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Approximation of Bernoulli measures for non-uniformly hyperbolic systems

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Abstract. An invariant measure is called a Bernoulli measure if the corresponding dynamics is isomorphic to a Bernoulli shift. We prove that for $C^{1+\alpha}$ diffeomorphisms any weak mixing hyperbolic measure could be approximated by Bernoulli measures. This also holds true for C^1 diffeomorphisms preserving a weak mixing hyperbolic measure with respect to which the Oseledets decomposition is dominated.

1. *Introduction*

In the ergodic theory of dynamical systems one way to study a measure is to approach it by well-understood measures. This is the case particularly in the differentiable ergodic theory, that is, ergodic theory of differentiable dynamical systems. One kind of well-understood measures is the Bernoulli measures, which are strong mixing measures in Bernoulli shifts. Bernoulli shifts have been discussed a lot in the history [**11**, **12**, **17**]. So, a natural question is when an invariant measure could be approximated by Bernoulli measures? Several works have been done related to this topic. In 1970, Bowen proved that a topologically mixing hyperbolic basic set admits a Bernoulli measure [**5**]. In 1972, Sigmund found that for an Axiom A diffeomorphism f, Bernoulli measures are dense in the set of all invariant measures supported on a basic set Ω as long as $f|_\Omega$ is topologically mixing [**18**]. His method is to use Markov partitions constructed by Bowen [**5**] through which every basic set of f would correspond to a subshift of finite type and the problem is reduced to finding Bernoulli measures on topologically mixing Markov chains. In 2015, Arbieto, Catalan and Santiago discussed this problem for diffeomorphisms beyond uniform hyperbolicity. They proved that for any generic diffeomorphism, if the dynamics restricted on an isolated homoclinic class is topologically mixing, then the Bernoulli measures are dense in the space of invariant measures supported on the class [**3**].

In the present paper we discuss this problem for non-uniformly hyperbolic dynamical systems. More precisely, we investigate a weak mixing hyperbolic measure for a $C^{1+\alpha}$ diffeomorphism or for a C^1 diffeomorphism with the Oseledec bundles of the weak mixing measure being dominated. We prove that in these cases the weak mixing hyperbolic measure could be approximated by Bernoulli measures.

2. *Definitions and main results*

2.1. *Basic concepts in ergodic theory.* Let X be a measurable space with σ-algebra $\mathcal{B}$ and μ be a probability measure on $(X, \mathcal{B})$. We call $(X, \mathcal{B}, \mu)$ a probability space and call $(X, \mathcal{B}, \mu, T)$ a probability system if $T : X \to X$ is a μ-preserving transformation. Now we introduce several basic concepts in ergodic theory, all of which could be found in [**20**].

Definition 2.1. Let $(X, \mathcal{B}, \mu, T)$ be a probability system. We call μ *strong mixing* if for each pair $A,\ B \in \mathcal{B}$,

$$\lim_{n\to+\infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B).$$

We call μ *weak mixing* if for each pair $A,\ B \in \mathcal{B}$, there is a subset $J = J(A, B)$ of $\mathbb{Z}^+$ with density 1 such that

$$\lim_{n\in J, n\to\infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B), \tag{1}$$

where J has density 1 means that $\lim_{n\to\infty} (\#\{J \cap \{0, 1, \ldots, n-1\}\}/n) = 1$.

Remark 2.2. The standard definition of weak mixing is the following:

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} |\mu(T^{-i}A \cap B) - \mu(A)\mu(B)| = 0 \quad \text{for all } A,\ B \in \mathcal{B}. \tag{2}$$

The expression (1) in Definition 2.1 is actually an equivalent characterization of (2) by [**20**, Theorem 1.21]. Since we are going to use (1) for the weak mixing assumption, we give the definition of weak mixing through (1) directly for convenience. In [**20**], an intuitive description of the strong mixing and weak mixing is given as follows: strong mixing means that for each $A \in \mathcal{B}$, the sequence of preimages $T^{-n}A$ becomes asymptotically independent of any other set B; for weak mixing, it means that the preimages $T^{-n}A$ become independent of $B \in \mathcal{B}$ provided a few instants of time are neglected (a subset of $\mathbb{Z}^+$ with zero density). Thus, it is easy to see that a strong mixing measure is weak mixing and a weak mixing measure is ergodic.

Taking a finite set $Y = \{0, 1, \ldots, k-1\}$, a σ-algebra $\mathcal{F} = 2^Y$, a probability vector $(p_0, p_1, \ldots, p_{k-1})$ such that $p_i > 0$ and $\sum_{i=0}^{k-1} p_i = 1$, we define a measure μ on $(Y, \mathcal{F})$

by $\mu(\{i\}) = p_i$. Then we obtain a probability space $(Y, \mathcal{F}, \mu)$. Write $(X, \mathcal{B}, m) = \prod_{-\infty}^{+\infty}(Y, \mathcal{F}, \mu)$, a product probability space which is defined as follows: let $X = \prod_{-\infty}^{+\infty} Y$. For $n \geq 0$ and $a_i \in Y$, let $[a_{-n}, \ldots, a_n] = \{(x_i)_{i=-\infty}^{+\infty} \in X \mid x_i = a_i, -n \leq i \leq n\}$. Consider a semi-algebra $\mathcal{S} = \{[a_{-n}, \ldots, a_n], n \geq 0\}$ of X and an additive function

$$m : \mathcal{S} \to [0, 1], \quad m([a_{-n}, \ldots, a_n]) = \prod_{i=-n}^{n} p_{a_i}.$$

Then $\mathcal{S}$ generates a unique σ-algebra $\mathcal{B}$ of X and m extends uniquely to a measure on $(X, \mathcal{B})$, which is still denoted by m. Define

$$T : X \to X, \quad T((y_n)_{n=-\infty}^{+\infty}) = (x_n)_{n=-\infty}^{+\infty} \quad \text{where } x_n = y_{n+1} \text{ for all } n \in \mathbb{Z}.$$

The probability system $(X, \mathcal{B}, m, T)$ is called a *Bernoulli shift*. It is well known that a Bernoulli shift is strong mixing [**20**].

Two probability systems $(X_i, \mathcal{B}_i, \mu_i, T_i)$, $i = 1, 2$, are *isomorphic* if there exist $M_i \in \mathcal{B}_i$ satisfying the following properties:

(i) $\mu_i(M_i) = 1$, $T_i M_i \subset M_i$;

(ii) there is an invertible measure-preserving transformation $\phi : M_1 \to M_2$ such that $\phi \circ T_1(x) = T_2 \circ \phi(x)$, for all $x \in M_1$.

Important dynamical properties like ergodicity, weak mixing, strong mixing and the measure-theoretic entropy are preserved by isomorphic systems [**20**].

2.2. *Main results.* A T-invariant measure μ is called a *Bernoulli measure* if the corresponding probability system $(X, \mathcal{B}, \mu, T)$ is isomorphic to a Bernoulli shift. In this paper we investigate for differentiable dynamical systems when an invariant measure could be approximated by Bernoulli measures. We call an invariant measure μ *hyperbolic* if it admits no zero Lyapunov exponents. Our first main result is the following.

THEOREM A. *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact Riemannian manifold M and μ be an f-invariant weak mixing hyperbolic measure. Then there exists a sequence of Bernoulli measures $\{\nu_n\}_{n\in\mathbb{N}}$ supported on topological mixing hyperbolic sets such that $\nu_n \to \mu$ as $n \to +\infty$ in weak*-topology.*

For $C^{1+\alpha}$ diffeomorphisms, Katok and Mendoza proved that an *ergodic* hyperbolic measure can be approximated by measures supported on *topological transitive* hyperbolic sets (see [**8**, Theorem S.5.9]). Theorem A says that a *weak mixing* hyperbolic measure can be approximated by measures supported on *topologically mixing* hyperbolic sets. Moreover, the approximation measures chosen in Theorem A are Bernoulli and thus strong mixing. Note that a topological transitive hyperbolic set which is not topologically mixing does not support any strong mixing measure, and this is actually the case of Katok and Mendoza's constructions in [**8**, Theorem S.5.9]. Thus, their construction is not applicable to Theorem A to obtain Bernoulli measures.

Since locally maximal hyperbolic sets in Theorem A are structurally stable, a direct application of Theorem A shows that weak mixing hyperbolic measures are persistent under C^1 perturbations.

COROLLARY B. *Let f be as in Theorem A. Let $f_n \to f$ in the C^1 topology and μ be an f-invariant weak mixing hyperbolic measure. Then there exists a sequence of f_n-invariant strong mixing hyperbolic measures μ_n such that $\mu_n \to \mu$ as $n \to +\infty$ in weak*-topology.*

Now we remove the 'Hölder' condition in Theorem A and consider C^1 non-uniformly hyperbolic systems with domination.

Definition 2.3. Let Λ be an f-invariant set. A continuous splitting $T_\Lambda M = E^s \oplus E^u$ is *dominated* if there are constants $C > 0$ and $\lambda \in (0, 1)$ such that for all $x \in \Lambda$ and all $n \geq 0$, we have $\|D_x f^n v\|/\|D_x f^n u\| \leq C\lambda^n$ for any $v \in E^s(x)$, $u \in E^u(x)$ with $\|v\| = 1$, $\|u\| = 1$.

A hyperbolic measure μ with domination means that the Oseledets decomposition $E^s \oplus E^u$ with respect to μ is dominated, where E^s and E^u are the direct sums of Oseledets bundles corresponding to negative and positive Lyapunov exponents of μ, respectively.

The classical non-uniformly hyperbolic theory always assumes that the derivative of a diffeomorphism is Hölder continuous, which could not be removed in general [**4**, **15**]. However, if the Hölder condition is replaced by the domination property between Oseledets decompositions E^s and E^u, some properties such as stable manifold theory and the Pesin entropy formula could still be preserved [**1**, **19**]. Liao, Sun and Wang [**9**] proved that upper semi-continuity of the entropy map holds for a C^1 non-uniformly hyperbolic system with domination but may fail for a $C^{1+\alpha}$ non-uniformly hyperbolic system without domination. So, under these two assumptions some results are parallel while some are not. For the topic of Bernoulli measure approximations, the results are parallel.

THEOREM C. *Let f be a C^1 diffeomorphism of a compact Riemannian manifold M and μ be an f-invariant weak mixing hyperbolic measure with domination. Then there exists a sequence of Bernoulli measures ν_n ($n \in \mathbb{N}$) supported on topologically mixing hyperbolic sets such that $\nu_n \to \mu$ as $n \to \infty$ in weak*-topology.*

In the following two sections we prove Theorem A and Theorem C, respectively. In §3, we first recall several basic facts about $C^{1+\alpha}$ non-uniformly hyperbolic dynamical systems where Lemma 3.2 is a little generalized version of the classical closing lemma in [**7**]. By the weak mixing property of the measure, we may choose two recurrent orbit arcs of two generic points of the measure with coprime lengths m_1, m_2, respectively. Then Lemma 3.2 enables us to obtain a periodic point z whose orbit 'closes' these two arcs. An interesting though elementary application of the coprime relation between m_1 and m_2 helps us to obtain the transverse intersection relation between z and $f^{m_1}z$ (Lemma 3.4), which is enough to get the Bernoulli measure we need by the argument in [**3**].

In §4, we deal with the case of C^1 non-uniformly hyperbolic systems for which the Hölder condition is replaced by the domination property. The weak mixing property is still used to obtain certain coprime relationships between recurrence times of recurrent orbits and Lemma 4.3 plays the role of Lemma 3.2 as in the proof of Theorem A.

3. *Proof of Theorem A*

Let M be a compact Riemannian manifold and $f : M \to M$ be a C^1 diffeomorphism. Let μ be an ergodic f-invariant hyperbolic measure; then for μ-almost every point the

Lyapunov exponents are constants and non-zero. Let λ_s be the norm of the largest negative Lyapunov exponent and λ_u be the smallest one of the positive Lyapunov exponents. Let ε be a positive constant with $0 < \varepsilon \ll \lambda_s, \lambda_u$. We define $\Lambda_k = \Lambda_k(\lambda_s, \lambda_u; \varepsilon)$, $k \geq 1$, to be the set of all points $x \in M$ for which there is a splitting $T_x M = E^s_x \oplus E^u_x$ with the invariant property $Df(E^s_x) = E^s_{fx}$ and $Df(E^u_x) = E^u_{fx}$ and satisfying:

(i) $\|Df^n|_{E^s_{f^m x}}\| \leq e^{\varepsilon k} e^{-(\lambda_s - \varepsilon)n} e^{\varepsilon|m|}$ for all $m \in \mathbb{Z},\ n \geq 1$;

(ii) $\|Df^{-n}|_{E^u_{f^m x}}\| \leq e^{\varepsilon k} e^{-(\lambda_u - \varepsilon)n} e^{\varepsilon|m|}$ for all $m \in \mathbb{Z},\ n \geq 1$.

Denote

$$\Lambda = \Lambda(\lambda_s, \lambda_u; \varepsilon) = \bigcup_{k=1}^{+\infty} \Lambda_k(\lambda_s, \lambda_u; \varepsilon),$$

which is called a Pesin set associated with μ. The Pesin set is an f-invariant set and is not necessarily compact in general.

3.1. *$C^{1+\alpha}$ non-uniformly hyperbolic theory.* In this section, we give some classical facts and results about $C^{1+\alpha}$ non-uniformly hyperbolic theory. For more details, readers may refer to [**7**, **13**, **14**].

Let f be a $C^{1+\alpha}$ diffeomorphism and $\Lambda = \Lambda(\lambda_s, \lambda_u; \varepsilon)$ be a Pesin set. Denote $\chi = \min\{\lambda_s, \lambda_u\}$. Let $\lambda' = \max\{\frac{1}{2}, e^{-99/100\chi}\}$ and $\gamma = (1 - \lambda')/20$. The following proposition describes the hyperbolic behavior of f in the neighborhoods of points in the Pesin set.

PROPOSITION 3.1. [**7**] *There exists $r_0 > 0$ such that for any point $x \in \Lambda$ there exist $0 < r(x) < r_0$, a neighborhood $C(x)$ and a diffeomorphism $\Phi_x : \mathbf{B}^{m-s}_{r(x)} \times \mathbf{B}^s_{r(x)} \longrightarrow C(x)$ (where $\mathbf{B}^i_r$ denotes the Euclidean ball in $\mathbf{R}^i$ with radius r around the origin) satisfying the following properties.*

(1) *For any $k > 0$, Φ_x is continuous on Λ_k and $r(x)$ has uniform positive lower bound r_k.*

(2) *Denote $f_x = \Phi^{-1}_{fx} \circ f \circ \Phi_x : \mathbf{B}^{m-s}_{r(x)} \times \mathbf{B}^s_{r(x)} \longrightarrow \mathbf{R}^{m-s} \times \mathbf{R}^s$. Then f_x has the form $f_x(u, v) = (A_x u, B_x v) + h_x(u, v) = (A_x u + h_{1x}(u, v), B_x v + h_{2x}(u, v))$, for all $(u, v) \in \mathbf{B}^{m-s}_{r(x)} \times \mathbf{B}^s_{r(x)}$, where $\|A_x\| < \lambda'$, $\|B_x^{-1}\| < \lambda'$, $h_{1x}(0, 0) = h_{2x}(0, 0) = (0, 0)$, $(dh_{1x})_{(0,0)} = (dh_{2x})_{(0,0)} = 0$, $\|(dh_x)_{(u,v)}\| \leq (1 - \lambda')^2/100$.*

(3) *For $0 < h \leq 1$ and $0 \leq \delta \leq \frac{1}{2} hr(x)$, denote $C(x, h) = \Phi_x(\mathbf{B}^{m-s}_{hr(x)} \times \mathbf{B}^s_{hr(x)})$. Denote by $U^{\gamma,\delta,h}_x$ the class of $(m - s)$-dimensional submanifolds in $C(x, h) : U^{\gamma,\delta,h}_x = \{\Phi_x(graph\ \varphi) | \varphi : \mathbf{B}^{m-s}_{hr(x)} \to \mathbf{B}^s_{hr(x)}$ is C^1, $\|\varphi(0)\| \leq \delta$, $\|d\varphi\| \leq \gamma\}$ and $S^{\gamma,\delta,h}_x$ the class of s-dimensional submanifolds in $C(x, h) : S^{\gamma,\delta,h}_x = \{\Phi_x(graph\ \varphi) | \varphi : \mathbf{B}^s_{hr(x)} \to \mathbf{B}^{m-s}_{hr(x)}$ is C^1, $\|\varphi(0)\| \leq \delta$, $\|d\varphi\| \leq \gamma\}$. Then, for $B = \Phi_x(graph\ \varphi) \in U^{\gamma,\delta,h}_x$, we have $fB \cap C(fx, h) \in U^{\lambda'\gamma,((1+\lambda')/2)\delta,h}_{fx}$; similarly for $A = \Phi_x(graph\varphi) \in S^{\gamma,\delta,h}_x$, $f^{-1}A \cap C(f^{-1}x, h) \in S^{\lambda'\gamma,((1+\lambda')/2)\delta,h}_{f^{-1}x}$.*

(4) *Denote by d'_x the distance generated by the Riemannian metric $<,>'_x$, which is the image of the Euclidean metric in $\mathbf{B}^{m-s}_{r(x)} \times \mathbf{B}^s_{r(x)}$ under Φ_x. Then for $B \in U^{\gamma,\delta,h}_x$ and any two points $y_1, y_2 \in B$, $d'_{fx}(fy_1, fy_2) > (\frac{1}{2} + 1/2\lambda')d'_x(y_1, y_2)$; similarly, for $A \in S^{\gamma,\delta,h}_x$ and any two points $y_1, y_2 \in A$, $d'_{f^{-1}x}(f^{-1}y_1, f^{-1}y_2) > (\frac{1}{2} + 1/2\lambda')d'_x(y_1, y_2)$.*

For $x \in \Lambda$, we call $C(x,h)$ the hyperbolic neighborhood of x. Under local coordinates Φ_x, $f_x = \Phi_{fx}^{-1} \circ f \circ \Phi_x$ in hyperbolic neighborhoods shows uniform hyperbolic behavior by Proposition 3.1(2). By Proposition 3.1(1), we could shrink hyperbolic neighborhoods of points in the same Λ_k to the same size $\frac{1}{2}r_k$: for $x \in \Lambda_k$ and $0 < h \le 1$, denote the neighborhood $\widehat{C}(x,h) = \Phi_x(\mathbf{B}^{m-s}_{(1/2)hr_k} \times \mathbf{B}^{s}_{(1/2)hr_k}) \subset C(x,h)$. For $B \in U_x^{\gamma,(1/4)hr_k,h}$, we call any manifold of the form $B \cap \widehat{C}(x,h)$ an *admissible* (u,h)*-manifold near* x and similarly $A \cap \widehat{C}(x,h)$ an *admissible* (s,h)*-manifold near* x when $A \in S_x^{\gamma,(1/4)hr_k,h}$.

The following lemma comes from the main lemma in [**7**], where the statement is about closing one arc, that is, the case $q=1$. We generalize it to q arcs to fit our situation in proving Theorem A (where we need $q=2$ actually) by applying the same mechanism. For the convenience of readers, we present a proof.

LEMMA 3.2. *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact Riemannian manifold M and $\Lambda = \Lambda(\lambda_s, \lambda_u; \varepsilon)$ be a Pesin set of f. Then for any $\delta > 0$ and positive integer k there exists $\psi = \psi(k,\delta) > 0$ such that if there exist finite points $\{x_i\}_{i=1}^q$ and finite positive integers $\{n_i\}_{i=1}^q$ such that $x_i, f^{n_i}(x_i) \in \Lambda_k(\lambda_s,\lambda_u;\varepsilon)$, $d(x_{i+1}, f^{n_i}x_i) < \psi$ for $1 \le i \le q-1$ and $d(x_1, f^{n_q}x_q) < \psi$, then there exists a periodic point z with period $n = \sum_{i=1}^q n_i$ satisfying:*

(i) $d(f^{n_0+\cdots+n_i+j}z, f^j x_{i+1}) < \delta, \quad i = 0,\ldots,q-1, \quad j = 0,\ldots,n_{i+1}-1,$ *where* $n_0 = 0$;

(ii) *z is hyperbolic and its local stable (unstable) manifold is an admissible $(s,1)$ $((u,1))$-manifold near x_1.*

Proof. Let $B^0_{1,0} = \Phi_{x_1}(\mathbf{B}^{m-s}_{(1/2)hr_k} \times \{\mathbf{0}\})$, which is an admissible (u,h)-manifold near x_1. For $1 \le i \le n_1$, let $B^0_{1,i} = f(B^0_{1,i-1} \cap C(f^{i-1}x_1,h))$. We extend $B^0_{1,0}$ to $\widetilde{B}^0_{1,0} = \Phi_{x_1}(\text{graph } \widetilde{\varphi}^0_{1,0}) \in U_{x_1}^{\gamma,(1/4)hr_k,h}$; see Proposition 3.1(3) for notation. For $1 \le i \le n_1$, let $\widetilde{B}^0_{1,i} = f(\widetilde{B}^0_{1,i-1} \cap C(f^{i-1}x_1,h))$. By Proposition 3.1(3), $\widetilde{B}^0_{1,n_1} \cap C(f^{n_1}x_1,h)$ could be written as $\Phi_{f^{n_1}x}\big(\text{graph}(\widetilde{\varphi}^0_{1,n_1})\big)$, where $\widetilde{\varphi}^0_{1,n_1} : \mathbf{B}^{m-s}_{hr(f^{n_1}x_1)} \to \mathbf{B}^{s}_{hr(f^{n_1}x_1)}$ is a C^1 map such that $\|\widetilde{\varphi}^0_{1,n_1}(0)\| \le ((1+\lambda')/2)^{n_1}(1/4)hr_k$ and $\|d\widetilde{\varphi}^0_{1,n_1}\| \le (\lambda')^{n_1}\gamma$. Since $B^0_{1,n_1} \cap C(f^{n_1}x_1,h)$ is a part of the manifold $\widetilde{B}^0_{1,n_1}$, there exists a neighborhood of zero $\mathbf{D}^0_{1,n_1} \subset \mathbf{B}^{m-s}_{(1/2)hr(f^{n_1}x_1)}$ such that $B^0_{1,n_1} = \Phi_{f^{n_1}x}(\text{graph}(\widetilde{\varphi}^0_{1,n_1} \mid \mathbf{D}^0_{1,n_1}))$. By the expansion property of the manifold in $U_x^{\gamma,\delta,h}$ and choice of γ, see Proposition 3.1(4), one shows easily that $\mathbf{D}^0_{1,n_1}$ contains a ball with radius $(hr_k/2)(\frac{1}{2} + 1/2\lambda')^{1/2} > hr_k/2$ around the origin.

Let $B^0_{2,0} = B^0_{1,n_1} \cap \widehat{C}(x_2,h)$. Since Φ_x is continuous on Λ_k (Proposition 3.1(1)), when $d(x_2, f^{n_1}x_1)$ is sufficiently small we have

$$\Phi_{x_2}^{-1}\Phi_{f^{n_1}x_1}(\text{graph}\widetilde{\varphi}^0_{1,n_1}|\mathbf{D}^0_{1,n_1}) = \text{graph}(\varphi^0_{2,0}),$$

where $\varphi^0_{2,0}(0) \le (1/4)hr_k$, $\|d\varphi^0_{2,0}\| \le \gamma$ and $\varphi^0_{2,0}$ could be defined on a ball $\mathbf{B}^{m-s}_{(1/2)hr_k}$. Thus, $B^0_{2,0}$ is an admissible (u,h)-manifold near x_2. Let $B^0_{2,i} = f(B^0_{2,i-1} \cap C(f^{i-1}x_2,h))$ for $1 \le i \le n_2$ and $B^0_{3,0} = B^0_{2,n_2} \cap \widehat{C}(x_3,h)$. We see that $B^0_{3,0}$ is an admissible (u,h)-manifold near x_3. The procedure keeps going until we obtain an admissible (u,h)-manifold $B^1_{1,0} = B^0_{q,n_q} \cap \widehat{C}(x_1,h)$ near x_1. Carrying out the above cyclic construction infinitely, we obtain a sequence of admissible (u,h)-manifolds $\{B^m_{1,0}\}_{m\ge 0}$ near x_1, which we denote as $\{B^m\}_{m\ge 0}$.

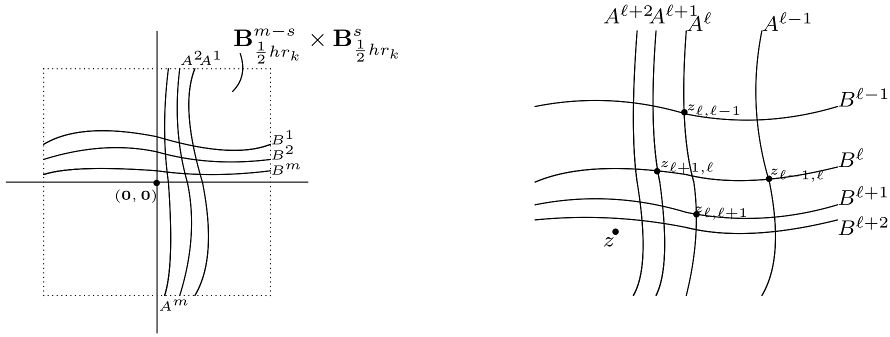

FIGURE 1. Admissible manifolds near x_1.

Similarly, applying f^{-1}, we obtain a sequence of admissible (s, h)-manifolds $\{A^m\}_{m\geq 0}$ near x_1. See the left-hand picture of Figure 1.

Let $z_{i,j}$ be the unique intersection between A^i and B^j in $\widehat{C}(x_1, h)$ whose constructions imply that $f^n z_{i,j} = z_{i-1,j+1}$ and in particular $f^n z_{\ell,\ell-1} = z_{\ell-1,\ell}$, where $n = \sum_{i=1}^{q} n_i$ for all $\ell \geq 1$. The expansion property of these admissible manifolds (Proposition 3.1(4)) implies that the sequences $\{z_{\ell,\ell-1}\}_{l\geq 0}$ and $\{z_{\ell-1,\ell}\}_{l\geq 0}$ converge exponentially to the same point z (see the right-hand picture of Figure 1), which is the shadowing periodic point we need. Observe that $f^{n_i+j}z$ always lies in the Lyapunov neighborhood $C(f^j x_{i+1}, h)$. For given $\delta > 0$ by choosing h sufficiently small we have $d(f^{n_i+j}z, f^j x_{i+1}) \leq \delta$. This proves (i). It is easy to show (ii), for stable and unstable manifolds are actually limits of $\{A^m\}_{m\geq 0}$ and $\{B^m\}_{m\geq 0}$ constructed in (i), respectively, in the C^0 topology. □

3.2. *Proof of Theorem A.* Denote by $\mathcal{M}_{\text{inv}}(M, f)$ the set of all f-invariant Borel probability measures with weak*-topology, which is metrizable: take a dense subset $\{\varphi_n\}_{n=1}^{\infty}$ of $C(M)$; then the weak*-topology could be given by the metric D: $\mathcal{M}_{\text{inv}}(M, f) \times \mathcal{M}_{\text{inv}}(M, f) \to \mathbb{R}$,

$$D(\mu, \nu) = \sum_{n=1}^{\infty} \frac{|\int \varphi_n d\mu - \int \varphi_n d\nu|}{2^n \|\varphi_n\|}.$$

By the triangle inequality, it is easy to verify the following affine property of D on $\mathcal{M}_{\text{inv}}(M, f)$.

LEMMA 3.3. *For any $\mu_1, \mu_2, \nu \in \mathcal{M}_{\text{inv}}(M, f)$ and $p \in [0, 1]$,*

$$D(p\mu_1 + (1-p)\mu_2, \nu) \leq pD(\mu_1, \nu) + (1-p)D(\mu_2, \nu).$$

Now we are prepared to prove Theorem A.

We will show that for given $\epsilon > 0$, there exists a Bernoulli measure ν such that $D(\mu, \nu) < \epsilon$. Let $L = L(\epsilon)$ be large enough such that $\sum_{j=L+1}^{\infty} (1/2^j) < \frac{1}{16}\epsilon$. Take $\delta > 0$ such that for any two points $w_1, w_2 \in M$ with $d(w_1, w_2) < \delta$, we have

$$|\varphi_j(w_1) - \varphi_j(w_2)| < \tfrac{1}{16}\epsilon\|\varphi_j\|, \quad j = 1, \ldots, L,$$

where $\{\varphi_n\}_{n=1}^{\infty}$ is a dense subset of $C(M)$ taken in the definition of D. Since the measure μ is ergodic and hyperbolic, we can take a Pesin set $\Lambda = \bigcup_{k\geq 0} \Lambda_k$ associated to μ.

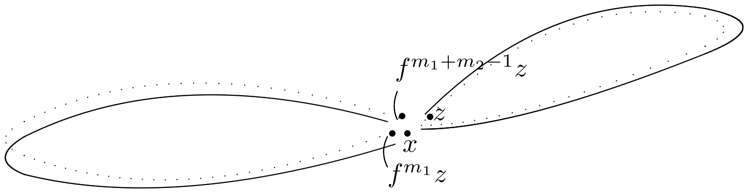

FIGURE 2. Periodic shadowing by z.

Since μ is weak mixing and thus ergodic, by the ergodic theorem,

$$G(\mu) := \left\{ x \in M : \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i x} \to \mu, n \to +\infty \right\}$$

has full measure and we call $x \in G(\mu)$ the generic point of μ. Denote

$$G_N(\mu) = \left\{ x \in G(\mu) \,\middle|\, D\left(\frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i x}, \mu\right) < \frac{1}{8}\epsilon, \forall n \geq N \right\}.$$

Choose N and k large enough such that $\mu(G_N(\mu) \cap \Lambda_k) > 0$. Fix $x \in \operatorname{supp}(\mu|_{\Lambda_k \cap G_N(\mu)})$ and denote $B(x, r) \cap \Lambda_k \cap G_N(\mu)$ as $B_{k,N}(x, r)$; then

$$\mu(B_{k,N}(x, r)) > 0 \quad \text{for all } r > 0.$$

For k and $\delta > 0$ taken as above, we take $\psi(k, \delta)$ as in Lemma 3.2 and let $r = \frac{1}{4}\psi(k, \delta)$.

Since μ is weak mixing, by Definition 2.1 there exist two *consecutive* integers m_1, $m_2 \geq N$ such that

$$\mu(f^{-m_i} B_{k,N}(x, r) \cap B_{k,N}(x, r)) > 0 \quad \text{for all } i = 1, 2.$$

It is clear that

$$(m_1, m_2) = 1.$$

Take two points y_1, y_2 such that y_i, $f^{m_i} y_i \in B_{k,N}(x, r)$, $i = 1, 2$. Note that

$$d(y_2, f^{m_1} y_1) < \psi(k, \delta), \quad d(y_1, f^{m_2} y_2) < \psi(k, \delta)$$

since $r = \frac{1}{4}\psi(k, \delta)$. Now the two points $\{y_i\}_{i=1,2}$ and the two integers $\{m_i\}_{i=1,2}$ satisfy the assumption of Lemma 3.2 and hence there is a hyperbolic periodic point z with period $m_1 + m_2$ satisfying

$$d(f^i z, f^i y_1) < \delta, 0 \leq i < m_1, \quad d(f^i z, f^{i-m_1} y_2) < \delta, m_1 \leq i < m_1 + m_2.$$

See Figure 2.

We split the rest of proof into two steps. The first step is to show that the periodic measure supported on the periodic orbit of z is close to μ in the weak*-topology. In the second step we will show that the stable manifold of z intersects transversely with the unstable manifold of fz, which implies that the 'horseshoe' associated with the transverse intersection is topologically mixing and thus there exists a Bernoulli measure supported on it. This Bernoulli measure is close to the periodic measure and thus close to the hyperbolic measure μ in the weak*-topology.

Step 1. For $\epsilon > 0$ given at the beginning of the proof of Theorem A, we show that

$$D\left(\frac{1}{m_1+m_2}\sum_{i=0}^{m_1+m_2-1}\delta_{f^i z},\,\mu\right) < \frac{1}{2}\epsilon. \tag{3}$$

Proof of Step 1. We reduce (3) to the following two inequalities:

$$D\left(\frac{1}{m_1}\sum_{i=0}^{m_1-1}\delta_{f^i z},\,\mu\right) < \frac{1}{4}\epsilon, \tag{4}$$

$$D\left(\frac{1}{m_2}\sum_{i=m_1}^{m_1+m_2-1}\delta_{f^i z},\,\mu\right) < \frac{1}{4}\epsilon. \tag{5}$$

We prove (4) and leave a similar proof of (5) to readers. Since $d(f^i z, f^i y_1) < \delta$, $0 \le i < m_1$,

$$|\varphi_j(f^i z) - \varphi_j(f^i y_1)| < \tfrac{1}{16}\epsilon\|\varphi_j\|,\, 0 \le i < m_1, \quad j = 1, \ldots, L.$$

By the choice of L, we have

$$\begin{aligned}
&D\left(\frac{1}{m_1}\sum_{i=0}^{m_1-1}\delta_{f^i z},\,\frac{1}{m_1}\sum_{i=0}^{m_1-1}\delta_{f^i y_1}\right)\\
&= \sum_{j=1}^{L}\frac{\left|\frac{1}{m_1}\sum_{i=0}^{m_1-1}(\varphi_j(f^i z)-\varphi_j(f^i y_1))\right|}{2^j\|\varphi_j\|} + \sum_{j=L+1}^{\infty}\frac{\left|\frac{1}{m_1}\sum_{i=0}^{m_1-1}(\varphi_j(f^i z)-\varphi_j(f^i y_1))\right|}{2^j\|\varphi_j\|}\\
&\le \left(\sum_{j=1}^{L}\frac{1}{2^j}\right)\frac{\epsilon}{16} + \frac{\epsilon}{16} < \frac{\epsilon}{16} + \frac{\epsilon}{16} = \frac{\epsilon}{8}.
\end{aligned} \tag{6}$$

Recall that $m_1 > N$ and thus $D((1/m_1)\sum_{i=0}^{m_1-1}\delta_{f^i y_1},\, \mu) < \frac{1}{8}\epsilon$. So, we have

$$D\left(\frac{1}{m_1}\sum_{i=0}^{m_1-1}\delta_{f^i z},\,\mu\right) \le D\left(\frac{1}{m_1}\sum_{i=0}^{m_1-1}\delta_{f^i z},\,\frac{1}{m_1}\sum_{i=0}^{m_1-1}\delta_{f^i y_1}\right) + D\left(\frac{1}{m_1}\sum_{i=0}^{m_1-1}\delta_{f^i y_1},\,\mu\right) < \frac{\epsilon}{4}.$$

This proves (4).

By Lemma 3.3, combined with (4) and (5), we directly obtain (3):

$$\begin{aligned}
&D\left(\frac{1}{m_1+m_2}\sum_{i=0}^{m_1+m_2-1}\delta_{f^i z},\,\mu\right)\\
&\le \frac{m_1}{m_1+m_2}D\left(\frac{1}{m_1}\sum_{i=0}^{m_1-1}\delta_{f^i z},\,\mu\right) + \frac{m_2}{m_1+m_2}D\left(\frac{1}{m_2}\sum_{i=m_1}^{m_1+m_2-1}\delta_{f^i z},\,\mu\right)\\
&< \frac{1}{4}\epsilon + \frac{1}{4}\epsilon = \frac{1}{2}\epsilon.
\end{aligned}$$

Thus, we complete the proof of Step 1. □

For the hyperbolic periodic point z with period $\pi(z)$, denote $W^s_{\text{loc}}(z)$ and $W^u_{\text{loc}}(z)$ as local stable and unstable manifolds of z, respectively, whose sizes are not uniquely determined. Denote $W^s(z) = \bigcup_{n\le 0} f^{n\pi(z)} W^s_{\text{loc}}(z)$ and $W^u(z) = \bigcup_{n\ge 0} f^{n\pi(z)} W^u_{\text{loc}}(z)$ as global stable and unstable manifolds of z. Write $W^s(z) \pitchfork W^u(f^i z) \neq \emptyset$ if $W^s(z)$ has transversal intersection with $W^u(f^i z)$.

Step 2. We show that $W^s(z) \pitchfork W^u(fz) \neq \emptyset$.

Proof of Step 2. By Lemma 3.2(ii), $W^u_{\text{loc}}(z)$ is an admissible $(u, 1)$-manifold near y_1. Denote $B(z) = W^u_{\text{loc}}(z) \cap \widehat{C}(y_1, 1)$. Following the method in the proof of Lemma 3.2(i), through iterations $B(f^i z) = f(B(f^{i-1}z) \cap \widehat{C}(f^{i-1}x_1, 1))$, $W^u_{\text{loc}}(f^{m_1}z)$ could be written as $\Phi_{f^{m_1}y_1}(\text{graph } \varphi|_{\mathbf{D}})$, where $\varphi : \mathbf{B}^{m-s}_{r_k} \to \mathbf{B}^s_{r_k}$ satisfies $\|\varphi(0)\| \le ((1+\lambda')/2)^{m_1} r_k/4 \le (1+\lambda')/2 \cdot r_k/4$, $\|d\varphi\| \le (\lambda')^{m_1}\gamma \le \lambda'\gamma$ and $\mathbf{D}$ contains a ball around the origin with radius $r_k/2(\frac{1}{2} + 1/2\lambda')^{1/2} > r_k/2$. Since both y_1 and $f^{m_1}y_1$ belong to Λ_k, shrinking r if necessary we have that $W^u_{\text{loc}}(f^{m_1}z) \cap \widehat{C}(y_1, 1)$ is an admissible $(u, 1)$-manifold near y_1. Since $W^s_{\text{loc}}(z)$ is an admissible $(s, 1)$-manifold near y_1, we obtain $W^u_{\text{loc}}(f^{m_1}z) \pitchfork W^s_{\text{loc}}(z) \neq \emptyset$.

We conclude the proof of Step 2 by the following lemma, which shows that the transverse intersection relation between $W^s(z)$ and $W^u(f^{m_1}z)$ could be transferred to an intersection between $W^s(z)$ and $W^u(fz)$†.

LEMMA 3.4. $W^s(z) \pitchfork W^u(fz) \neq \emptyset$.

Proof. Since $(m_1, m_2) = 1$, there exist $p, q \in \mathbb{Z}^+$ such that

$$m_1 p - m_2 q = 1. \tag{7}$$

Note that $W^s(z) \pitchfork W^u(f^{m_1}z) \neq \emptyset$ implies that $W^s(f^{m_1}z) \pitchfork W^u(f^{2m_1}z) \neq \emptyset$ and hence $W^s(z) \pitchfork W^u(f^{2m_1}z) \neq \emptyset$. In fact, let B be a small disk contained in $W^u(f^{2m_1}z)$ with the same dimension that intersects transversely with $W^s(f^{m_1}z)$. By the inclination lemma [**16**], for any n sufficiently large we have $f^{\pi(z)n}B$, which, still contained in $W^u(f^{2m_1}z)$, is C^1 close to $W^u_{\text{loc}}(f^{m_1}z)$. Thus, $f^{\pi(z)n}D \pitchfork W^s(z) \neq \emptyset$ since $W^u(f^{m_1}z) \pitchfork W^s(z) \neq \emptyset$. So, we obtain $W^s(z) \pitchfork W^u(f^{2m_1}z) \neq \emptyset$. By induction, we have $W^s(z) \pitchfork W^u(f^{m_1(p+q)}z) \neq \emptyset$, which means that $W^s(f^{(m_1+m_2)q}z) \pitchfork W^u(f^{m_1(p+q)}z) \neq \emptyset$ since $\pi(z) = m_1 + m_2$. Iterating backwards $(m_1+m_2)q$ steps, we obtain $W^s(z) \pitchfork W^u(f^{m_1p - m_2q}z) \neq \emptyset$. By (7), we directly get $W^s(z) \pitchfork W^u(fz) \neq \emptyset$. □

Now we complete the proof of Theorem A. Let $w \in W^s(z) \pitchfork W^u(fz)$; by [**16**, Theorem 4.5], there is a small neighborhood U of $\text{orb}(z) \cup \text{orb}(w)$ such that $f|_{\Lambda_U}$ is a hyperbolic invariant set which is conjugate to a subshift of finite type, where $\Lambda_U = \bigcap_{i\in\mathbb{Z}} f^i U$. By [**3**, Proposition 4.7], $f|_{\Lambda_U}$ is topologically mixing and, by [**5**, Theorem 34], f admits a Bernoulli measure ν on Λ_U. Choosing U small enough, we have $D(\nu, (1/(m_1+m_2)) \sum_{i=0}^{m_1+m_2-1} \delta_{f^i z}) < \epsilon/2$. Combining this with (3), we obtain $D(\mu, \nu) < \epsilon$. Thus, we complete the proof of Theorem A. □

† It was pointed out to us by the referee that this is an automatic corollary of [**2**, Proposition 1], which implies that $W^s(z) \pitchfork W^u(f^m z) \neq \emptyset$ if and only if $m \in \ell\mathbb{Z}$, where ℓ is a positive integer dividing $\pi(z)$. Readers may refer to [**2**] for more details. In particular in our case, $\ell = 1$ by the coprime relationship between m_1 and m_2. However, we still present our more elementary proof here, which is much shorter, and only makes use of the inclination lemma [**16**].

4. *Proof of Theorem C*

We prove Theorem C in this section. Recall that in the proof of Theorem A, Lemma 3.2 plays a key role in which the Hölder condition of derivatives is crucial since it determines the sizes of hyperbolic neighborhoods at different points. When the Hölder condition is removed and replaced by the domination property of the hyperbolic measure, we have another closing lemma of Liao [**10**] and its generalization by Gan [**6**].

4.1. *Closing lemma for C^1 non-uniformly hyperbolic systems with domination.*

Definition 4.1. [**10**] Given positive integers $n_1, n_2, \ldots, n_r$ and a real $\lambda \in (0, 1)$, an orbit arc

$$(x, \{n_j\}_{j=1}^r) = (x, fx, \ldots, f^{n_1}x, f^{n_1+1}x, \ldots, f^{n_1+n_2}x, \ldots, f^{n_1+\cdots+n_r}x)$$

is called λ-*quasi hyperbolic* with gap T with respect to a splitting $T_xM = E_x \oplus F_x$ if $0 < n_j \le T$, $j = 1, \ldots, r$, and the following three conditions are satisfied.

(1) $\prod_{j=0}^{k-1} \|Df^{n_{j+1}}|_{E_j}\| \le \lambda^k$, $1 \le k \le r$, where $E_j = Df^{n_0+\cdots+n_j}E_x$ and $n_0 = 0$.

(2) $\prod_{j=k}^{r-1} m(Df^{n_{j+1}}|_{F_j}) \ge \lambda^{k-r}$, $0 \le k \le r-1$, where $F_j = Df^{n_0+\cdots+n_j}F_x$ and $n_0 = 0$.

(3) $\|Df^{n_{j+1}}|_{E_j}\|/m(Df^{n_{j+1}}|_{F_j}) \le \lambda^2$, $0 \le j \le r-1$.

Here $m(Df|F_j)$ is the minimum norm of $Df|F_j$, i.e.

$$m(Df|F_j) = \inf\{\|Dfv\| : v \in F_j, \|v\| = 1\}.$$

In 1979, Liao [**10**] gave a closing lemma which asserts the existence of a hyperbolic periodic orbit near a quasi hyperbolic arc if its starting point and ending point are close, that is, tracing a hyperbolic arc by a hyperbolic periodic orbit. In 2002, Gan [**6**] generalized the closing lemma to a shadowing lemma, that is, tracing 'well-arranged' countably many hyperbolic arcs by a real orbit.

Definition 4.2. Let λ, ρ, T, d be positive reals, where $\lambda \in (0, 1)$, $T \ge 1$. A sequence of orbit arcs $(x_i, \{n_{i_j}\}_{j=1}^{r(i)})_{i=-\infty}^{\infty}$ is called a λ-*quasi hyperbolic ρ-pseudo orbit* with gap T if for any i, $(x_i, \{n_{i_j}\}_{j=1}^{r(i)})$ is λ-quasi hyperbolic with gap T with respect to the splitting $T_{x_i}M = E_{x_i} \oplus F_{x_i}$ and $d(f^{n_{i_1}+\cdots+n_{i_{r(i)}}}x_i, x_{i+1}) \le \rho$.

A point z d-*shadows* $(x_i, \{n_{i_j}\}_{j=1}^{r(i)})_{i=-\infty}^{\infty}$ provided that $d(f^kz, f^{k-N_i}x_i) \le d$, $k = N_i + \sum_{j=1}^{\ell} n_{i_j}$, $\ell = 1, \ldots, r(i)$, where

$$N_i = \begin{cases} 0, & i = 0, \\ \displaystyle\sum_{k=0}^{i-1}\left(\sum_{j=1}^{r(k)} n_{k_j}\right), & i > 0, \\ \displaystyle\sum_{k=-1}^{i}\left(\sum_{j=1}^{r(k)} n_{k_j}\right), & i < 0. \end{cases}$$

LEMMA 4.3. [**6**] *Let f be a C^1 diffeomorphism on a compact Riemannian manifold M and Λ be an f-invariant closed subset of M. Assume that there exists a continuous invariant splitting $T\Lambda = E \oplus F$ on Λ. Then for any $\lambda \in (0, 1)$ and $T > 0$ there exist $L > 0$ and $d_0 > 0$ such that for any $d \in (0, d_0]$ and any λ-quasi hyperbolic d-pseudo* *orbit* $(x_i, \{n_{i_j}\}_{j=1}^{r(i)})_{i=-\infty}^{\infty}$ *with gap* T *with respect to the splitting* $E \oplus F$*, there exists a point* z *on* M *which Ld-shadows* $(x_i, \{n_{i_j}\}_{j=1}^{r(i)})_{i=-\infty}^{\infty}$*. Moreover, if* $(x_i, \{n_{i_j}\}_{j=1}^{r(i)})_{i=-\infty}^{\infty}$ *is periodic, i.e. there exists* $c \in \mathbb{Z}^+$ *such that for any* $i \in \mathbb{Z}$, $x_i = x_{i+c}$, $r(i) = r(i+c)$ *and* $n_{i_j} = n_{(i+c)_j}$, $1 \le j \le r(i)$*, then* z *is also periodic with period* $\sum_{i=1}^{c}(n_{i_1} + \cdots + n_{i_{r(i)}})$.

Remark 4.4.

(i) For the convenience of our application, the statement here is a little different from [**6**], where in a quasi hyperbolic orbit arc all n_i are taken as 1.

(ii) Since the gap T is finite, for any $\rho > 0$ by choosing d sufficiently small the shadowing point z actually ρ-shadows the whole quasi hyperbolic orbit arc $(x_i, \{n_{i_j}\}_{j=1}^{r(i)})$, i.e. not only shadows the points $f^{n_{i_j}}(x_i)$.

4.2. *Proof of Theorem C.* For given $\epsilon > 0$, we show that there exists a Bernoulli measure ν such that $D(\mu, \nu) < \epsilon$.

Since μ is weak mixing and hyperbolic, by the ergodic theorem, the Lyapunov exponents are μ-almost everywhere constant and non-zero. Let λ_s be the norm of the maximal negative Lyapunov exponent and λ_u be the minimal positive exponent of μ, respectively. By [**1**, Lemma 8.4], for any $\delta > 0$ there exist an integer $N_\delta > 0$ and a measurable set X with $\mu(X) = 1$ such that for any $x \in X$ and $N \ge N_\delta$, we have

$$\lim_{k\to+\infty} \frac{1}{kN} \sum_{i=0}^{k-1} \log \|Df^N|_{E^s(f^{iN}x)}\| < -\lambda_s + \delta, \tag{8}$$

$$\lim_{k\to+\infty} \frac{1}{kN} \sum_{i=0}^{k-1} \log \|Df^{-N}|_{E^u(f^{-iN}x)}\| < -\lambda_u + \delta. \tag{9}$$

Thus, for any $x \in X$ and $N \ge N_\delta$, there exists $C(x, N) \ge 1$ such that for any $k \ge 1$, we have

$$\prod_{i=0}^{k-1} \|Df^N|_{E^s(f^{iN}x)}\| \le C(x, N) e^{kN(-\lambda_s+\delta)}, \tag{10}$$

$$\prod_{i=0}^{k-1} \|Df^{-N}|_{E^u(f^{-iN}x)}\| \le C(x, N) e^{kN(-\lambda_u+\delta)}. \tag{11}$$

Denote

$$G_N(\mu) = \left\{ x \in G(\mu) \,\middle|\, D\left(\frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i x}, \mu\right) < \frac{1}{8}\epsilon, \forall n \ge N \right\}.$$

Take $N_0 > N_\delta$ such that

$$\mu(G_{N_0}(\mu)) > \frac{1}{2}, \quad \frac{\|Df^N|_{E^s}\|}{m(Df^N|_{E^u})} \le \frac{1}{2} \quad \text{for all } N \ge N_0.$$

Now fix two prime integers $N_1, N_2 \ge N_0$, which implies in particular that $(N_1, N_2) = 1$. Denote $X_C = \{x \in X \mid C(x, N_1) \le C, \ C(x, N_2) \le C\}$. Fix $x \in \mathrm{supp}(\mu|_{X_C \cap G_{N_0}(\mu)})$ for C large enough and write $B_{C,N_0}(x, r) = B(x, r) \cap X_C \cap G_{N_0}(\mu)$, where $B(x, r)$ denotes the ball of diameter r centered at x.

Denote $J = J(B_{C,N_0}(x, r), B_{C,N_0}(x, r))$ as in Definition 2.1, which is of density 1. For the moment we denote the density of a subset $I \subset \mathbb{Z}^+$ as $\rho(I)$. As in the proof of Theorem A, we could obtain the coprime relationship from the weak mixing property.

LEMMA 4.5. *For any $M_0 > 0$, there exist integers $m_1, m_2 > M_0$ such that*

$$(m_1 N_1, m_2 N_2) = 1.$$

Proof. Let

$$J_1 = \{iN_1 | i \geq 0\}, \quad J_2 = \{iN_2 | i \geq 0\}.$$

Obviously, $\rho(J_1) = 1/N_1$, $\rho(J_2) = 1/N_2$. Let

$$J_1' = \{i \geq 0 \mid iN_1 \in J\}, \quad J_2' = \{i \geq 0 \mid iN_2 \in J\}.$$

Since

$$\begin{aligned}\lim_{n\to\infty} \frac{\#\{0 \leq i < n | iN_1 \in J\}}{n} &= \lim_{n\to\infty} \frac{\#\{0 \leq j < nN_1 | j \in J_1 \cap J\}}{nN_1} \cdot N_1 \\ &= \rho(J_1 \cap J) \cdot N_1 = 1,\end{aligned}$$

we have $\rho(J_1') = 1$. Similarly, $\rho(J_2') = 1$. Also note that

$$\rho(J_1' \cap J_2') = 1, \quad \rho(J_1' \cap J_2' \setminus J_1 \cup J_2) = 1 - \frac{1}{N_1} - \frac{1}{N_2} > \frac{1}{2}.$$

Hence, we could find sufficiently large integers $m_1, m_2 \in J_1' \cap J_2' \setminus J_1 \cup J_2$ such that $(m_1, m_2) = 1$.

Since N_1, N_2 are primes, by the choice of m_1, m_2 it is easy to see that $m_1 N_1$ and $m_2 N_2$ are coprime, that is, $(m_1 N_1, m_2 N_2) = 1$. □

As in the proof of Theorem A, we will make use of recurrent points of $B_{C,N_0}(x, r)$ to find a periodic measure close to μ.

LEMMA 4.6. *There exists a periodic point z such that*

$$D\left(\frac{1}{\pi(z)} \sum_{j=0}^{\pi(z)-1} \delta_{f^j z}, \mu\right) < \frac{1}{2}\epsilon, \tag{12}$$

where $\pi(z) = (m_1 N_1 + m_2 N_2)$ is the period of z.

Proof. Take $x_j \in B_{C,N_0}(x, r) \cap f^{-m_j N_j} B_{C,N_0}(x, r)$ for $j = 1, 2$. Let $\chi = \min\{\lambda_s, \lambda_u\} > 0$ and $0 < \delta \ll \chi$. Enlarging N_1, N_2 if necessary, we have

$$\prod_{i=0}^{k-1} \|Df^{N_j}|_{E^s(f^{iN_j} x_j)}\| \leq \lambda^k, \quad k = 1, \ldots, m_j, \tag{13}$$

$$\prod_{i=k}^{m_j-1} m(Df^{N_j}|_{E^u(f^{iN_j} x_j)}) \geq \lambda^{k-m_j}, \quad k = 0, \ldots, m_j - 1, \tag{14}$$

where $\lambda = \max\{e^{N_i(-\chi+2\delta)}, 1/\sqrt{2}\} \in (0, 1)$.

By the choice of N_1, N_2, we have

$$\frac{\|Df^{N_j}|_{E^s}\|}{m(Df^{Nj}|_{E^u})} \leq \lambda^2, \quad j \in \{1, 2\}.$$

Thus, we get a λ-quasi hyperbolic $2r$-pseudo orbit with gap $\max\{N_1, N_2\}$:

$$x_1, \ldots, f^{N_1}x_1, \ldots, f^{2N_1}x_1, \ldots, f^{m_1N_1}x_1,$$
$$x_2, \ldots, f^{N_2}x_2, \ldots, f^{2N_2}x_2, \ldots, f^{m_2N_2}x_2.$$

This is a periodic pseudo orbit. By Lemma 4.3, for any $d > 0$, by choosing $r > 0$ sufficiently small, we obtain a periodic point z such that $\mathrm{orb}(z, f)$ d-shadows this quasi hyperbolic pseudo periodic orbit. Then similarly as in Step 1 in the proof of Theorem A, we obtain (12). □

Shrinking d if necessary, we assume that z is a hyperbolic periodic point satisfying estimations as (13) and (14). More precisely, we acquire a constant $\widetilde{\lambda} \in (\lambda, 1)$ such that

$$\|Df^{\pi(z)}|_{E^s(z)}\| \leq \widetilde{\lambda}^{m_1+m_2}, \quad m(Df^{\pi(z)}|_{E^u(z)}) \geq \widetilde{\lambda}^{m_1+m_2},$$

where $\pi(z) = m_1N_1 + m_2N_2$ is the period of z. Thus, local stable and unstable manifolds of z and $f^{m_1N_1}z$ have uniform sizes as m_1, m_2 go to infinity. Shrinking d further if necessary, we obtain $W^s(z) \pitchfork W^u(f^{m_1N_1}z) \neq \emptyset$.

Since $(m_1N_1, m_2N_2) = 1$, by the same argument in the proof of Lemma 3.4, we obtain $W^s(z) \pitchfork W^u(fz) \neq \emptyset$. Thus, we could find a Bernoulli measure ν supported on a topologically mixing hyperbolic set with

$$D\left(\frac{1}{m_1N_1 + m_2N_2}\sum_{j=0}^{m_1N_1+m_2N_2-1}\delta_{f^jz}, \nu\right) < \frac{\epsilon}{2}.$$

Therefore, $D(\mu, \nu) < \epsilon$ and we complete the proof of Theorem C. □

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