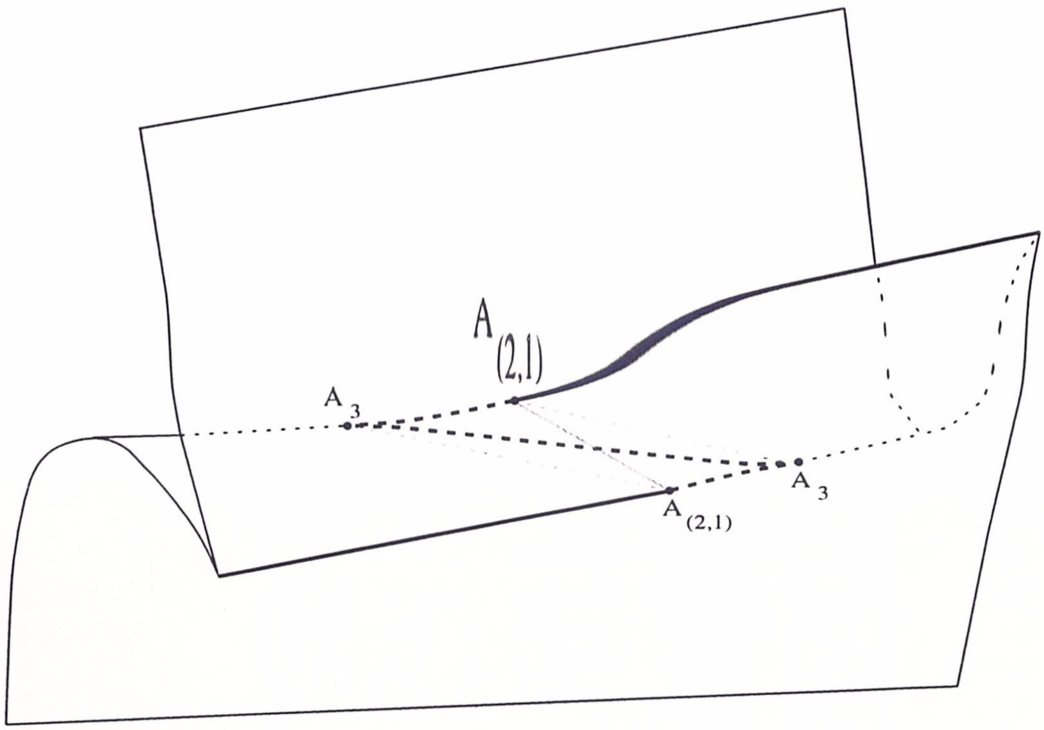

Figure 2: Discriminant of F_ε ($\varepsilon < 0$) — the butterfly
(thick line is cuspidal edge, grey line is self-intersection, broken lines are hidden)

Moreover, any stable corank 1 map-germ is an A_k for some natural number k. As is easily seen from the normal form, the set of points in $\mathbf{C}^n$ where a stable map has an A_k singularity is a submanifold of codimension k (given by $x_1 = \cdots = x_{k-1} = y = 0$). The image of this set is also a smooth submanifold of codimension k. It turns out (Mather-Gaffney geometric critierion [10]) that a map with only corank 1 singularities is stable if and only if these submanifolds in the discriminant are in general position.

Suppose the map $G : \mathbf{C}^n \to \mathbf{C}^n$ is stable (and defined on some open subset of $\mathbf{C}^n$). Let z be in the image of G, and put $S = G^{-1}(z) = \{s_1, \ldots, s_d\}$. Suppose G has an A_{r_j} singularity ($r_j \geq 0$) at s_j (for $j = 1, \ldots, d$). Then z represents a *zero-scheme* if and only if $r_1 + \cdots + r_d = n$. That is, after suppressing those r_j equal to zero, $\mathcal{P} = (r_1, \ldots, r_\ell)$ is a partition of n. We call such a multi-singularity an $A_\mathcal{P}$-singularity. In the case $n = 2$, the two possibilities are a cusp, with $\mathcal{P} = (2)$, and a double-fold, with $\mathcal{P} = (1, 1)$; for $n = 3$ the three possibilities are a swallowtail, with $\mathcal{P} = (3)$, a fold-cusp, with $\mathcal{P} = (2, 1)$ and a triple fold, with $\mathcal{P} = (1, 1, 1)$ — as in the examples above.

The question is, given an $\mathcal{A}$-finite (i.e. of finite $\mathcal{A}$-codimension or equivalently $\mathcal{A}$-finitely determined) map-germ $F : (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$, and a partition $\mathcal{P}$ of n, how many $A_\mathcal{P}$ singularities are there in a stabilization of F, in a suitably small neighbourhood of 0? This number is independent of the particular stabilization chosen, and we denote it $\sharp A_\mathcal{P} = \sharp A_\mathcal{P}(F)$.

We consider corank-1 map-germs from $X = (\mathbf{C}^n, 0)$ to $Y = (\mathbf{C}^n, 0)$. Choosing linearly adapted coordinates, we write

$$\begin{array}{rcl} F : \mathbf{C}^{n-1} \times \mathbf{C} & \to & \mathbf{C}^{n-1} \times \mathbf{C} \\ (x, y) & \mapsto & (x, f(x, y)). \end{array}$$

When F is weighted homogeneous, we put,

$$\begin{array}{rclcrcl} w_0 & = & \mathtt{wt}(y), & & w_i & = & \mathtt{wt}(x_i), \\ d & = & \text{degree}(f), & & w & = & \prod_{i=1}^{n-1} w_i. \end{array} \tag{1}$$

Let $\mathcal{P} = (r_1, \dots, r_\ell)$ be a partition of n, with $r_1 \geq r_2 \geq \cdots$, and call ℓ the *length* of $\mathcal{P}$. Define $N(\mathcal{P})$ to be the order of the subgroup of S_ℓ which fixes $\mathcal{P}$. Here S_ℓ acts on $\mathbf{R}^\ell$ by permuting the coordinates. For example, for $\mathcal{P} = (4, 4, 2, 2, 2, 1, 1, 1)$ we have $N(\mathcal{P}) = (2!)(3!)^2 = 72$.

Theorem 1 *Let $F : (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ be a corank-1 weighted-homogeneous $\mathcal{A}$-finite map-germ, with weights and degrees as above. For any stabilization of F, and any partition $\mathcal{P}$ of n,*

$$\sharp A_{\mathcal{P}} = \frac{w_0^{n-1}}{N(\mathcal{P}) w} \prod_{j=1}^{n+\ell-1} \left(\frac{d}{w_0} - j \right),$$

where ℓ is the length of $\mathcal{P}$.

1 Intermediate results

Associated to $X = \mathbf{C}^{n-1} \times \mathbf{C}$ and $\mathcal{P}$ we will be considering various spaces. In particular,

$$\begin{array}{rcl} X_\ell & = & \mathbf{C}^{n-1} \times \mathbf{C}^\ell, \\ X^\ell & = & \mathbf{C}^{n-1} \times \mathbf{C}^{\ell+n}. \end{array}$$

We will also be considering a versal deformation $\tilde{F}$ of F, with base $\mathbf{C}^d$, and then we denote $\tilde{X}_\ell = \mathbf{C}^d \times X_\ell$, and similarly $\tilde{X}^\ell = \mathbf{C}^d \times X^\ell$.

Let $\tilde{F} : \tilde{X} \to \tilde{Y}$ be an $\mathcal{A}_e$-versal unfolding of F (with base $\mathbf{C}^d$), so that

$$\tilde{F}(u, x, y) = (u, x, \tilde{f}(x, y, u)) = (u, \tilde{F}_u(x, y)).$$

For each partition $\mathcal{P} = (r_1, \dots, r_\ell)$ of n we consider (following ideas of Gaffney [3]) the subscheme $\tilde{V}(\mathcal{P})$ of $\tilde{X}_\ell := \mathbf{C}^d \times \mathbf{C}^{n-1} \times \mathbf{C}^\ell$, where $\ell = \text{length}(\mathcal{P})$, defined by

$$\tilde{V}(\mathcal{P}) := \text{clos} \left\{ (u, x, y_1, \dots, y_\ell) \in \tilde{X}_\ell \;\middle|\; \begin{array}{l} \bullet\ y_i \neq y_j, \\ \bullet\ F(u, x, y_i) = F(u, x, y_j), \text{ and} \\ \bullet\ \tilde{F}_u \text{ has a singularity of type } A_{r_j} \\ \quad \text{at } (u, x, y_j) \end{array} \right\},$$

where 'clos' means the analytic closure in $\tilde{X}_\ell$. Let $\pi = \pi(\mathcal{P}) : \tilde{V}(\mathcal{P}) \to \mathbf{C}^d$ be the restriction to $\tilde{V}(\mathcal{P})$ of the Cartesian projection $\tilde{X}_\ell \to \mathbf{C}^d$. For generic $u \in \mathbf{C}^d$, the fibre $\pi^{-1}(u)$ consists of those 'multi-points' (also known as 'sets') where F has an $A_\mathcal{P}$ multi-germ. We are thus interested in the degree of $\pi(\mathcal{P})$.

Proposition 2 *If $\mathcal{P} = (r_1, \ldots, r_\ell)$ is a partition of n, then*

$$\sharp A_\mathcal{P} = \frac{1}{N(\mathcal{P})}\text{degree}(\pi(\mathcal{P})).$$

PROOF Let $\mathbf{y} = (y_1, \ldots, y_\ell) \in \tilde{V}(\mathcal{P})$ and $\sigma \in S_\ell$. We have

$$\mathbf{y}^\sigma := (y_{\sigma(1)}, \ldots, y_{\sigma(\ell)}) \in \tilde{V}(\mathcal{P})$$

if and only if $r_{\sigma(j)} = r_j$ for each $j = 1, \ldots, \ell$. There are $N(\mathcal{P})$ such σ. The points $\mathbf{y}$ and $\mathbf{y}^\sigma$ are distinct, but the corresponding sets are the same, and it is the sets that are counted in $\sharp A_\mathcal{P}$. □

Let $\tilde{\mathcal{I}}(\mathcal{P})$ be the ideal defining $\tilde{V}(\mathcal{P})$, and put

$$\mathcal{I}(\mathcal{P}) = (\tilde{\mathcal{I}}(\mathcal{P}) + \langle u_1, \ldots, u_d \rangle)/\langle u_1, \ldots, u_d \rangle \subset \mathcal{O}_{X_\ell},$$

corresponding to the intersection of $\tilde{V}(\mathcal{P})$ with $\{0\} \times X_\ell = X_\ell$. The main theorem follows from the following results, which are proved in §2.

Proposition 3 *$\tilde{V}(\mathcal{P})$ is smooth of dimension d, and $\pi(\mathcal{P}) : \tilde{V}(\mathcal{P}) \to \mathbf{C}^d$ is finite-1 and flat.*

Proposition 4 *$\mathcal{I}(\mathcal{P})$ is a zero-dimensional complete intersection and hence the degree of $\pi(\mathcal{P})$ coincides with $\dim_\mathbf{C} \mathcal{O}_{X_\ell}/\mathcal{I}(\mathcal{P})$.*

Proposition 5 *If F is weighted homogeneous, with weights and degree as in (1), the projection $\pi(\mathcal{P}) : \tilde{V}(\mathcal{P}) \to \mathbf{C}^d$ has degree*

$$d(\mathcal{P}) = \frac{1}{w_0^\ell w} \prod_{j=1}^{n+\ell-1} (d - jw_0).$$

2 Proofs

Nearby the $(A_{r_1} + \cdots + A_{r_\ell})$ multi-germs, there are points in the target with $(r_1+1) + (r_2+1) + \cdots + (r_\ell + 1)$ preimages (i.e. $n + \ell$ preimages). We shall define an $(n+\ell)$-tuple scheme in $X^\ell = \mathbf{C}^{n-1} \times \mathbf{C}^{n+\ell}$, which on the appropriate diagonal specializes to the ideal defining $(A_{r_1} + \cdots + A_{r_\ell})$ multi-germs (Lemma 6 below).

We denote the coordinates of X^ℓ by

$$(x, \mathbf{y}) = (x, y_0^1, \ldots, y_{r_1}^1, y_0^2, \ldots, y_{r_2}^2, \ldots, y_0^\ell, \ldots, y_{r_\ell}^\ell).$$

In X^ℓ there is a diagonal of particular interest, namely,

$$\Delta(\mathcal{P}) = \{(x, \mathbf{y}) \in X^\ell \mid y_i^k = y_j^k, \ \forall i, j = 1, \ldots, r_k, \ \forall k = 1, \ldots, \ell\},$$

which can be parametrized in the obvious way by $(x, y^1, \ldots, y^\ell)$:

$$(x, \mathbf{y}) = (x, y^1, \ldots, y^1, y^2, \ldots, y^2, \ldots, y^\ell, \ldots, y^\ell), \tag{2}$$

with y^i repeated $r_i + 1$ times. This corresponds to an embedding of X_ℓ in X^ℓ.

Let $\mathcal{I}_{\Delta(\mathcal{P})}$ be the ideal defining $\Delta(\mathcal{P})$, that is

$$\mathcal{I}_{\Delta(\mathcal{P})} = \left\langle y_i^k - y_0^k, \ \forall i = 1, \ldots, r_k, \ \forall k = 1, \ldots, \ell \right\rangle.$$

A generic point of $\Delta(\mathcal{P})$ is one of the form (2) with $y^i \neq y^j$, for $i \neq j$.

We define a sheaf of ideals $\mathcal{J}(f, \mathcal{P}) \subset \mathcal{O}_{X^\ell}$ by

$$\mathcal{J}(f, \mathcal{P}) = \langle h_i \mid i = 1, \ldots, n + \ell - 1 \rangle,$$

with

$$h_i = V^{-1}. \begin{vmatrix} 1 & y_0^1 & \cdots & (y_0^1)^{i-1} & f_0^1 & (y_0^1)^{i+1} & \cdots & (y_0^1)^{n+l-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & y_{r_1}^1 & \cdots & (y_{r_1}^1)^{i-1} & f_{r_1}^1 & (y_{r_1}^1)^{i+1} & \cdots & (y_{r_1}^1)^{n+l-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & y_0^\ell & \cdots & (y_0^\ell)^{i-1} & f_0^\ell & (y_0^\ell)^{i+1} & \cdots & (y_0^\ell)^{n+l-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & y_{r_\ell}^\ell & \cdots & (y_{r_\ell}^\ell)^{i-1} & f_{r_\ell}^\ell & (y_{r_\ell}^\ell)^{i+1} & \cdots & (y_{r_\ell}^\ell)^{n+l-1} \end{vmatrix},$$

where $V = V(y_0^1, \cdots, y_{r_1}^1, \cdots, y_0^\ell, \cdots, y_{r_\ell}^\ell)$ is the Vandermonde determinant and

$$f_i^k = f(x, y_i^k).$$

Lemma 6 ([6, Lemma 2.7]) *At a generic point of $\Delta(\mathcal{P})$ we have,*

$$\begin{aligned} \mathcal{J}(f, \mathcal{P}) + \mathcal{I}_{\Delta(\mathcal{P})} &= \left\langle (\partial_y f)_1, \ldots, (\partial_y^{r_1} f)_1, \ldots, (\partial_y f)_\ell, \ldots, (\partial_y^{r_\ell} f)_\ell \right\rangle \\ &\quad + \left\langle f(x, y^i) - f(x, y^1); 2 \leq i \leq \ell \right\rangle + \mathcal{I}_{\Delta(\mathcal{P})}. \end{aligned}$$

PROOF OF PROPOSITION 3 It is clear that $\tilde{V}(\mathcal{P})$ is an analytically closed set of dimension d. In fact, it follows from [5, Cor. 4.3.3] that $\tilde{V}(\mathcal{P})$ is smooth of dimension d. The projection $\pi(\mathcal{P}) : \tilde{V}(\mathcal{P}) \to \mathbf{C}^d$ is a finite mapping. Indeed, f is finitely $\mathcal{A}$-determined and $A_{\mathcal{P}}$ is a 0-stable multigerm, so by the Mather-Gaffney geometric criterion [10], the fibre over zero is either the origin or empty. Moreover, the ideal $\mathcal{I}(\mathcal{P})$ defining the fibre of π over zero is obtained as the specialization of $\mathcal{J}(f,\mathcal{P})$ to the principal diagonal Δ_0, whose generators are $y^i - y^j$, for $i, j = 1 \ldots l$. In fact, by Lemma 6,

$$\mathcal{J}(f,\mathcal{P}) + \mathcal{I}_{\Delta_0} = \left\langle (\partial_y f), \ldots, (\partial_y^{n+l-1} f) \right\rangle + \mathcal{I}_{\Delta_0}.$$

The generators $(\partial_y f), \ldots, (\partial_y^{n+l-1} f)$ form a regular sequence in $\mathbf{C}^{n-1+\ell}$. So, by [7] the projection $\pi(\mathcal{P}) : \tilde{V}(\mathcal{P}) \to \mathbf{C}^d$ is flat. □

PROOF OF PROPOSITION 4 If it were not a complete intersection, the projection $\pi(\mathcal{P})$ would not be finite. □

PROOF OF PROPOSITION 5 Note that

$$\text{degree}(h_i) = d - iw_0,$$

so that the product of all the degrees of the h_i is

$$\prod_{j=1}^{n+\ell-1} (d - jw_0).$$

From Bezout's theorem applied to the ideal $\mathcal{J}(F,\mathcal{P}) + \mathcal{I}_{\Delta(\mathcal{P})}$, it follows that its degree is

$$\frac{1}{w_0^{\ell+n} w} \prod_{j=1}^{n+\ell-1} (d - jw_0) w_0^n = \frac{1}{w_0^{\ell} w} \prod_{j=1}^{n+\ell-1} (d - jw_0).$$

Therefore, by Lemma 6 above, this is also $d(\mathcal{P})$. □

3 Multiplicities of strata in A_k discriminants

In this final section, we use Theorem 1 to give a simple formula for the local multiplicity of the closure of each stratum in the discriminant of an A_k singularity.

Consider the stable A_k map $f : \mathbf{C}^k \to \mathbf{C}^k$,

$$f(x_1, \ldots, x_{k-1}, y) = (X_1, \ldots, X_{k-1}, Y) = (x_1, \ldots, x_{k-1}, y^{k+1} + x_1 y^{k-1} + \cdots + x_{k-1} y).$$

This map is clearly weighted homogeneous, with weights $\text{wt}(x_i) = \text{wt}(X_i) = i + 1$, $\text{wt}(y) = 1$ and $\text{wt}(Y) = k + 1$. The discriminant $\Delta(f)$ is stratified by the various

$A_{\mathcal{P}}$ multi-germs, where $\mathcal{P} = (r_1, \dots, r_\ell)$ is a partition of $n \leq k + 1 - \ell$. Denote this stratum by $\Delta_{\mathcal{P}}$ and its closure by $Z_{\mathcal{P}}$. $Z_{\mathcal{P}}$ is an algebraic subvariety of $\mathbf{C}^k$ of dimension $D = k - n$. Note that Goryunov [5, §4.3] shows that if $n > k + 1 - \ell$ then $\Delta_{\mathcal{P}}$ is empty (his $D(\mu_1, \dots, \mu_k)$ corresponds to our $\Delta_{\mathcal{P}}$ for $\mathcal{P} = (\mu_1 + 1, \dots, \mu_k + 1)$).

Theorem 7 *The multiplicity of $Z_{\mathcal{P}}$ at the origin is given by,*

$$\frac{1}{N(\mathcal{P})}(D+1)D(D-1)\dots(D-\ell+2),$$

where $D = \dim(Z_{\mathcal{P}})$ and $N(\mathcal{P})$ is defined in the introduction.

To prove this, we first need a lemma on the geometric structure of A_k discriminants.

Lemma 8 *Let $Z_{\mathcal{P}}$ be as above, and let (z_i) be any sequence of points in $Z_{\mathcal{P}}$ converging to 0. Then*

$$T_0 Z_{\mathcal{P}} := \lim_{i\to\infty} T_{z_i} Z_{\mathcal{P}} = \{(\mathbf{X}, Y) \mid Y = X_{k-n+1} = X_{k-n+2} = \dots = X_{k-1} = 0\}.$$

PROOF As is well-known and easy to see, the discriminant of f coincides with the discriminant of the orbit map $\sigma_0 : \mathbf{C}^k_s \to \mathbf{C}^k_t$ for the action of the permutation group S_{k+1}, where $\mathbf{C}^k_s$ is identified with the subspace of $\mathbf{C}^{k+1}$ the sum of whose coordinates vanishes, and S_{k+1} acts on $\mathbf{C}^{k+1}$ by permuting the coordinates. Consider the extension σ of σ_0 to $\mathbf{C}^{k+1}$ defined as usual by,

$$\begin{array}{rcl} \sigma : \mathbf{C}^{k+1} & \longrightarrow & \mathbf{C}^{k+1} \\ (y_1, \dots, y_{k+1}) & \mapsto & (\sum_i y_i, \sum_{i<j} y_i y_j, \dots, y_1 \dots y_{k+1}). \end{array}$$

Clearly, $\mathbf{C}^k_t$ is to be identified with the subspace of $\mathbf{C}^{k+1}$ with vanishing first coordinate. It will be more convenient for computations to change coordinates in the target of σ so that σ takes the form

$$\tilde{\sigma}(y_1, \dots, y_{k+1}) = (\sum_i y_i, \sum_i y_i^2, \sum_i y_i^3, \dots, \sum_i y_i^{k+1}).$$

Note that the linear subspaces of the form $T_0 Z_{\mathcal{P}}$ are preserved by the differential at the origin of this change of coordinates; indeed the differential is a diagonal matrix. Denote by $\tilde{\Delta}$ the discriminant of $\tilde{\sigma}$.

Given the partition $\mathcal{P} = (r_1, \dots, r_\ell)$ of n, the stratum $\tilde{\Delta}_{\mathcal{P}}$ is the image under $\tilde{\sigma}$ of $\Sigma_{\mathcal{P}} \subset \mathbf{C}^{k+1}$ parametrized by

$$(y_1, \dots, y_\ell) \mapsto (y_1, \dots, y_1, y_2, \dots, y_2, \dots, y_\ell, \dots, y_\ell),$$

where y_j occurs with multiplicity r_j, and the y_j are distinct. Write $\tilde{\sigma}_{\mathcal{P}}$ for the restriction of $\tilde{\sigma}$ to $\Sigma_{\mathcal{P}}$. Using this parametrization of $\Sigma_{\mathcal{P}}$, $\tilde{\sigma}_{\mathcal{P}}$ has the form,

$$\tilde{\sigma}_{\mathcal{P}}(y_1, \ldots, y_\ell) = (\sum_i r_i y_i, \sum_i r_i y_i^2, \ldots, \sum_i r_i y_i^{k+1}).$$

Thus, at a point $y \in \Sigma_{\mathcal{P}}$, the differential of $\tilde{\sigma}_{\mathcal{P}}$ is

$$d\tilde{\sigma}_{\mathcal{P}}(y) = \begin{bmatrix} r_1 & \cdots & r_\ell \\ 2r_1y_1 & \cdots & 2r_\ell y_\ell \\ \vdots & & \vdots \\ (k+1)r_1y_1^k & \cdots & (k+1)r_\ell y_\ell^k \end{bmatrix}.$$

Notice that the top $\ell \times \ell$ minor is equal to $\ell!\,(\prod r_i)\,V(y_1, \ldots, y_\ell)$, where V is the Vandermonde determinant, which is non-vanishing on $\tilde{\Delta}_{\mathcal{P}}$. Consequently, at points of $\tilde{\Delta}_{\mathcal{P}}$, the tangent space to $\tilde{\Delta}_{\mathcal{P}}$ projects isomorphically onto $\mathbf{C}^\ell$ (defined by the vanishing of the last $k-\ell$ coordinates).

Finally, since the $k-\ell$ components of $\tilde{\sigma}$ are of strictly higher degree than the first ℓ, it follows that in the limit as $(y_1, \ldots, y_\ell) \to (0, \ldots, 0)$, the tangent space to $\tilde{\Delta}_{\mathcal{P}}$ tends to $\mathbf{C}^\ell$. Intersecting source and target with $\mathbf{C}^k_s$ and $\mathbf{C}^k_t$ respectively shows that the same is true of the tangent space to $\Delta_{\mathcal{P}}$, as required.

□

PROOF OF THEOREM 7 It follows from this lemma that the multiplicity at 0 of $Z_{\mathcal{P}}$ is given by the intersection multiplicity of $Z_{\mathcal{P}}$ with the n-dimensional subspace

$$\{(\mathbf{X}, Y) \mid X_1 = \cdots = X_{k-n} = 0\},$$

which is transverse to this unique limiting tangent space $T_0Z_{\mathcal{P}}$, and it remains for us to compute this multiplicity.

To this end, consider the map $g : \mathbf{C}^n \to \mathbf{C}^n$ defined by

$$g(u_1, \ldots, u_{n-1}, y) = (u_1, \ldots, u_n, y^{k+1} + u_1y^{n-1} + \cdots + u_{n-1}y),$$

which is induced from f by the immersion $\gamma : \mathbf{C}^n \to \mathbf{C}^k$,

$$\gamma(u_1, \ldots, u_{n-1}, y) = (0, \ldots, 0, u_1, \ldots, u_{n-1}, y),$$

in the sense that $f \circ \gamma = \gamma \circ g$.

By the lemma, this inclusion is transverse to $\Delta(f)$ away from the origin, so that it is $\mathcal{K}_{\Delta(f)}$-finite, and consequently, g is $\mathcal{A}$-finite (Damon [1]). Moreover, a stabilization g_ε of g is obtained by perturbing the embedding γ to an embedding γ_ε transverse to $\Delta(f)$, and *a fortiori* transverse to $Z_{\mathcal{P}}$. If γ_ε is transverse to $Z_{\mathcal{P}}$, then $\mathrm{im}(\gamma_\varepsilon) \cap Z_{\mathcal{P}} = \mathrm{im}(\gamma_\varepsilon) \cap \Delta_{\mathcal{P}}$ is a finite set (for dimensional reasons).

The points of this intersection are precisely the image under γ_ε of the points in $\mathbf{C}^n$ (the image of g_ε) over which g_ε has $A_{\mathcal{P}}$ an singularity. Since g is weighted homogeneous, the number of such points is given by Theorem 1. A simple computation then proves Theorem 7. □

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