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Estimation of the density of point processes on $\mathbb{R}^m$ via wavelets

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Abstract

In this article we consider the problem of estimating the density of the expectation measure with respect to Lebesgue measure of point processes on $\mathbb{R}^m$. We present an estimator of the density for a very wide class of point processes. The only requirement that is made upon the point processes is that the density is locally square integrable. The estimation is made by wavelet expansions.

keywords. Density, intensity, non-internally correlated point processes, point processes, Poisson processes, sure inference, threshold, wavelets.

1. Introduction

In this work we consider the problem of estimating the density of point processes on $\mathbb{R}^m$. Both simple and non-simple point processes are considered for estimation purposes. If a point process is simple then its intensity is equal to its density so that the estimation of the intensity of simple point processes follows as a particular case.

We propose the following methodology: the restriction of the density function to the observation region, which is initially assumed to be an $\mathbb{R}^m$-interval where we know the points of occurrence of events of a trajectory of the process, is expanded in a wavelet series. Unbiased estimators for the wavelet coefficients as well as unbiased estimators of the variance of the former estimators are given. We also obtain inferential sequences for the wavelets coefficients in case of non-internally correlated point processes (and for non-homogeneous Poisson Process as a particular case). Using the coefficient estimators we obtain an unbiased estimator of the density by a syntheses procedure. Inferential sequences for the density are also given.

The plan of this article is the following. In section 2 we define classes of point processes to be considered for estimation purposes,for which it is possible to calculate confidence bands for the density function,and present some useful properties as well as the basic concepts of "sure inference". Section 3 is devoted to solve the problem of estimation of the density of point processes on $\mathbb{R}^m$ when the observation region is an $\mathbb{R}^m$ interval. In section 4 comments are made upon the estimation of the density for observation regions other then intervals and we close the work presenting a conclusion in section 5.

2. Classes of processes, working properties and sure inference analysis.

2.1. Notation. We first introduce some notation that will be necessary. We will work with Lebesgue measurable functions, $h : \mathbb{R}^m \to \mathbb{R}$ which are bounded in bounded intervals of $\mathbb{R}^m$ or, equivalently, which are integrable in the sense of Lebesgue and bounded on bounded intervals of $\mathbb{R}^m$. Let us call this class of functions $\mathcal{L}^m$. Denote by $\overline{\mathcal{L}}^m$ the class of functions which are Lebesgue integrable over bounded intervals of $\mathbb{R}^m$.

We will use the notation $|a,b|$, $a=(a_1,\ldots,a_m)$, $b=(b_1,\ldots,b_m)$ to represent any of the 4^m possible intervals of $\mathbb{R}^m$ which can be written in the form $\prod_{i=1}^m |a_i,b_i|$, where $|a_i,b_i|$ represents one of the intervals (a_i,b_i), $(a_i,b_i]$, $[a_i,b_i)$ or $[a_i,b_i]$ of the real line. We also use the notation χ_C for the characteristic function (or indicator) of a set C ($\chi_C(x)=1 \leftrightarrow x\in C \,\wedge\, \chi_C(x)=0 \leftrightarrow x\notin C$). Lebesgue measure on $\mathbb{R}^m$ will be indicated simply by ℓ independently of the dimension m. If it is necessary to emphasize the dimension we will write ℓ_m. The σ-algebra of Lebesgue measurable sets in $\mathbb{R}^m$ is denoted by $\Lambda_{\mathbb{R}^m}$. $B_{\mathbb{R}^m}$ is used for the σ-algebra of Borel sets. Functions that differ over zero measure subsets of their common domain or of a common extension of their domains are, naturally, considered identical. All functions that we consider are assumed to be measurable.

Point processes are usually denoted by N. We remind that a point processes is a measurable function from a probability space (Ω, A, P) into $(\hat{N}_{\mathbb{R}^m}, B_{\hat{N}_{\mathbb{R}^m}})$ where $\hat{N}_{\mathbb{R}^m}$ stands for the set of boundedly finite integer valued measures defined on the Borel sets of $\mathbb{R}^m$ and $B_{\hat{N}_{\mathbb{R}^m}}$ is the class of Borel sets of $\hat{N}_{\mathbb{R}^m}$. We will also denote by N the realizations of a point process, i.e., elements of $\hat{N}_{\mathbb{R}^m}$. We remaind that they are typically written as $N=\sum_{i\in I} k_i\delta_{x_i}$ where $k_i\in\mathbb{N}^*$, $\sharp I\le\sharp\mathbb{N}$, and δ_{x_i} is the Dirac measure with atom at $x_i\in\mathbb{R}^m$. The expectation measure is denoted by EN and the moment measures by $M_k=E(\prod_{i=1}^{k} N)$. (See Daley and Vere-Jones, 1988 for further information on point processes).

2.2. **Classes of processes.** We will work with classes of point processes that satisfy at least one of the following.

Definition 2.1. *Assumption $\mathcal{B}$. A point process satisfies assumption $\mathcal{B}$ if and only if $EN \ll \ell$.*

Definition 2.2. *Assumption $\mathcal{A}$. A point process satisfies assumption $\mathcal{A}$ if and only if it satisfies assumption $\mathcal{B}$ and $E(N\times N)(A\cap D)=EN\pi_1(A\cap D)$ holds for all $A\in B_{\mathbb{R}^{2m}}$ where D is the diagonal set of $\mathbb{R}^{2m}=\mathbb{R}^m\times\mathbb{R}^m$, i.e., $D=\{(x,x)\in\mathbb{R}^{2m}|x\in\mathbb{R}^m\}$ and π_1 is the projection $\pi_1:\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}^m$, $\pi_1(x,y)=x$.*

Definition 2.3. *A point process is called non-internally correlated (NIC) if and only if for all A and B disjoint Borel sets of $\mathbb{R}^m$ we have* $\mathrm{Cov}\,(N(A),N(B))=0$.

Definition 2.4. *A point process N on $\mathbb{R}^m$ is called n-th order non-internally correlated (NIC ^{n}) if and only if for all A_k and B_k disjoint Borel sets of $\mathbb{R}^{2^k m}$ we have* $\mathrm{Cov}\,(\prod_{i=1}^{2^k} N, \prod_{i=1}^{2^k} N)(A_k,B_k)=0$ *for all k, $0\le k\le n$. If a point process is NIC ^{n} for all $n\in N$ we say it is an infinite order NIC point process, NIC $^\infty$.*

We will denote by $\mathcal{B}$, $\mathcal{A}$, NIC, NIC ^{n} or NIC $^\infty$ the classes of point processes that satisfy, respectively, definitions 2.1 to 2.4.

2.3. **Working properties.**

Proposition 2.1. *If N satisfies Assumption $\mathcal{B}$ then, for all EdN-integrable function, φ, we have $\int\varphi dEN=\int\varphi\nu_N d\ell$.*

Proof Immediate, since $\nu_N = \mathrm{d}EN/\mathrm{d}\ell$ a.e.[ℓ]. ∎

Let us denote by D_k the diagonal set of $\mathbb{R}^{km} = \mathbb{R}^m \times \ldots \times \mathbb{R}^m$, i.e., $D_k = \{(x,\ldots,x) \in \mathbb{R}^{km} | x \in R^m\}$.

Proposition 2.2. *If N satisfies Assumption $\mathcal{B}$ then, for all functions φ integrable with respect to the covariance measure* $\mathrm{Cov}\,(N,N)$, *we have:*

$$\int \varphi \mathrm{d}\mathrm{Cov}\,(N,N) = \int_{\mathbb{R}^{2m}-D} \varphi \mathrm{d}\mathrm{Cov}\,(N,N) + \int_D \varphi \mathrm{d}M_2.$$

Proof It is enough to prove that $\int_D \varphi \mathrm{d}\mathrm{Cov}\,(N,N) = \int_D \varphi \mathrm{d}M_2$.

$$\begin{aligned}\int_D \varphi \mathrm{d}\mathrm{Cov}\,(N,N) &= \int_D \varphi \mathrm{d}(E(N\times N) - EN \times EN) \\ &= \int_D \varphi \mathrm{d}M_2 - \int_D \varphi \frac{\mathrm{d}EN}{\mathrm{d}\ell} \otimes \frac{\mathrm{d}EN}{\mathrm{d}\ell} \mathrm{d}\ell \times \mathrm{d}\ell = \int_D \varphi \mathrm{d}M_2\end{aligned}$$

since $\ell_{2m}(D) = (\ell_m \times \ell_m)(D) = 0$. ∎

Proposition 2.3. *If N satisfies Assumption $\mathcal{A}$ then, for all functions φ_1 integrable with respect to the covariance measure* $\mathrm{Cov}\,(N,N)$, *we have:*

$$\int \varphi_1 \mathrm{d}\mathrm{Cov}\,(N,N) = \int_{\mathbb{R}^{2m}-D} \varphi_1 \mathrm{d}\mathrm{Cov}\,(N,N) + \int_{\mathbb{R}^m} \varphi \nu_N \mathrm{d}x, \;\; \varphi(x) = \varphi_1(x,x).$$

Proof For all $N \in \mathcal{A}$ we have

$$\int_D \varphi_1 \mathrm{d}M_2 = \int_D \varphi\pi_1 \mathrm{d}E(N\times N) = \int_D \varphi\pi_1 \mathrm{d}(EN\pi_1) = \int_{\pi_1(D)} \varphi \mathrm{d}EN = \int_{\mathbb{R}^m} \varphi\nu_N \mathrm{d}x$$

and the result follow from proposition 2.2. ∎

We will also write

$$\int \varphi(x)\nu_N(x)\mathrm{d}x = \int \varphi(x)\mathrm{Var}\,(\mathrm{d}N(x))$$

where the right hand side means $\iint_{D_2} \varphi_1(u,v)\mathrm{Cov}\,(\mathrm{d}N(u),\mathrm{d}N(v))$, D_2 the diagonal set of $\mathbb{R}^{2m}$ and $\varphi(x) = \varphi_1(x,x)$.

The following proposition is useful for the calculation of covariances of random variables associated to point processes that are written as integrals.

Proposition 2.4. *Let X and Y be random variables defined by the stochastic integrals $X = \int_A f\mathrm{d}N$ and $Y = \int_B g\mathrm{d}N$, D_2 diagonal set of $\mathbb{R}^{2m}$, π_1 the projection $\pi_1(x,y) = x$ and $A, B \in \mathcal{B}_{\mathbb{R}^m}$ such that* $(\mathrm{supp}\, f \cap A) \times (\mathrm{supp}\, g \cap B)$ *is bounded. For N under Assumption $\mathcal{B}$ we have*

$$\mathrm{Cov}\,(X,Y) = \int_{(A\times B)-D_2} f\otimes g \;\; \mathrm{Cov}\,(\mathrm{d}N,\mathrm{d}N) + \int_{(A\times B)\cap D_2} f\otimes g\mathrm{d}M_2.$$

If $\mathrm{Cov}\,(\mathrm{d}N,\mathrm{d}N) \ll \mathrm{d}\ell\times\mathrm{d}\ell$ *on* $(A\times B)-D_2$, *i.e., there exists* $q_2 \in \overline{\mathcal{L}}^{2m}$, $\mathrm{d}\mathrm{Cov}\,(N,N) = q_2(u,v)\mathrm{d}u\mathrm{d}v$,

$$\mathrm{Cov}\,(X,Y) = \int_{(A\times B)-D_2} f(u)g(v)q_2(u,v)\mathrm{d}u\mathrm{d}v + \int_{(A\times B)\cap D_2} f\otimes g\mathrm{d}M_2.$$

If N is NIC then

$$\mathrm{Cov}\,(X,Y) = \int_{(A\times B)\cap D_2} f\otimes g \mathrm{d}M_2.$$

For N under Assumption $\mathcal{A}$ we have

$$\mathrm{Cov}\,(X,Y) = \int_{(A\times B)-D_2} f\otimes g \;\; \mathrm{Cov}\,(\mathrm{d}N,\mathrm{d}N) + \int_{\pi_1((A\times B)\cap D_2)} fg\nu_N \mathrm{d}\ell.$$

If $\mathrm{Cov}\,(\mathrm{d}N,\mathrm{d}N) \ll \mathrm{d}\ell\times\mathrm{d}\ell$ *on* $(A\times B)-D_2$, *i.e., there exists* $q_2 \in \overline{\mathcal{L}}^{2m}$, $\mathrm{dCov}\,(N,N) = q_2(u,v)\mathrm{d}u\mathrm{d}v$,

$$\mathrm{Cov}\,(X,Y) = \int_{(A\times B)-D_2} f(u)g(v)q_2(u,v)\mathrm{d}u\mathrm{d}v + \int_{\pi_1((A\times B)\cap D_2)} f(x)g(x)\nu_N(x)\mathrm{d}x.$$

If N is NIC then

$$\mathrm{Cov}\,(X,Y) = \int_{\pi_1((A\times B)\cap D_2)} f(x)g(x)\nu_N(x)\mathrm{d}x.$$

Proof Since

$$E(XY) = E\left(\iint_{A\times B} f(u)g(v)\mathrm{d}N(u)\mathrm{d}N(v)\right) = \iint_{A\times B} f(u)g(v)E(\mathrm{d}N(u)\mathrm{d}N(v))$$

and also

$$E(X)E(Y) = \int_A f(u)E\mathrm{d}N(u)\int_B g(v)E\mathrm{d}N(v) = \iint_{A\times B} f(u)g(v)E\mathrm{d}N(u)E\mathrm{d}N(v),$$

we have

$$\begin{aligned}\mathrm{Cov}\,(X,Y) &= \iint_{A\times B} f(u)g(u)\left[E(\mathrm{d}N(u)\mathrm{d}N(v)) - E\mathrm{d}N(u)E\mathrm{d}N(v)\right] \\ &= \iint_{A\times B} f(u)g(v)\mathrm{Cov}\,(\mathrm{d}N(u),\mathrm{d}N(v)).\end{aligned}$$

Now, the proposition is established using Propositions 2.2 and 2.3 and noting that if N is NIC then $q_2(u,v) = 0$. ∎

Observe that, since Poisson processes are special cases of NIC point processes, the sixth equality above is fulfilled for Poisson processes.

2.4. Sure Inference Analysis.

Now we present the basics of sure inference analysis. We remark that this is an analysis of inference that can be successfully used on distribution free settings.

Definition 2.5. *The triple* $(X,(V_n)_{n\in\mathbb{N}^*},(\hat{V}_n)_{n\in\mathbb{N}^*})$ *formed by a random variable* $X:\Omega\to\mathbb{R}$, *a sequence of positive numbers* $(V_n)_{n\in\mathbb{N}^*}$ *and a sequence of random variables* $(\hat{V}_n:\Omega\to\mathbb{R})_{n\in\mathbb{N}^*}$, *is an* **inferential sequence** *for* $x\in\mathbb{R}$ *if and only if the following are valid:*

(i) $EX = x$, $V_1 = \mathrm{Var}\,(X)$,
(ii) $\forall n\in\mathbb{N}^*\;\; V_{n+1} = \mathrm{Var}\,(\hat{V}_n)$,
(iii) $\forall n\in\mathbb{N}^*\;\; E\hat{V}_n = V_n$,
(vi) $\forall n\in\mathbb{N}^*\;\; \hat{V}_n(\Omega)\subset\mathbb{R}_+$.

We will use the notation $(X,\ V_n,\ \hat{V}_n)$ to represent an inferential sequence and, occasionally, we will simply say that the sequences V_n and $\hat{V}_n$ form an inferential sequence for x.

Theorem 2.1. **(On the inferential sequence of random variables.)** *Let $(X,\ V_n,\ \hat{V}_n)$ be an inferential sequence for $x \in \mathbb{R}$. If*

$$L_m(\omega, \lambda_1, \ldots, \lambda_m) = \lambda_1 \sqrt{\hat{V}_1(\omega) + \lambda_2 \sqrt{\cdots + \lambda_{m-1} \sqrt{\hat{V}_{m-1}(\omega) + \lambda_m \sqrt{V_m}}}},$$

$\lambda_i \in \mathbb{R}_+^*$ for $1 \leq i \leq m$, $m \in \mathbb{N}^*$, then

$$P\{x \in [X(\omega) - L_m(\omega, \lambda_1, \ldots, \lambda_m), X(\omega) + L_m(\omega, \lambda_1, \ldots, \lambda_m)]\} \geq 1 - \sum_{i=1}^{m} \frac{1}{\lambda_i^2}.$$

We can develop a sure inference analysis to obtain "at least probability p" confidence bands in a completely similar way to that presented above for random variables.

From now on, I is simply an arbitrary set.

Definition 2.6. *The triple $(X, (V_n)_{n\in\mathbb{N}^*}, (\hat{V}_n)_{n\in\mathbb{N}^*})$ formed by a stochastic process $X : \Omega \times I \to \mathbb{R}$, a sequence of functions $(V_n : I \to \mathbb{R})_{n\in\mathbb{N}^*}$ and a sequence of stochastic processes $(\hat{V}_n : \Omega \times I \to \mathbb{R})_{n\in\mathbb{N}^*}$ is said to be an* **inferential sequence** *for $x : I \to \mathbb{R}$ if and only if:*

(i) $EX = x$, $V_1 = Var(X)$,

(ii) $\forall n \in \mathbb{N}^*$ $V_{n+1} = Var(\hat{V}_n)$,

(iii) $\forall n \in \mathbb{N}^*$ $E\hat{V}_n = V_n$,

(vi) $\forall n \in \mathbb{N}^*$ $\hat{V}_n(\Omega \times I) \subset \mathbb{R}_+$.

Theorem 2.2. **(On the inferential sequence of stochastic processes.)** *Let $(X, V_n, \hat{V}_n)$ be an inferential sequence for $x : I \to \mathbb{R}$. Let for all $m \in \mathbb{N}^*$, $L_m : \Omega \times I \times (\mathbb{R}_+^*)^m \to \mathbb{R}_+$ be given by*

$$L_m(\omega, t, \lambda_1, \ldots, \lambda_m) = \lambda_1 \sqrt{\hat{V}_1(\omega, t) + \lambda_2 \sqrt{\ldots + \lambda_{m-1} \sqrt{\hat{V}_{m-1}(\omega, t) + \lambda_m \sqrt{V_m(t)}}}},$$

then, for all $t \in I$ and all $m \in \mathbb{N}^*$, we have

$$P\{x(t) \in [X(\omega, t) - L_m(\omega, t, \lambda_1, \ldots, \lambda_m), X(\omega, t) + L_m(\omega, t, \lambda_1, \ldots, \lambda_m)]\} \geq 1 - \sum_{i=1}^{m} \frac{1}{\lambda_i^2}.$$

See de Miranda (2004) for a detailed presentation of this subject.

3. Estimation of the Density

Let N be a point process over $\mathbb{R}^m$, with unknown density ν_N.

Let $\{\psi_{i,j} : i, j \in \mathbb{Z}\}$ be an orthonormal wavelet basis of $L^2(\mathbb{R})$ of the form $\psi_{i,j}(t) = 2^{j/2}\psi(2^j t - i)$ or $\psi_{i,j}(t) = 2^{j/2}\psi(2^j(t - t_1) + t_1 - iT)$ for some mother wavelet ψ obtained, if necessary by the composition of a standard wavelet with an affine transformation, such that its support is $[t_1, t_2]$ with $T = t_2 - t_1$. Let ϕ be the father wavelet corresponding to ψ.

Similarly, let $\{\phi_{k,\ell i}, \psi_{i,j} : i, k \in \mathbb{Z},\ j \geq \ell i, j,\ \ell i \in \mathbb{Z}\}$ be an orthonormal wavelet basis that contains all the scales beyond some fixed integer ℓi.

It is extremely pleasant to adopt the following notation. Let ${}_d\mathbb{Z} = \{z \in \mathbb{Z} : z \geq d\}, d \in \mathbb{Z} \cup \{-\infty\}$ and $Ze(\ell i) = \mathbb{Z} \cup (\mathbb{Z} \times_{\ell i} \mathbb{Z})$ if $\ell i \in \mathbb{Z}$. If $\ell i = -\infty$, then $Ze(\ell i) = \mathbb{Z}^2$.

Let us use Greek letters for indexes in $Ze(\ell i)$ and we shall write $\psi_\eta = \phi_{\eta,\ell i}$ if and only if $\eta \in \mathbb{Z}$ and $\psi_\eta = \psi_{i,j}$ if and only if $\eta = (i, j) \in \mathbb{Z}^2$.

Thus, the wavelet expansions $f(t) = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\delta_{ij}\psi_{i,j}(t)$ and $f(t) = \sum_{k\in\mathbb{Z}}\gamma_k\phi_{k,\ell i}(t) + \sum_{i\in\mathbb{Z}}\sum_{j\in\ell_i\mathbb{Z}}\delta_{ij}\psi_{i,j}(t)$ will be simply written

$$f = \sum_{\eta\in Ze(\ell i)} \alpha_\eta\psi_\eta,$$

for α_η given by $\int_{-\infty}^{\infty} f\psi_\eta dt = \int_{\mathbb{R}}(\sum_\xi \alpha_\xi\psi_\xi)\psi_\eta dt = \sum_\xi \int_{\mathbb{R}} \alpha_\xi\psi_\xi\psi_\eta dt = \sum_\xi \alpha_\xi < \psi_\xi, \psi_\eta >= \alpha_\eta$.

Let for all n, $1 \le n \le m$, $\{\psi_{n,i,j} : i,j \in \mathbb{Z}\}$, $\psi_{n,i,j}(t) = 2^{j/2}\psi_n(2^j t - i)$ or $\psi_{n,i,j}(t) = 2^{j/2}\psi_n(2^j(t - a_n) + a_n - iT_n)$ and $\{\phi_{n,k,\ell i}, \psi_{n,i,j} : i,k \in \mathbb{Z},\ j \ge \ell i_n, j,\ \ell i_n \in \mathbb{Z}\}$ be orthonormal wavelet bases of $L^2(\mathbb{R})$ as above where $\operatorname{supp}\psi_n = [a_n, b_n]$ and $T_n = b_n - a_n$. For easy of notation we write $(Ze(\ell i))_n = Ze(\ell i_n)$. These bases are simply written as $\{\psi_{n,\eta_n} | \eta_n \in (Ze(\ell i))_n\}$ and they are also orthonormal bases of $L^2[a_n, b_n]$, $1 \le n \le m$. Taking tensor products we form the orthonormal base $\{\psi_\eta | \psi_\eta = \otimes_{n=1}^m \psi_{n,\eta_n}, \eta = (\eta_1, \ldots, \eta_m) \in \prod_{n=1}^m (Ze(\ell i))_n\}$ of $L^2(\mathbb{R}^m)$ and also of $L^2(\prod_{n=1}^m [a_n, b_n])$. In this way if $f \in L^2(\mathbb{R}^m)$ we have

$$f = \sum_{\eta\in \prod_{n=1}^m (Ze(\ell i))_n} \alpha_\eta\psi_\eta$$

with $\alpha_\eta = \int_{\mathbb{R}^m} f\psi_\eta d\ell$.

Our aim is to obtain the restriction of ν_N to $\prod_{n=1}^m [a_n, b_n] = [a, b]$ based on the points of a trajectory of the process that are contained in this $\mathbb{R}^m$ interval. Define

$$\nu = \begin{cases} \nu_N & \text{if } x \in [a, b], \\ 0 & \text{otherwise.} \end{cases}$$

From now on we assume that $\nu \in L^2([a, b])$. Therefore for the wavelet expansion of ν we have

$$\nu = \sum_\eta \beta_\eta\psi_\eta, \tag{1}$$

with

$$\beta_\eta = \int_{\mathbb{R}^m} \nu\psi_\eta d\ell = \int_{[a,b]} \nu\psi_\eta d\ell. \tag{2}$$

The main purpose is to estimate ν through the expansion (1) and for this we need to estimate the wavelet coefficients β_η given by (2).

We set $q_2 = d\mathrm{Cov}\,(N, N)/d\ell_{2m}$ if $\mathrm{Cov}\,(N, N) \ll \ell_{2m}$. If we do not have $\mathrm{Cov}\,(N, N) \ll \ell_{2m}$, we may replace $q_2(u, v)dudv$ by $d\mathrm{Cov}\,(N, N)$ in the statements of the theorems and propositions that follow.

3.1. Estimation of the Wavelet Coefficients.

We propose the following estimator of β_η:

$$\hat{\beta}_\eta = \int_{[a,b]} \psi_\eta dN.$$

The main properties of this estimator are given in the following theorem.

Theorem 3.1. *For all η and ξ we have:*
If N satisfies Assumption $\mathcal{B}$, then

(i) *the estimator $\hat{\beta}_\eta$ is unbiased.*
(ii) *Letting $C = [a,b]^2 - \{(x,x) \in \mathbb{R}^{2m} : x \in [a,b]\}$, we have*

$$Cov(\hat{\beta}_\eta, \hat{\beta}_\xi) = \int\int_C \psi_\eta(u)\psi_\xi(v)q_2(u,v)dudv + \int_{[a,b]^2\cap D_2} \psi_\eta(u)\psi_\xi(u)dM_2(u,v). \tag{3}$$

(iii) *In particular,*

$$Var(\hat{\beta}_\eta) = \int\int_C \psi_\eta(u)\psi_\eta(v)q_2(u,v)dudv + \int_{[a,b]^2\cap D_2} \psi_\eta(u)\psi_\eta(u)dM_2(u,v). \tag{4}$$

If N satisfies Assumption $\mathcal{A}$ then
(iv) *Similarly,*

$$Cov(\hat{\beta}_\eta, \hat{\beta}_\xi) = \int\int_C \psi_\eta(u)\psi_\xi(v)q_2(u,v)dudv + \int_{[a,b]} \psi_\eta(u)\psi_\xi(u)\nu(u)du. \tag{5}$$

(v) *In particular,*

$$Var(\hat{\beta}_\eta) = \int\int_C \psi_\eta(u)\psi_\eta(v)q_2(u,v)dudv + \int_{[a,b]} \psi_\eta^2(u)\nu(u)du. \tag{6}$$

Proof. (i) Since

$$E(\hat{\beta}_\eta) = E\int_{[a,b]} \psi_\eta dN = \int_{[a,b]} \psi_\eta \nu_N d\ell = \int_{[a,b]} \psi_\eta \nu d\ell = \beta_\eta,$$

$\hat{\beta}_\eta$ is unbiased.
(ii) and (iv) Apply proposition 2.4 for $X = \hat{\beta}_\eta$, $Y = \hat{\beta}_\xi$ and $A = B = [a,b]$.
(iii) Immediate from (ii).
(v) Immediate from (iv). ∎

Assume that N is a NIC point process. In this case $q_2(u,v) = 0$ and (3) and (5) become

$$\text{Cov}(\hat{\beta}_\eta, \hat{\beta}_\xi) = \int\int_{[a,b]^2\cap D_2} \psi_\eta \otimes \psi_\xi dM_2 = E\int\int_{[a,b]^2\cap D_2} \psi_\eta \otimes \psi_\xi d(N\times N) \quad \text{for } N \in \mathcal{B}. \tag{7}$$

$$\text{Cov}(\hat{\beta}_\eta, \hat{\beta}_\xi) = \int_{[a,b]} \psi_\eta\psi_\xi \nu d\ell = E\int_{[a,b]} \psi_\eta\psi_\xi dN \quad \text{for } N \in \mathcal{A} \tag{8}$$

and (4) and (6) reduce to

$$\text{Var}(\hat{\beta}_\eta) = \int\int_{[a,b]^2\cap D_2} \psi_\eta \otimes \psi_\eta dM_2 = E\int\int_{[a,b]^2\cap D_2} \psi_\eta \otimes \psi_\eta d(N\times N) \quad \text{for } N \in \mathcal{B}. \tag{9}$$

$$\text{(10)} \qquad \mathrm{Var}(\hat\beta_\eta) = \int_{[a,b]} \psi_\eta^2 \nu \mathrm{d}\ell = \int_{[a,b]} \psi_\eta^2 E dN = E\int_{[a,b]} \psi_\eta^2 dN \quad \text{for } N \in \mathcal{A}.$$

This leads us to propose the following expressions as estimators of (7) to (10),

$$\widehat{\mathrm{Cov}}(\hat\beta_\eta, \hat\beta_\xi) = \int\int_{[a,b]^2\cap D_2} \psi_\eta \otimes \psi_\xi \mathrm{d}(N\times N) \text{ and } \widehat{\mathrm{Var}}(\hat\beta_\eta) = \int\int_{[a,b]^2\cap D_2} \psi_\eta \otimes \psi_\eta \mathrm{d}(N\times N), \text{ for } N\in\mathcal{B}$$

and

$$\widehat{\mathrm{Cov}}(\hat\beta_\eta, \hat\beta_\xi) = \int_{[a,b]} \psi_\eta \psi_\xi dN \text{ and } \widehat{\mathrm{Var}}(\hat\beta_\eta) = \int_{[a,b]} \psi_\eta^2 dN, \text{ for } N \in \mathcal{A},$$

which are obviously unbiased.

Let us denote by $I(\omega)$ an appropriate set of indices such that $N_\omega = \sum_{i\in I(\omega)} k_i\delta_{x_i}$ and $I(\omega, A) = \{i \in I(\omega) | x_i \in A\}$.

Theorem 3.2. (Inferential sequence for the wavelet coefficients - $\mathcal{B}$.)

If N is a NIC ∞ point process, satisfying Assumption B, then for all $\xi \in \prod_{n=1}^{m}(Ze(\ell i))_n$, $\hat\beta_\xi = \int_{[a,b]} \psi_\xi \mathrm{d}N$, $V_{\xi,n} = \int_{[a,b]^{2^n}\cap D_{2^n}} (\otimes_{i=1}^{2^n}\psi_\xi)\mathrm{d}M_{2^n}$, and $\hat V_{\xi,n} = \int_{[a,b]^{2^n}\cap D_{2^n}} (\otimes_{i=1}^{2^n}\psi_\xi)\mathrm{d}(\prod_{i=1}^{2^n} N) = \sum_{i\in I(\cdot,[a,b])} \psi_\xi^{2^n}(x_i)(N(\{x_i\}))^{2^n}$, is an inferential sequence for β_ξ.

Proof Note that $\hat V_{\xi,n}(\omega) = \sum_{j\in I(\omega,[a,b])} \{(\otimes_{i=1}^{2^n}\psi_\xi)(x_j,\ldots,x_j)(\prod_{i=1}^{2^n}(\sum_{l\in I(\omega,[a,b]} k_l\delta_{x_l}))(x_j,\ldots,x_j)\} = \sum_{j\in I(\omega,[a,b])} (\psi_\xi(x_j))^{2^n} k_j^{2^n} = \sum_{i\in I(\omega,[a,b])} \psi_\xi^{2^n}(x_i)(N_\omega(\{x_i\}))^{2^n}$.

(i) $E\hat\beta_\xi = E\int_{[a,b]} \psi_\xi \mathrm{d}N = \int_{[a,b]} \psi_\xi \nu_N \mathrm{d}\ell = \int_{\mathbb{R}^m} \psi_\xi \sum_\eta \beta_\eta \psi_\eta \mathrm{d}\ell = \sum_\eta \beta_\eta \int_{\mathbb{R}^m} \psi_\xi\psi_\eta \mathrm{d}\ell = \sum_\eta \beta_\eta < \psi_\xi, \psi_\eta > \; = \beta_\xi$; $\mathrm{Var}\,(\hat\beta_\xi) = V_{\xi,1}$ is immediate.

(ii) By proposition 2.4 we write $\mathrm{Var}\,(\hat V_{\xi,n}) = \mathrm{Var}\,(\int_{[a,b]^{2^n}\cap D_{2^n}} (\otimes_{i=1}^{2^n}\psi_\xi)\mathrm{d}(\prod_{i=1}^{2^n} N)) =$

$$\int\int_{([a,b]^{2^n}\cap D_{2^n})^2 - D_{2^{n+1}}} (\otimes_{i=1}^{2^{n+1}}\psi_\xi)\mathrm{dCov}\,(\prod_{i=1}^{2^n} N, \prod_{i=1}^{2^n} N) + \int\int_{([a,b]^{2^n}\cap D_{2^n})^2\cap D_{2^{n+1}}} (\otimes_{i=1}^{2^{n+1}}\psi_\xi)\mathrm{d}E(\prod_{i=1}^{2^{n+1}} N) =$$

$$\int\int_{[a,b]^{2^{n+1}}\cap D_{2^{n+1}}} (\otimes_{i=1}^{2^{n+1}}\psi_\xi)\mathrm{d}M_{2^{n+1}} = V_{\xi,n+1} \text{ since } \frac{\mathrm{dCov}\,(\prod_{i=1}^{2^n} N, \prod_{i=1}^{2^n} N)}{\mathrm{d}\ell_{2^{n+1}m}} = 0 \text{ for NIC } \infty \text{ point processes}$$

and $([a,b]^{2^n}\cap D_{2^n})^2 \cap D_{2^{n+1}} = [a,b]^{2^{n+1}} \cap D_{2^{n+1}}$.

(iii) $E\hat V_{\xi,n} = E\int_{[a,b]^{2^n}\cap D_{2^n}} (\otimes_{i=1}^{2^n}\psi_\xi)\mathrm{d}(\prod_{i=1}^{2^n} N) = \int_{[a,b]^{2^n}\cap D_{2^n}} (\otimes_{i=1}^{2^n}\psi_\xi)\mathrm{d}M_{2^n} = V_{\xi,n}$.

(iv) Since $\forall \omega \in \Omega\ \hat{V}_{\xi,n}(\omega) = \left(\int\limits_{[a,b]^{2^n} \cap D_{2^n}} (\otimes_{i=1}^{2^n} \psi_\xi) \mathrm{d}(\prod_{i=1}^{2^n} N) \right)(\omega) = \int\limits_{[a,b]^{2^n} \cap D_{2^n}} (\otimes_{i=1}^{2^n} \psi_\xi) \mathrm{d}(\prod_{i=1}^{2^n} N_\omega) \geq 0$ we have $\forall \xi \in \prod_{i=1}^{m} (Ze(\ell i))_i\ \forall n \in \mathbb{N}^*\ \hat{V}_{\xi,n}(\Omega) \subset \mathbb{R}_+$ and the theorem follows. ■

Theorem 3.3. (Inferential sequence for the wavelet coefficients - $\mathcal{A}$.)
If N is a NIC point process, satisfying Assumption $\mathcal{A}$, then for all $\xi \in \prod_{n=1}^{m} (Ze(\ell i))_n$, $\hat{\beta}_\xi = \int\limits_{[a,b]} \psi_\xi \mathrm{d}N$, $V_{\xi,n} = \int\limits_{[a,b]} \psi_\xi^{2^n} \nu \mathrm{d}N$ and $\hat{V}_{\xi,n} = \int\limits_{[a,b]} \psi_\xi^{2^n} \mathrm{d}\ell = \sum\limits_{i \in I(\cdot,[a,b])} \psi_\xi^{2^n}(x_i)$, is an inferential sequence for β_ξ.

Proof (i) Immediate.
(ii) Using Proposition 2.4 with $f = g = \psi_\xi^{2^n}$, we write

$$\operatorname{Var}(\hat{V}_{\xi,n}) = \operatorname{Var}\left(\int_{[a,b]} \psi_\xi^{2^n} dN \right) = \int \int_C \psi_\xi^{2^n}(u) \psi_\xi^{2^n}(v) q_2(u,v) du dv + \int_0^T \psi_\xi^{2^{n+1}} \nu d\ell.$$

Since $q_2(u,v) = 0$, we obtain

$$\operatorname{Var}(\hat{V}_{\xi,n}) = \int_{[a,b]} \psi_\xi^{2^{n+1}} \nu \mathrm{d}\ell = V_{\xi,n+1}.$$

■

(iii) $E\hat{V}_{\xi,n} = \int_{[a,b]} \psi_\xi^{2^n} E \mathrm{d}N = V_{\xi,n}$.
(iii) Immediate. ■

We remark that for all n and ξ, $V_{\xi,n+1}$ is finite, due to the essentially boundedness of ψ_ξ as well as compactness of its support.

Therefore, in the case of a NIC point process N under Assumption $\mathcal{A}$, the estimators for β_ξ and the respective and successive variances are easy to compute, being all of the form $\int_{[a,b]} \psi_\xi^{2^n} dN$, and for a particular trajectory with m occurrences in the interval $[a,b]$, at points $x_0, x_1, \ldots, x_{m-1}$, this expression reduces to $\sum_{i=0}^{m-1} \psi_\xi^{2^n}(x_i)$.

3.2. Estimation of the Density Function. We are now in position to estimate the density function ν through a synthesis procedure using the estimates of the wavelet coefficients. For easy of notation we will write $\prod_{i=1}^{m} (Ze(\ell i))_i = \mathrm{Ze}(\ell i)$.

Theorem 3.4. *Let $\hat{\nu} = \sum_{\eta \in \mathrm{Ze}(\ell i)} \hat{\beta}_\eta \psi_\eta$.*

If N satisfies Assumption $\mathcal{B}$, then

(i) *the function $\hat{\nu}$ is an unbiased estimator for the density function ν.*

(ii) *The variance of $\hat{\nu}$ is given by*

$$Var(\hat{\nu}) = \sum_{\eta,\xi}\left(\int\int_C \psi_\eta(u)\psi_\xi(v)q_2(u,v)dudv + \int\int_{[a,b]^2\cap D_2} \psi_\eta(u)\psi_\xi(v)\mathrm{d}M_2(u,v)\right)\psi_\eta\psi_\xi \text{ for } N\in\mathcal{B}.$$

$$Var(\hat{\nu}) = \sum_{\eta,\xi}\left(\int\int_C \psi_\eta(u)\psi_\xi(v)q_2(u,v)dudv + \int_{[a,b]} \psi_\eta\psi_\xi\nu\mathrm{d}\ell\right)\psi_\eta\psi_\xi \text{ for } N\in\mathcal{A}.$$

If N is a NIC point process, then

(iii)

$$\mathrm{Var}\,(\hat{\nu}) = \sum_{\eta,\xi}\left(\int\int_{[a,b]^2\cap D_2} \psi_\eta(u)\psi_\xi(v)\mathrm{d}M_2(u,v)\right)\psi_\eta\psi_\xi, \text{ for } N\in\mathcal{B}.$$

$$\mathrm{Var}\,(\hat{\nu}) = \sum_{\eta,\xi}\left(\int_{[a,b]} \psi_\eta\psi_\xi\nu\mathrm{d}\ell\right)\psi_\eta\psi_\xi, \text{ for } N\in\mathcal{A}.$$

(iv) *and an unbiased estimator for $Var(\hat{\nu})$ is*

$$\widehat{Var}(\hat{\nu}) = \sum_{\eta,\xi}\left(\int\int_{[a,b]^2\cap D_2} \psi_\eta(u)\psi_\xi(v)\mathrm{d}(N\times N)\right)\psi_\eta\psi_\xi, \text{ for } N\in\mathcal{B}.$$

$$\widehat{Var}(\hat{\nu}) = \sum_{\eta,\xi}\left(\int_{[a,b]} \psi_\eta\psi_\xi dN\right)\psi_\eta\psi_\xi, \text{ for } N\in\mathcal{A}.$$

Proof (i) Since E is a continuous linear functional,

$$E(\hat{\nu}) = E(\sum_\eta \hat{\beta}_\eta\psi_\eta) = \sum_\eta \beta_\eta\psi_\eta = \nu.$$

(ii) Note that $\mathrm{Var}(\hat{\nu}) = E(\sum_\eta(\hat{\beta}_\eta - \beta_\eta)\psi_\eta)^2 = E\left(\sum_\xi\sum_\eta(\hat{\beta}_\eta - \beta_\eta)(\hat{\beta}_\xi - \beta_\xi)\psi_\eta\psi_\xi\right) =$

$= \sum_\xi\sum_\eta \mathrm{Cov}(\hat{\beta}_\eta, \hat{\beta}_\xi)\psi_\eta\psi_\xi$ and apply Theorem 3.1.

(iii) For a NIC point process, since $q_2(u,v) = 0$, the expressions in (ii) reduce to the sums of the second terms inside the parentheses.

(iv) Immediate, since $\nu\mathrm{d}\ell = E\mathrm{d}N$ and $\mathrm{d}M_2 = E\mathrm{d}(N\times N)$. ∎

Inferential sequences for ν can be obtained using the result of the following theorem.

Theorem 3.5. (Inferential Sequence for the Density - $\mathcal{A}$.) *Let $\eta = (\eta_1, \ldots, \eta_{2^n}) \in (\mathrm{Ze}(\ell\mathrm{i}))^{2^n}$ be an element of the cartesian product of $\mathrm{Ze}(\ell\mathrm{i})$ by itself 2^n times, and N a NIC point process that satisfies Assumption $\mathcal{A}$. Let*

$$V_n(\hat{\nu}) = \sum_{\eta\in(\mathrm{Ze}(\ell\mathrm{i}))^{2^n}}\left(\int_{[a,b]}\prod_{\ell=1}^{2^n}\psi_{\eta_\ell}\nu dx\right)\prod_{\ell=1}^{2^n}\psi_{\eta_\ell}$$

and

$$\hat{V}_n(\hat{\nu}) = \sum_{\eta \in (\mathrm{Ze}(\ell i))^{2^n}} \left(\int_{[a,b]} \prod_{\ell=1}^{2^n} \psi_{\eta_\ell} dN \right) \prod_{\ell=1}^{2^n} \psi_{\eta_\ell}, \text{ for all } n \geq 1.$$

Then $V_n(\hat{\nu})$ and $\hat{V}_n(\hat{\nu})$ are sequences of variances and estimators, respectively, such that:

(i) $E(\hat{\nu}) = \nu$, $V_1(\hat{\nu}) = \mathrm{Var}\,(\hat{\nu})$.

(ii) $\forall n \in \mathbb{N}^*$ $V_{n+1}(\hat{\nu}) = Var(\hat{V}_n(\hat{\nu}))$.

(iii) $\forall n \in \mathbb{N}^*$ *$\hat{V}_n(\hat{\nu})$ is an unbiased estimator for $V_n(\hat{\nu})$.*

(iv) $\forall n \in \mathbb{N}^*$ $\hat{V}_n(\hat{\nu})(\Omega \times [a,b]) \subset \mathbb{R}_+$

That is, $(\hat{\nu},\ V_n(\hat{\nu}),\ \hat{V}_n(\hat{\nu})\ )$ is an inferential sequence of stochastic processes for the density ν.

Proof (i) Immediate.
(ii) Since E is a continuous linear functional, we have

$$\mathrm{Var}(\hat{V}_n(\hat{\nu})) = \mathrm{Var}\left(\sum_{\eta \in (\mathrm{Ze}(\ell i))^{2^n}} \left(\int_{[a,b]} \prod_{\ell=1}^{2^n} \psi_{\eta_\ell} dN \right) \prod_{\ell=1}^{2^n} \psi_{\eta_\ell} \right) =$$

$$\sum_{\eta,\xi \in (\mathrm{Ze}(\ell i))^{2^n}} \mathrm{Cov}\left(\int_{[a,b]} \prod_{\ell=1}^{2^n} \psi_{\eta_\ell} dN, \int_{[a,b]} \prod_{m=1}^{2^n} \psi_{\xi_m} dN \right) \prod_{\ell=1}^{2^n} \psi_{\eta_\ell} \prod_{m=1}^{2^n} \psi_{\xi_m}.$$

Using Proposition 2.4 we have

$$\mathrm{Var}(\hat{V}_n(\hat{\nu})) = \sum_{\eta,\xi \in (\mathrm{Ze}(\ell i))^{2^n}} \left(\int_{[a,b]} \prod_{\ell=1}^{2^n} \psi_{\eta_\ell} \prod_{m=1}^{2^n} \psi_{\xi_m} \nu \mathrm{d}x \right) \prod_{\ell=1}^{2^n} \psi_{\eta_\ell} \prod_{m=1}^{2^n} \psi_{\xi_m} =$$

$$= \sum_{\mu \in (\mathrm{Ze}(\ell i))^{2^{n+1}}} \left(\int_{[a,b]} \prod_{\ell=1}^{2^{n+1}} \psi_{\mu_\ell} \nu \mathrm{d}x \right) \prod_{\ell=1}^{2^{n+1}} \psi_{\mu_\ell} = V_{n+1}(\hat{\nu}).$$

(iii) Equality $E\hat{V}_n(\hat{\nu}) = V_n(\hat{\nu})$ follows from the linearity and continuity of E, Campbell's theorem and Proposition 2.1.
(iv) Since $\forall n \in \mathbb{N}^*$, $\forall \omega \in \Omega$, $\forall x \in [a,b]$,

$$\hat{V}_n(\hat{\nu})(\omega, x) = \sum_{\eta \in (\mathrm{Ze}(\ell i))^{2^n}} \left(\int_{[a,b]} \prod_{\ell=1}^{2^n} \psi_{\eta_\ell} dN_\omega \right) \prod_{\ell=1}^{2^n} \psi_{\eta_\ell}(x) =$$

$$= \sum_{\eta \in (\mathrm{Ze}(\ell i))^{2^n}} \int_{[a,b]} \left(\prod_{\ell=1}^{2^n} \psi_{\eta_\ell} \right) \left(\prod_{j=1}^{2^n} \psi_{\eta_j}(x) \right) dN_\omega = \sum_{\eta \in (\mathrm{Ze}(\ell i))^{2^n}} \int_{[a,b]} \left(\prod_{\ell=1}^{2^n} \psi_{\eta_\ell} \psi_{\eta_\ell}(x) \right) dN_\omega =$$

$$= \int_{[a,b]} \sum_{\eta \in (\mathrm{Ze}(\ell i))^{2^n}} \left(\prod_{\ell=1}^{2^n} \psi_{\eta_\ell} \psi_{\eta_\ell}(x) \right) dN_\omega =$$

$$= \int_{[a,b]} \sum_{\eta,\xi \in (\mathrm{Ze}(\ell i))^{2^{n-1}}} \left(\prod_{\ell=1}^{2^{n-1}} \psi_{\eta_\ell} \psi_{\eta_\ell}(x) \right) \left(\prod_{m=1}^{2^{n-1}} \psi_{\xi_m} \psi_{\xi_m}(x) \right) dN_\omega =$$

$$= \int_{[a,b]} \sum_{\eta \in (\mathrm{Ze}(\ell i))^{2^{n-1}}} \left(\prod_{\ell=1}^{2^{n-1}} \psi_{\eta_\ell}(x) \psi_{\eta_\ell} \right) \sum_{\xi \in (\mathrm{Ze}(\ell i))^{2^{n-1}}} \left(\prod_{m=1}^{2^{n-1}} \psi_{\xi_m}(x) \psi_{\xi_m} \right) dN_\omega =$$

$$= \int_{[a,b]} \left(\sum_{\eta \in (\mathrm{Ze}(\ell i))^{2^{n-1}}} \prod_{\ell=1}^{2^{n-1}} \psi_{\eta_\ell}(x) \psi_{\eta_\ell} \right)^2 dN_\omega \geq 0 \,,$$

the theorem is proved. ■

Theorem 3.6. (Inferential Sequence for the Density - B.) *Let $\eta = (\eta_1, \ldots, \eta_{2^n}) \in (\mathrm{Ze}(\ell i))^{2^n}$ be an element of the cartesian product of* Ze(ℓi) *by itself 2^n times, and N a NIC $^\infty$ point process that satisfies Assumption B. Let*

$$V_n(\hat{\nu}) = \sum_{\eta \in (\mathrm{Ze}(\ell i))^{2^n}} \left(\int_{[a,b]^{2^n} \cap D_{2^n}} (\bigotimes_{\ell=1}^{2^n} \psi_{\eta_\ell}) \mathrm{d}M_{2^n} \right) \prod_{\ell=1}^{2^n} \psi_{\eta_\ell}$$

and

$$\hat{V}_n(\hat{\nu}) = \sum_{\eta \in (\mathrm{Ze}(\ell i))^{2^n}} \left(\int_{[a,b]^{2^n} \cap D_{2^n}} (\bigotimes_{\ell=1}^{2^n} \psi_{\eta_\ell}) \mathrm{d}(\prod_{j=1}^{2^n} N) \right) \prod_{\ell=1}^{2^n} \psi_{\eta_\ell}, \text{ for all } n \geq 1.$$

Then $(\hat{\nu}, V_n(\hat{\nu}), \hat{V}_n(\hat{\nu}))$ is an inferential sequence of stochastic processes for the density ν.

Proof Analogous to Theorem's 3.5 proof. ■

Theorems 3.4 and 3.6 are the main results of this work. Note that using Theorem 3.4 we can estimate the density of virtually every point process since the only requirement that is made is that the expectation measure exists and is a locally square integrable function. Inference bands are also estimated for NIC point processes. In case sure inference is recommended, Theorem 3.6 furnishes an inferential sequence for ν for NIC ∞ point processes. Clearly, for NIC ^{n} this theorem has a direct analogous statement for an inferential for an n-th order inferential set that allows us to calculate up till the n-th order sure inference bands for ν. Note also that the estimate are easy to perform.

4. Some Comments of Practical Content

In this section we discuss some of the difficulties that may arise in estimating the density of point processes and we present, in an informal way, some possible solutions.

In practical situations where we want to estimate the density of a point process that occurs in $\mathbb{R}^m$ we have access to a collection of points in $\mathbb{R}^m$ that represent the occurrence of events within a region $\mathcal{O} \in \mathbb{R}^m$. We will call this region an observation region. It will always be assumed that this region is a Borel set of $\mathbb{R}^m$. In addition, in practice this region is bounded and for estimating purposes we can assume that $\mathcal{O}$ is such that there is no proper affine subspace of $\mathbb{R}^m$, i.e., with dimension $d < m$, that contains $\mathcal{O}$, because if there were such a subspace we could consider the point process to occur in $\mathbb{R}^d$ for estimating purposes. Note that observation regions $\mathcal{O}$, with this properties only, may be still extremely complicated.

Given an observation region $\mathcal{O}$, we now pose the problem of estimating the density. One possible solution is to fit the region $\mathcal{O}$ inside an $\mathbb{R}^m$-interval, $\mathcal{I}$, and then estimate an extension of the density to this $\mathbb{R}^m$-interval, which can be made by the methods presented in section 3, and then take the restriction of this estimate to $\mathcal{O}$ as the estimate of the density on $\mathcal{O}$. Let us call $\{x_i | i \in I\} \subset \mathcal{O}$ the observed points of occurrence of N. If $\mathcal{O}$ is a proper subset of $\mathcal{I}$ then there are infinitely many extensions and we have to choose among them. One way to extend the density is to assume that it is zero outside $\mathcal{O}$. Another possibility is to consider that the density is constant outside $\mathcal{O}$. The estimation of the density is made on $\mathcal{I}$ considering, in the first case, that only the points x_i, $i \in I$, have occurred in $\mathcal{I}$ and, for example, generating an homogeneous Poisson process on $\mathcal{I} - \mathcal{O}$ with appropriate intensity, for example $\lambda = (\sum_{i \in I} k_i)/\ell_m(\mathcal{O})$ where k_i is the multiplicity of occurrence of the point x_i and when $\ell_m(\mathcal{O}) \neq 0$, in the seconde case. These procedures for extending the density may cause "boundary effect" to appear but they have the great advantage of being general procedures which is a desired feature since the regions $\mathcal{O}$ may be extremely complicated. Alternatively, the choice of an extension will be guided by the regularity of the region $\mathcal{O}$ and by some kind of exploratory or preliminary analysis of the point process data. For example, an extending procedure that will depend on the information given by the data set is the following. For open star shaped domains with not so irregular border, $\mathcal{O}$, for which there is a center p such that there are balls $B(p, r_1) \subset \mathcal{O}$ and $B(p, r_2) \supset \mathcal{O}$ for which the ratio r_2/r_1 is "not so big", divide the sphere S^{m-1} in $\ell = \prod_{i=1}^{m-1} \ell_i$ regions, Q_i, with the same area, induced from $\mathbb{R}^m$ volume, by partitioning the domain of the canonical spherical coordinates $\varphi : [0, 2\pi] \times [0, \pi]^{m-2} \to S^{m-1}$ in an appropriate product partition and then choose $\mathcal{O}_i$, for each i, $1 \le i \le \ell$, an open set contained in the solid angle corresponding to Q_i that contains the intersection of the border of $\mathcal{O}$ with this solid angle. Let $\mathcal{O}_i^* = \mathcal{O}_i \cap \mathcal{O}$. Now calculate the mean density on $\mathcal{O}_i^*$, summing the multiplicities of occurrence inside $\mathcal{O}_i^*$ and dividing by $\ell(\mathcal{O}_i^*)$, and generate an homogeneous Poisson process which intensity is equal to this mean density, for each i, on the intersection of the i-th solid angle with $\mathcal{I} - \mathcal{O}$. Finally, estimate the density on $\mathcal{I}$. This procedure will probably reduce border effects.

If the observation region is good enough, for example, it is an open set such that its border $\partial\mathcal{O}$ is an $m-1$ dimensional differentiable smooth manifold, we can "mirror the point process with respect to this boundary", that is, we can choose a distance $\varepsilon > 0$ and for each point x_i such that its distance to the border $\partial\mathcal{O}$ is less than ε we mirror this point with respect to the border, that is, we take, on the normal to $\partial\mathcal{O}$ that posses through x_i, a point x_i' outside $\mathcal{O}$ such that its distance to $\partial\mathcal{O}$ is that of x_i to $\partial\mathcal{O}$. Now we can take an interval $\mathcal{I}$ that contains $\mathcal{O}^+ = \{x \in \mathbb{R}^+ | d(x, \mathcal{O}) \le \varepsilon\}$ and, for example, generate an homogeneous Poisson on $\mathcal{I} - \mathcal{O}^+$. This procedure will, provided ε is not too small, reduce the border effects as they will now appear at the proximity of $\partial\mathcal{O}^+$ so that the restriction of the estimated density on $\mathcal{I}$ to $\mathcal{O}$ will exhibit a better behavior. We observe that a similar "mirror" procedure is recommended for the very estimation of densities on $\mathbb{R}^m$-intervals.

Another possibility of solving the problem of estimation on $\mathcal{O}$ is to choose a convenient cover of $\mathcal{O}$ by disjoint $\mathbb{R}^m$-intervals and then estimate the density on these intervals.

5. Conclusion

In this work we dealt with the problem of estimating the density of a non-homogeneous point process on $\mathbb{R}^m$. The assumptions that are made upon the point process in order for it to be suitable for wavelet estimation of its density are extremely mild. It is only required that the density exists and is a locally square integrable function. Inferential sequence for the density function are obtained both for $N \in (\mathcal{B} \cap \text{NIC}^{\infty})$ and for $N \in (\mathcal{A} \cap \text{NIC})$ which permit us to calculate sure inference bands for the density function. Point process under $\mathcal{B}$ need not be quasi-simple, as a matter of fact, they admit multiplicity of occurrence. On the odder hand, if N is under $\mathcal{A}$ it is quasi-simple, that is, there are no multiplicities a.s.$[P]$. (See de Miranda, 2003.) It is interesting to observe that multiplicity permits a wider spectrum of possibilities of internal dependence probability structures and this is reflected on the fact that n-th order inference bands (see de Miranda, 2003) are obtained for $\mathcal{B}$ point processes under NIC ^{n} or NIC $^{\infty}$, a strong requirement then NIC as for $\mathcal{A}$ point processes. It is also important to note that we can choose different families of wavelets to form $\psi_\eta = \psi_{1,\eta_1} \otimes \ldots \otimes \psi_{m,\eta_m}$, i.e., ψ_i's may belong to different families and the choice may be guided by practical instances. Finally we observe that as we have inferential sequences for wavelets coefficients, we can perform thresholding procedures upon these coefficients in much the same way as was done in de Miranda and Morettin, 2003 and de Miranda, 2003.

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References

[1] Besbeas, P., De Feis, I. and Sapatinas, T. (2002). A comparative simulation study of wavelet shrinkage estimators for Poisson counts. *Rapporto Tecnico*, 240/02. Istituto per le Applicazioni del Calcolo "Mauro Picone"-Sezione di Napoli, CNR, Italy.

[2] Brillinger, D.R. (1975). Statistical inference for stationary point processes. *Stochastic Processes and Related Topics*, 1, 55–99.

[3] Brillinger, D.R. (1978). Comparative aspects of the study of ordinary time series and point processes. *Developments in Statistics*, 1, 33–133.

[4] Brillinger, D.R. (1997). Some wavelet analyses of point processes data. The Third First Asilomar Conference on Signals, Systems and Computers. IEEE Computer Society, 1087–1091.

[5] Cramér, H. and Leadbetter,M.R. (1967). *Stationary and Related Stochastic Processes*. New York: Wiley.

[6] Daley, D.J. and Vere-Jones, D. (1988) *An Introduction to the Theory of Point Processes*. New York: Springer.

[7] Daubechies, I. (1992). *Ten Lectures on Wavelets*. SIAM.

[8] de Miranda, J.C.S. (2003a). On the estimation of the intensity of point processes via wavelets. PhD thesis. University of São Paulo. São Paulo, Brazil. (In Portuguese.)

[9] de Miranda, J.C.S. (2003b). Some wide classes of point Processes - Interrelationships and three working properties. Preprint.

[10] de Miranda, J.C.S. (2003c). Sure inference analysis. Preprint.

[11] de Miranda, J.C.S. and Morettin, P.A. (2003a). Characterization of the intensity of point process. Preprint.

[12] Donoho, D.L., Johnstone, I.M., Kerkyacharian, G. and Picard, D. (1996). Density estimation by wavelet shrinkage. *The Annals of Statistics*, 24, 508–539.

[13] Helmers, R. and Zitikis, R. (1999). On estimation of Poisson intensity functions. *Annals of the Institute of Statistical Mathematics*, 51, 265–280.

[14] Kolaczyk, E.D. (1999a). Bayesian multiscale models for Poisson processes. *Journal of the American Statistical Association*, **94**, 92–933.
[15] Kolaczyk, E.D. (1999b). Wavelet shrinkage estimation of certain Poisson intensity signals using corrected thresholds. *Statistica Sinica*, 9, 119–135.
[16] Meyer, Y. (1992). *Ondelettes et Algorithmes Concurrents*. Hermann.
[17] Morettin, P.A. (1999). *Waves and Wavelets*. São Paulo: University of São Paulo Press.
[18] Priestley, M.B. (1996). Wavelets and time-dependent spectral analysis. *Journal of Time Series Analysis*, **17**, 85–103.
[19] Rathbun, S.L. and Cressie, N. (1994). Asymptotic properties of estimators for the parameters of spatial inhomogeneous Poisson point processes. *Advances in Applied Probability*, **26**, 122–154.
[20] Snyder, D.L. (1975). *Random Point Processes*. New York: Wiley.
[21] Timmermann, K.E. and Nowak, R.D. (1997). Multiscale modelling and estimation of Poisson processes with application to photon-limited imaging. *IEEE Transactions on Information Theory*, **45**, 846–862.

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