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PATTERN'S RELIABILITY IMPORTANCE FOR DISCRETE LIFETIMES DISTRIBUTIONS.

by

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PATTERN'S RELIABILITY IMPORTANCE FOR DISCRETE LIFETIMES DISTRIBUTIONS

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ABSTRACT. Using martingale methods in reliability theory we develop a discrete version of the Barlow and Proschan reliability importance of a pattern to the system reliability under dependence conditions. We give an estimator for such a measure.
keywords: Martingale methods in reliability theory, compensator process, reliability importance measure, empiric distribution function.

INTRODUCTION

For a coherent system of independent components with absolutely continuous distribution, Barlow and Proschan (1975) defined the reliability importance of component i to the system reliability as the probability that the i-th component causes system failure. Iyer (1993) generalized this concept to the case where the component lifetimes are jointly continuous, but no necessarily independent.

Using martingale methods in reliability theory Bueno and Menezes (2004), developed the Barlow and Proschan reliability importance of a pattern to the system reliability under dependence conditions. Their definition generalizes the Iyer concept allowing simultaneous failure of a set of components. They specialize in the case where the component lifetimes are totally inaccessible stopping time, in which case their compensator process, relative to the component failure's counting, are continuous. In this paper we developed the reliability importance of patterns to the system reliability in the case where the component lifetimes have discrete distributions, with supports in the set $\{1, 2, ...\} = \mathbf{N}$ and, are not necessarily independent.

1. PRELIMINARIES

This section serves as background for next sections. For detailed presentations of this framework we recommend Dellacherie and Meyer (1982).

We are given a discrete probability space $(\Omega, \Im, P)$ with an increasing family of σ-fields $(\Im_n)_{n\in\mathbf{Z}^+}$. A process $(V_n)_{n\in\mathbf{Z}^+}$ is called $\Im_n$-predictable if V_0 is $\Im_0$-measurable and V_n is $\Im_{n-1}$-measurable for all $n \in \mathbf{Z}^+$. If X is an adapted process and V a predictable process, we shall denote $\int VdX$ the process $(\int_{[0,n]} VdX_k)_{n\in\mathbf{Z}^+}$ defined by:

$$(1.0.1)\quad \int_{[0,n]} VdX_k = V_0X_0 + V_1(X_1 - X_0) + ... + V_n(X_n - X_{n-1}).$$

Key words and phrases. Martingale methods in reliability theory, compensator process, reliability importance measure, empiric distribution function.

The process (1.0.1) is sometime called the transform of X by V. It is just the discrete form of the stochastic integral $\int VdX$. We use the following notation:

$$\int_{[0,\infty]} VdX_k = \sum_{k=0}^{\infty} V_k(X_k - X_{k-1}).$$

Theorem 1.0.1. *Let $X = (X_n)_{n\in\mathbf{Z}^+}$ be a martingale and $V = (V_n)_{n\in\mathbf{Z}^+}$ a predictable process. If the random variables $\int_{[0,n]} VdX_k$ are integrable, them $\int VdX$ is a martingale.*

Theorem 1.0.2. *(Doob- Meyer decomposition, discrete case) Let $X = (X_n)_{n\in\mathbf{Z}^+}$ be a submartingale with respect to a filtration $(\Im_n)_{n\in\mathbf{Z}^+}$. Then we define*

$$X_n = A_n + M_n \tag{1.0.2}$$

where

$$\begin{aligned} A_n &= A_{n-1} + E[X_n|\Im_{n-1}] - X_{n-1}, \\ M_n &= X_n - A_n, n \in \mathbf{Z}^+. \end{aligned}$$

The process M is a $\Im_n$-martingale and A is nondecreasing and $\Im_n$-predictable process, with $A_0 = 0$.

Remark 1.0.3. The decomposition in the theorem above is unique and the process A is often called compensator.

2. Mathematical Formulation:

We consider a collection of components, $C_1, C_2, ..., C_n$. These are often assumed to form a large system Φ. Each component C_i has a positive lifetime T_i after 0, where 0 can be thought of as the time at which Φ is installed. We let $T_i, i = 1, ..., n$, random variables in a discrete probability space $(\Omega, \Im, P)$, whose distribution $F_i, i = 1, ..., n$ has support contained in the set $\{0, 1, ...\} = \mathbf{Z}^+$. Our aim is to derive the failure rate process for the lifetime

$$T = inf\{k \in \mathbf{Z}^+ : \Phi(\mathbf{X}_k) = 0\}$$

with respect to the filtration $\mathbf{F}$ given by $\Im_k = \sigma(\mathbf{X}_s, s = 0, 1, ..., k)$, where $\mathbf{X}_s = (X_s(1), X_s(2), ..., X_s(n))$ and $X_s(i) = I(T_i > s), i = 1, ...n$. We call this filtration the complete information filtration or filtration on the component level. Each $\Im_k$ is assumed to be completed with the null sets of $(\Im_k)_{k\in\mathbf{Z}^+}$.

We next describe the failures of $C_1, C_2, ..., C_n$ as they appear in advancing time, as a discrete stochastic process. This is conveniently done in terms of a discrete multivariate (or market) point process (see Bremaud (1982), pg 69). The failure a of system consisting of $C_1, C_2, ..., C_n$ can then be thought of as a simple point process derived from the multivariate process.

For any outcome $T_1(\omega), T_2(\omega), ..., T_n(\omega)$ of the lifetimes of $C_1, C_2, ..., C_n$, let $q(\omega)$ be the number of distinct values in the set $\{T_i(\omega); i = 1, ..., n\}$. We denote the strictly increasing order statistics of this set by $T_{(k)}$, having then

$$T_{(1)} < T_{(2)} < ... < T_{(q(w))}.$$

Also let

$$J_{(k)}(w) = \{i : T_i(w) = T_{(k)}(w), i = 1, ..., n\}$$

be the index set of the components failing at the kth smallest failure time $T_{(k)}$. If there are no multiples failures, the value of $J_{(k)}$ is one of the singletons $\{i\}, i = 1, ..., n$. In general, however, $J_{(k)}$ is a Λ-valued random variable, where Λ is the power set of $\{1, 2, ..., n\}$. We call $T_{(k)}$ the kth failure time and $J_{(k)}$ the kth failure pattern.

The random sequence $(T_{(k)}, J_{(k)})_{k=1,...,q}$ (of random length q)describes completely how the components $C_1, ..., C_n$ fail. We let

$$T_{(q+1)} = T_{(q+2)} = ... = \infty$$

and

$$J_{(q+1)} = J_{(q+2)} = ... = \emptyset$$

and call the discrete multivariate point process $(T_{(k)}, J_{(k)})_{k\in\mathbf{N}}$ the failure process of $C_1, ..., C_n$.

Another equivalent way to describe the failures is by a discrete multivariate counting process: For each fixed $J \in \Lambda$, let T_J and $N_J(w; k)$ be defined by

$$T_J = \inf\{T_{(k)} : J_{(k)} = J\},$$

where $\inf \emptyset = \infty$ and

$$N_J(w, k) = I(T_J \leq k) = \begin{cases} 0 & \text{if } k < T_J(w) \\ 1 & \text{if } k \geq T_J(w). \end{cases}$$

Each $(T_i(w))_{1\leq i\leq n}$ determines a sample path of the process $(N_J(k); J \in \Lambda)_{k\in\mathbf{Z}^+}$ and conversely. Therefore

$$\Im_k = \sigma(N_J(s); s = 0, ..., k, J \in \Lambda)$$

is equivalent to

$$\Im_k = \sigma(\mathbf{X}_s, s = 0, 1, ..., k).$$

The stoping times T_i or T_J are rarely of direct concern in reliability theory. One is more interested in system failures times, which depend on the cumulative pattern of failed components. In more detail, let Φ a monotone (or coherent) system with lifetime T. For $\mathbf{x} \in \{0, 1\}^n$ and $J = \{j_1, j_2, ..., j_r\} \in \Lambda$, the vectors $(\mathbf{1_J}, \mathbf{x})$ and $(\mathbf{0_J}, \mathbf{x})$ denote those n-dimensional state vectors in which the components $x_{j_1}, ..., x_{j_r}$ of $\mathbf{x}$ are replaced by $1'$s and $0'$s, respectively. Let $D(k)$ be the set of components that have failed up to time k, formally

$$D(k) = \begin{cases} J_{(1)} \cup ... \cup J_{(l)}, & \text{if } T_{(l)} = k \\ \emptyset & \text{if } k < T_{(1)}. \end{cases}$$

We can therefore think that the point process with its only point at T, or equivalently the counting process

$$N_\Phi(k) = I(T \leq k), k \in \mathbf{Z}^+,$$

has been derived from the multivariate point process $(T_{(k)}, J_{(k)})_{k\in\mathbf{N}}$.

Then we define a pattern J to be critical at time k, if

$$I(J \cap D(k) = \emptyset)(\Phi(\mathbf{1_J}, \mathbf{X_k}) - \Phi(\mathbf{0_J}, \mathbf{X_k})) = 1$$

and denote by

$$\Gamma_\Phi(k) = \{J \in \Lambda : I(J \cap D(k) = \emptyset)(\Phi(\mathbf{1_J}, \mathbf{X_k}) - \Phi(\mathbf{0_J}, \mathbf{X_k})) = 1\}$$

the collection of all such patterns critical at k. We see that $k \to \Gamma_\Phi(k)$ is increasing in the natural partial order of Λ for $k \leq T$.

Changing the point of view slightly, we can fix a time $k \in \mathbb{Z}^+$ and then look what immediate failure patterns in k would result in a system failure. Note that,

$$\{T = k\} = \bigcup_{J \in \Gamma_\Phi(k)} \{T_J = k\}, \tag{2.0.3}$$

where the events $\{T_J = k\}$ are disjoint.

It will be convenient to define the critical level of the failure pattern J, the time from which onwards a failure of pattern J leads to system failure, that is, the $\Im_t$-stopping time $Y_\Phi(J)$ by

$$\begin{aligned} Y_\Phi(J) &= \inf\{k \in \mathbb{Z}^+ : J \in \Gamma_\Phi(k)\} \\ &= \inf\{k \in \mathbb{Z}^+ : D(k) \cup J \in \Lambda_\Phi\}, \end{aligned}$$

where Λ_Φ is the collection the cut sets of Φ.

We clearly see that

$$\begin{aligned} \{J \in \Gamma_\Phi(k)\} &= \{Y_\Phi(J) < k \leq T\} \\ &= \{\omega : k \leq T_J(\omega) = T(\omega)\}. \end{aligned} \tag{2.0.4}$$

Having treated the component and system failures as discrete point processes, a reader familiar with martingale methods in point process theory already expects that the notion of hazard will be in terms of the stochastic intensity of such processes, or, in an integral form, the compensator of the counting process $(N_J(k); J \in \Lambda)_{k \in \mathbb{Z}^+}$ (or of $(N_\Phi(k))_{k \in \mathbb{Z}^+}$). This notion of hazard is developed in the following. We remark that little of what follows depends on the particular structure we have postulated for the mark space Λ, the family of all subsets of $\{1, 2, ..., n\}$.

We start by considering a fixed pattern $J \in \Lambda$ and introduce the corresponding pattern-specific hazard process. The family of such processes, indexed by $J \in \Lambda$, is called the multivariate hazard process. We then go on by studying an arbitrary system lifetime and derive the connection between its hazard process and the multivariate process.

The $\Im_t$- compensator $(A_J(k))_{k \in \mathbb{Z}^+}$ of the univariate discrete counting process $(N_J(k))_{k \in \mathbb{Z}^+}$ is the a.s. unique increasing predictable process such that, for each $l \in \mathbb{N}$, the difference process stopped at $T_{(l)}$,

$$A_J(k \wedge T_{(l)})) - N_J(k \wedge T_{(l)}),$$

is an $(\Im_k)$-martingale (see theorem 1.0.2). In view of the fact that $T_{(q+1)} = \infty$ then we have that $N_J(k) - A_J(k)$, is an $(\Im_k)$-martingale, in particular

$$M_J(k) = N_J(k \wedge T) - A_J(k \wedge T)$$

is an $(\Im_k)$-martingale.

The compensator, when understood as a measure on the real line, is well known to have the interpretation

$$dA_J(k) = P(T_J = k | \Im_{k-1}).$$

Considering then $\Im_k = \sigma(N_J(s); s = 0, 1, ..., k \text{ and } J \in \Lambda)$, (see Arjas 1989), on the set $\{T \geq k\}$, we write:

$$\lambda_k(J) I(T_J \geq k) = dA_J(k) = \frac{P(T_J = k, J \in \Gamma_\Phi(k))}{P(T_J \geq k, J \in \Gamma_\Phi(k))} I(T_J \geq k),$$

and

$$A_J(k) = \int_{[0,k]} dA_J(s) = \sum_{s=1}^{k} \frac{P(T_J = s, J \in \Gamma_\Phi(s))}{P(T_J \geq s, J \in \Gamma_\Phi(s))} I(T_J \geq s),$$

where $(\lambda_k(J))$ be the **F**-failure rate process corresponding to T_J. Intuitively, this corresponds to predicting if T_J is going to occur "now", based on all observations available up to the present, but not including it. Motivated by this we call $(A_J(k))_{k\in \mathbf{Z}^+}$ the hazard process of failure pattern J and $(A_J(k); J \in \Lambda)_{k\in \mathbf{Z}^+}$ the multivariate hazard process.

We now go on by studying the $\Im_k$-compensator of the counting process $(N_\Phi(k))_{k\in \mathbf{Z}^+}$ of system failure, denoting it by $(A_\Phi(k))_{k\in \mathbf{Z}^+}$. For obvious reasons we call this compensator the system hazard process.

It is natural to ask what is the contribution of the failure's component propensity for predicting the system's failure propensity. To answer this question, in Theorem 2.0.4 below, Aven and Jensen (1999)(discrete version) characterize the relationship between the component's $\Im_k$- compensator and the system's $\Im_k$-compensator processes.

Theorem 2.0.4. *Let $(\lambda_k(J))$ be the* **F***-failure rate process corresponding to $T_J, J \in \Lambda$. Then for all $k \in \mathbf{N}$ on $\{T \geq k\}$:*

$$\lambda_k = \sum_{J\in\Lambda} I(J \cap D(k) = \emptyset)(\Phi(\mathbf{1_J}, \mathbf{X_k}) - \Phi(\mathbf{0_J}, \mathbf{X_k}))\lambda_k(J) = \sum_{J\in\Gamma_\Phi(k)} \lambda_k(J), \text{ a.e.}$$

where λ_k be the failure process corresponding to T.

Proof. We show that the condition

$$E\left[\int_{[0,\infty)} g(k) dN_\Phi(k)\right] = E\left[\sum_{k=1}^{\infty} g(k) \sum_{J\in\Gamma_\Phi(k)} \lambda_k(J)\right] \tag{2.0.5}$$

holds for all predictable process $g : \Omega \times \mathbf{Z}^+ \to \mathbf{Z}^+$. By general results of point process theory this then certifies that $\sum_{J\in\Gamma_\Phi(k)} dA_J(k)$ is the compensator measure of $N_\Phi(k)$.

By specializing in Theorem 1.0.1, we see that condition

$$\begin{aligned} E\left[\int_{[0,\infty)} \sum_{J\in\Gamma_\Phi(k)} f(k,J) dN_J(k)\right] &= E\left[\int_{[0,\infty)} \sum_{J\in\Gamma_\Phi(k)} f(k,J) dA_J(k)\right] \\ &= E\left[\sum_{k=1}^{\infty} \sum_{J\in\Gamma_\Phi(k)} f(k,J)\lambda_k(J)\right] \end{aligned} \tag{2.0.6}$$

is valid for all predictable process $f : \Omega \times \mathbf{Z}^+ \times \Lambda \to \mathbf{Z}^+$. Choosing in particular a predictable g and

$$f(k, J) = g(k) I(J \in \Gamma_\Phi(k))$$

and using (2.0.3) we see that (2.0.6) reduces to (2.0.5) (f is predictable since g is predictable and $I(J \in \Gamma_\Phi(k))$ is predictable). We write

$$\lambda_k = \sum_{J\in\Gamma_\Phi(k)} \lambda_k(J). \tag{2.0.7}$$

□

Remark 2.0.5.

i: As $\{J \in \Gamma_\Phi(k)\} \subseteq \{T \geq k\} \in \Im_{k-1}$ and $\{T \geq k\}$ is an atom of $\Im_{k-1}$ (see Aven and Jensen (1999), pg. 71) we have $I(J \in \Gamma_\Phi(k))$ is $\Im_{k-1}$-mensurable and therefore predictable.

ii: Considering (2.0.4) and (2.0.7) we write:

$$A_\Phi(k) = \sum_{J \in \Lambda} [A_J(k \wedge T) - A_J(Y_\Phi(J))]^+ \text{ a.e..}$$

3. IMPORTANCE OF PATTERNS.

Under the assumption previous of section follows that a pattern J is critical to the system at time k if

$$I(J \in D(k))(\Phi(\mathbf{1_J}, \mathbf{X_k}) - \Phi(\mathbf{0_J}, \mathbf{X_k})) = 1.$$

Clearly, we have the equivalence

$$\{\omega : k \leq T_J(\omega), J \in \Gamma_\Phi(k)\} = \{\omega : k \leq T(\omega), J \in \Gamma_\Phi(k)\} \tag{3.0.8}$$

and an the set $\{T_J = k\}$ we can define:

Definition 3.0.6. The reliability importance of the pattern J (on the set $\{T_J = k\}$), to the system's reliability at time k is the probability that J is critical to the system at time k, that is

$$I_C^B(J, k) = P(J \in \Gamma_\Phi(k) | T_J = k).$$

Remark 3.0.7. In the case where simultaneous failure are ruled out, the values of $J_{(k)}$ is one of the singletons $\{i\}, i = 1, ..., n,$ and

$$I_C^B(\{i\}, k) = P(\Phi(\mathbf{1_i}, \mathbf{X_k}) - \Phi(\mathbf{0_i}, \mathbf{X_k}) = 1 | T_i = k),$$

which is a discrete version of the result from Iyer(1992). If the components are independents, $I_C^B(\{i\}, k)$ is the Birnbaum reliability importance.

We apply the notion of criticality to generalize the Barlow and Proschan importance of component's reliability for the system's reliability in one discrete case.

Definition 3.0.8. Let T be the lifetime of a coherent system and let T_J be the pattern lifetime, with $\Im_k$-compensator processes $A_J(k) = \sum_{s=1}^k I(T_J \geq s)\lambda_s(J)$. The reliability importance of pattern J for the system reliability is defined by

$$I^\Phi(J) = \sum_{k=1}^{\infty} I_C^B(J, k) P(T_J = k).$$

In other words, the reliability importance of pattern J for the system reliability is the probability that the pattern J causes system failure, i.e:

$$I^\Phi(J) = \sum_{k=1}^{\infty} P(T_J = k, J \in \Gamma_\Phi(k)).$$

Proposition 3.0.9. *Under the assumption above, we have:*

$$\sum_{J \in \Lambda} I^\Phi(J) = 1,$$

this is, $I^\Phi(J), J \in \Lambda$ *is a probability measure.*

Proof. Not that, from Theorem (2.0.4), we have:

$$\begin{aligned}
P(T \leq k) &= E(A_\Phi(k)) \\
&= \sum_{J\in\Lambda} E[\int_{[1,k]} I(J \in \Gamma_\Phi(s))dA_J(s)] \\
&= \sum_{J\in\Lambda} E[\sum_{s=1}^{k} I(J \in \Gamma_\Phi(s))\lambda_s(J)] \\
&= \sum_{J\in\Lambda}\sum_{s=1}^{k} E[I(J \in \Gamma_\Phi(s))\lambda_s(J)] \\
&= \sum_{J\in\Lambda}\sum_{s=1}^{k} P(T_J = s, J \in \Gamma_\Phi(s)).
\end{aligned}$$

Therefore

$$\begin{aligned}
E(A_\Phi(T)) &= \sum_{J\in\Lambda}\sum_{k=1}^{\infty} P(T_J = k, J \in \Gamma_\Phi(k)) \\
&= \sum_{J\in\Lambda} I^\Phi(J) = 1.
\end{aligned}$$

□

Remark 3.0.10.

i: For a coherent system Φ the reliability importance of the component i to system reliability, at time k, is the probability that the component i causes the system failure, i.e.:

$$I(i,k) = P(system\ failure\ in\ time\ k\ due\ to\ componet\ i).$$

In other words:

$$I(i,k) = \sum_{\{i\}\in J} I_C^B(J,k)P(T_J = k).$$

ii: For a coherent system Φ the reliability importance of the component i to system reliability is the probability that the component i causes the system failure, i.e.:

$$I^\Phi(i) = P(system\ failure\ due\ to\ componet\ i).$$

In other words:

$$I^\Phi(i) = \sum_{\{i\}\in J} I^\Phi(J).$$

Example 1:

We analyze a system of two dependent components C_1, C_2. To this we consider "the two-dimensional geometric distribution of Marshall and Olkin" with parameters given by $0 < p_1, p_2 < 1$ and $0 \leq p_{12} < 1$. Let T_1 and T_2 component's lifetimes. An interpretation of this distribution is as follows. Three independent geometric random variables Z_1, Z_2, Z_{12} with corresponding parameters p_1, p_2 and p_{12} describe the time points when a shock causes failure of component 1 or 2 or all intact components at the same time respectively. Then the components lifetimes are given by $T_1 = Z_1 \wedge Z_{12}$ and $T_2 = Z_2 \wedge Z_{12}$. The three different patterns to distinguish are $\{1\}$, $\{2\}$, $\{12\}$. Note that $T_{\{1\}} \neq T_1$ as we have for example $T_{\{1\}} = \infty$ on $\{T_1 = T_2\}$, i.e., on $\{Z_{12} \leq Z_1 \wedge Z_2\}$.

Series System: $(T = T_1 \wedge T_2)$

On the set $\{T \geq k, T_J \geq k\}$

$$\lambda_{\{12\}}(k) = \frac{(p_{12} + p_1p_2(1-p_{12}))[(1-p_1)(1-p_2)(1-p_{12})]^{k-1}}{1-(p_{12}+p_1p_2(1-p_{12}))\left[\frac{1-[(1-p_1)(1-p_2)(1-p_{12})]^{k-1}]}{1-(1-p_1)(1-p_2)(1-p_{12})}\right]};$$

$$\lambda_{\{1\}}(k) = \frac{(p_1(1-p_{12})(1-p_2)[(1-p_1)(1-p_2)(1-p_{12})]^{k-1}}{1-(p_1(1-p_{12})(1-p_2))\left[\frac{1-[(1-p_1)(1-p_2)(1-p_{12})]^{k-1}]}{1-(1-p_1)(1-p_2)(1-p_{12})}\right]};$$

$$\lambda_{\{2\}}(k) = \frac{(p_2(1-p_{12})(1-p_1)[(1-p_1)(1-p_2)(1-p_{12})]^{k-1}}{1-(p_2(1-p_{12})(1-p_1))\left[\frac{1-[(1-p_1)(1-p_2)(1-p_{12})]^{k-1}]}{1-(1-p_1)(1-p_2)(1-p_{12})}\right]};$$

$$\Gamma_\Phi(k) = \begin{cases} \{12\}, & \text{in } \{Z_{12} \geq k, Z_1 \wedge Z_2 \geq k\} \bigcup \{Z_{12} > k, k \leq Z_1 = Z_2 \leq Z_{12}\} \\ \{1\}, & \text{in } \{Z_1 \geq k, Z_2 \wedge Z_{12} > Z_1\} \\ \{2\}, & \text{in } \{Z_2 \geq k, Z_1 \wedge Z_{12} > Z_2\}. \end{cases}$$

Therefore :

$$I^\Phi(\{12\}) = \frac{p_{12} + p_1p_2(1-p_{12})}{1-(1-p_1)(1-p_2)(1-p_{12})}$$

$$I^\Phi(\{1\}) = \frac{p_1(1-p_{12})(1-p_2}{1-(1-p_1)(1-p_2)(1-p_{12})}$$

$$I^\Phi(\{2\}) = \frac{p_2(1-p_{12})(1-p_1}{1-(1-p_1)(1-p_2)(1-p_{12})}.$$

Parallel System : $(T = T_1 \vee T_2)$

On the set $\{T \geq k, T_J \geq k\}$

$$\lambda_{\{12\}}(k) = \frac{(p_{12} + p_1p_2(1-p_{12}))[(1-p_1)(1-p_2)(1-p_{12})]^{k-1}}{1-(p_{12}+p_1P_2(1-p_{12}))\left[\frac{1-[(1-p_1)(1-p_2)(1-p_{12})]^{k-1}]}{1-(1-p_1)(1-p_2)(1-p_{12})}\right]};$$

$$\begin{aligned} \lambda_{\{1\}}(k) &= \frac{(p_1 + p_{12} - p_1p_{12})[(1-p_1)(1-p_{12})]^{k-1}}{1-B} \\ &+ \frac{(p_1p_{12} - p_1p_{12})[(1-p_1)(1-p_{12})(1-p_2)]^{k-1}}{1-B}; \end{aligned}$$

where :

$$B = \sum_{s=1}^{k-1}[(p_1+p_{12}-p_1p_{12})[(1-p_1)(1-p_{12})]^{s-1}] + \sum_{s=1}^{k-1}[(p_1p_{12}-p_1p_{12})[(1-p_1)(1-p_{12})(1-p_2)]^{s-1}].$$

$$\lambda_{\{2\}}(k) = \frac{(p_2+p_{12}-p_2p_{12})[(1-p_2)(1-p_{12})]^{k-1}}{1-C} + \frac{(p_2p_{12}-p_2p_{12})[(1-p_1)(1-p_{12})(1-p_2)]^{k-1}}{1-C};$$

where :

$$C = \sum_{s=1}^{k-1}[(p_2+p_{12}-p_2p_{12})[(1-p_2)(1-p_{12})]^{s-1}] + \sum_{s=1}^{k-1}[(p_2p_{12}-p_2p_{12})[(1-p_1)(1-p_{12})(1-p_2)]^{s-1}].$$

$$\Gamma_\Phi(k) = \begin{cases} \{12\}, & \text{in } \{Z_{12}\geq k, Z_1\wedge Z_2\geq k\}\bigcup\{Z_{12}>k, k\leq Z_1=Z_2\leq Z_{12}\} \\ \{1\}, & \text{in } \{Z_1\wedge Z_{12}\geq k, Z_2<k\} \\ \{2\}, & \text{in } \{Z_2\wedge Z_{12}\geq k, Z_1<k\}. \end{cases}$$

Therefore :

$$I^\Phi(\{12\}) = \frac{p_{12}+p_1p_2(1-p_{12})}{1-(1-p_1)(1-p_2)(1-p_{12})}$$

$$I^\Phi(\{1\}) = \frac{(p_1+p_{12}-p_1P_{12})}{1-(1-p_1)(1-p_{12})} + \frac{(p_1p_{12}-p_1-P_{12})}{1-(1-p_1)(1-p_2)(1-p_{12})}$$

$$I^\Phi(\{2\}) = \frac{(p_2+p_{12}-p_1P_{12})}{1-(1-p_2)(1-p_{12})} + \frac{(p_2p_{12}-p_2-P_{12})}{1-(1-p_1)(1-p_2)(1-p_{12})}.$$

4. A ESTIMATOR FOR IMPORTANCE OF PATTERNS .

We beginning by considering the random variable (r.v.) $\bar{T} = \sum_{J\in\Lambda} T_J I(T_J = T)$. Fortunately $\{T = k\} = \bigcup_{J\in\Gamma_\Phi(k)} \{T_J = k\}$, and the events $\{T_J = k\}$ are disjoint, that is, for almost all $\omega \in \Omega$ fixed there exist only one pattern $J \in \Lambda$ where $T_J(\omega) = \bar{T}(\omega)$. Therefore $\bar{T} = T$ a.e. and $F(x) = P(T \leq x) = P(\bar{T} \leq x)$, for all x.

Let $\{\bar{T}_n, n \geq 1\}$ be a sequence of independent and identically distributed r.v.'s with the common $d.f.F$. This is sometimes referred to as the "underlying" or "theoretical distribution" and is regarded as "unknown" in statistical lingo. For each ω the values $\bar{T}(\omega)$ are called "samples" or "observed values", and the idea is to get information on F by looking at the samples. For each n, and each $\omega \in \Omega$, let, either $n \in \mathbf{N}$ and $\{\bar{T}_k, k = 1, ..., n\}$ be arranged in increasing order as

$$Y_{n1}(\omega) \leq Y_{n2}(\omega) \leq ... \leq Y_{nn}(\omega).$$

Now define a discrete $d.f.F_n(.,\omega)$ as follows:

$$F_n(x,\omega) = \begin{cases} 0, & \text{if } 0 < x < Y_{n1}(\omega), \\ \dfrac{k}{n}, & \text{if } Y_{nk}(\omega) \le x < Y_{n,k+1}(\omega), \\ 1, & \text{if } x \ge Y_{nn}(\omega). \end{cases}$$

For each x, $nF_n(x,\omega)$ is the number of values of $k, 1 \le k \le n$, for which $\bar{T}_k(\omega) \le x$; or in other words, $F_n(x,\omega)$ is the observed frequency of sample values not exceeding x. The function $F_n(.,\omega)$ is called "empiric distribution function based on n samples from F". For each x, we have

$$\begin{aligned} F_n(x,\omega) &= \frac{1}{n}\sum_{k=1}^{n} I(\bar{T}_k(\omega) \le x) \\ &= \frac{1}{n}\sum_{k=1}^{n}\sum_{J\in\Lambda} I(T_J^k(\omega) = T(\omega) \le x), \end{aligned} \tag{4.0.9}$$

where $\{T_J^k, k \ge 1\}$ be a sequence of independent, identically distributed r.v.'s with the common $d.f.$ F_J. Note what the strong law of large number applies, and we conclude that

$$F_n(x,\omega) \to F(x) \quad a.e.. \tag{4.0.10}$$

Under the above assumptions, Glivenko and Cantelli (see Chung (1968)) proved:

Theorem 4.0.11. *We have as $n \to \infty$*

$$\sup_{0<x<\infty} |F_n(x,\omega) - F(x)| \to 0 \quad a.e.$$

Definition 4.0.12. Under the above assumptions, we defined the estimate of $I^\Phi(J)$ as

$$\hat{I}_n^\Phi(J) = \frac{1}{n}\sum_{k=1}^{n} I(T_J^k(\omega) = T(\omega)), \tag{4.0.11}$$

where $\{T_J^k, k \ge 1\}$ be a sequence of independent, identically distributed r.v.'s with the common $d.f.$ F_J.

Remark 4.0.13.

- **i:** By (4.0.9) and Theorem (4.0.10) is easy see what:
$$\lim_{n\to\infty} |\hat{I}_n^\Phi(J) - I^\Phi(J)| = 0 \quad a.e$$
for all $J \in \Lambda$.
- **ii:** If we are working with a model where the positive lifetimes T_i's not necessarily independents, but $T = \Phi(T_1, ..., T_n) = \Phi(Z_1, Z_2, ..., Z_m)$, where Z_i are independents, then $\hat{I}_n^\Phi(J)$ is easy to implement.

Example 2: (In continuation of Example 1).

Considering in the model of Example 1, the parallel system $T = T_1 \vee T_2$ we compute the values of $\hat{I}_n^\Phi(J), J \in \{\{1\},\{12\},\{2\}\}$. We take o value of $n = 10000$ and for some values of p_1, p_2, p_{12}. See the table below:

p_1, p_2, p_{12}	$I^{\Phi}(\{12\})$	$I^{\Phi}(\{1\})$	$I^{\Phi}(\{2\})$	$\hat{I}_n^{\Phi}(\{12\})$	$\hat{I}_n^{\Phi}(\{1\})$	$\hat{I}_n^{\Phi}(\{2\})$
0.5,0.3,0.5	0.697	0.0909	0.2121	0.6963	0.0871	0.2146
0.5,0.3,0.1	0.3431	0.1971	0.4599	0.3441	0.1946	0.4613
0.1,0.2,0.2	0.5094	0.3396	0.1509	0.5045	0.3410	0.1545
0.3,0.9,0.05	0.3283	0.6411	0.0305	0.3326	0.6360	0.0314

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