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A Poisson algebra on the Hida Test functions and a quantization using the Cuntz algebra

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Abstract
In this note, we define one more way of quantization of classical systems. The quantization we consider is an analogue of classical Jordan–Schwinger map which has been known and used for a long time by physicists. The difference, compared to Jordan–Schwinger map, is that we use generators of Cuntz algebra $\mathcal{O}_\infty$ (i.e. countable family of mutually orthogonal partial isometries of separable Hilbert space) as a "building blocks" instead of creation–annihilation operators. The resulting scheme satisfies properties similar to Van Hove prequantization, i.e. exact conservation of Lie brackets and linearity.

1 Introduction

In this note, we define one more way of quantization (see review [1] and references therein) of classical systems. The quantization we consider is an analogue of classical Jordan–Schwinger map which has been known and used for a long time by physicists ([2]). The difference, compared to Jordan–Schwinger map, is that we use generators of Cuntz algebra $\mathcal{O}_\infty$ (i.e. countable family of mutually orthogonal partial isometries of separable Hilbert space) as a "building blocks" instead of creation–annihilation oper-

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ators. The resulting scheme satisfies properties similar to Van Hove prequantization, i.e. exact conservation of Lie brackets and linearity. The second result of the paper is a construction of representation of Heisenberg algebra through Cuntz generators (Remark 24). Other way of construction of canonical commutation relations through isometries has been presented in paper [3]. The difference is that our construction is through an explicit formula while they construct the operators through recursive process. Furthermore, their iterative process results in polynomials of Cuntz generators of arbitrarily high degree while in our case we have quadratic dependence upon Cuntz generators.

The theory of representations of the algebra $\mathcal{O}_\infty$ (see [4] and references therein) seems to be much richer than the theory of representations of canonical commutation relations. In particular, there is no analogue of Stone–von Neumann theorem and classification of classes of irreducible representations is connected with completely different areas such as the theory of modular classes [5] and wavelet theory [6]. Furthermore, as shown in [7] a classification of all irreducible representations is in a certain sense impossible.

The article hence reaches out to several areas such as representations of the Cuntz algebras, Van Hove prequantization, canonical commutation relations, infinite-dimensional Lie algebras and stochastic analysis. A variant of the Jordan–Schwinger map is used to connect operators on the Cuntz algebra to Poissonian manifolds. In that case, the Poisson brackets are "mapped" to the commutator. For a suitable choice of operators, the Heisenberg algebra in finite and infinite dimensions is obtained, based on Cuntz algebras. Since representations of $\mathcal{O}_\infty$ show more variety in representations [4], new quantum systems could be identified. An algebraic analogue of Jordan–Schwinger map first described in the paper [8] has been used there to construct explicitly representations of finite- and infinite-dimensional algebras and derive spectral theorem for the class of self-adjoint (in certain sense) operators in locally convex spaces. An important feature of the transformation is a transfer of the operator defined on sufficiently arbitrary locally convex topological vector space to the operator on Hilbert space. This allows reduce study of , for instance, spectral properties of operators on topological vector spaces to the corresponding theory of operators in Hilbert spaces. Further applications of algebraic analogue of Jordan–Schwinger map to the operator algebras, i.e. Leavitt path algebras, spectral theory and theory of representations, will be considered in the forthcoming works of authors. The hope of the authors is that these ideas could be connected to the quantization theory and result in new insights. The main results of this article are the quantization of both finite-dimensional and infinite-dimensional system via Cuntz algebras. For this purpose, we use the algebraic analogue of the Jordan–Schwinger map. The resulting objects are operators on the Cuntz algebra $\mathcal{O}_\infty$. For the infinite-dimensional case, a Poisson structure is constructed in white noise analysis. Infinite-dimensional Heisenberg algebras are already studied in white noise analysis, see, for example, [9, 10] and the references therein. The infinite-dimensional case gives rise to an infinite-dimensional Heisenberg Lie group, which has various applications in non-commutative analysis, e.g. [11], and quantum physics, e.g. [12].

2 Quantization of finite-dimensional systems via Cuntz algebras

Let $(C^\infty(M), \{\cdot,\cdot\})$ be a Poisson manifold with Poisson brackets $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, H an auxiliary separable Hilbert space and $\{T_i\}_{i=1}^\infty : H \to H$ generators of Cuntz algebra $\mathcal{O}_\infty$, i.e. mutually orthogonal isometries of H [6, 13, 14]. We can assume without loss of generality that

$$\sum_{k=1}^\infty T_k T_k^* = Id.$$

For $h \in C^\infty(M)$, we define $Q, R \in \mathcal{L}(C^\infty(M), \mathcal{O}_\infty)$ as follows:

$$Q(h) := \sum_{i,j=1}^\infty < \{h, e_j\}, f_i >_{C^\infty(M),(C^\infty(M))^*} T_i T_j^*,$$

$$R(h) := \sum_{i,j=1}^\infty < h e_j, f_i >_{C^\infty(M),(C^\infty(M))^*} T_i T_j^*,$$

where $\{e_i\}_{i=1}^\infty, \{f_i\}_{i=1}^\infty$ is a biorthogonal system in $C^\infty(M)$ (with some fixed dual $(C^\infty(M))^*$). Then we have

Lemma 21 *Let $f, g \in C^\infty(M)$, then*

$$[Q(f), Q(g)] = Q(\{f, g\}), \tag{1}$$
$$[Q(f), R(g)] = R(\{f, g\}), \tag{2}$$
$$Q(g)R(f) + Q(f)R(g) = Q(fg), \tag{3}$$
$$R(f)R(g) = R(fg). \tag{4}$$

Proof It immediately follows from commutation properties of operators $\{T_i, T_j^*\}_{i,j=1}^\infty$ and Poisson brackets properties. □

Definition 22 Define the quantization $\widehat{Q} := R - 2iQ$, with $\widehat{Q} \in \mathcal{L}(C^\infty(M), \mathcal{O}_\infty)$.

Theorem 23 *$\widehat{Q} \in \mathcal{L}(C^\infty(M), \mathcal{O}_\infty)$ satisfies*

$$\widehat{Q}(1) = Id, \tag{5}$$
$$[\widehat{Q}(f), \widehat{Q}(g)] = -2i\widehat{Q}(\{f, g\}), \tag{6}$$
$$[\widehat{Q}(q_k), \widehat{Q}(q_j)] = [\widehat{Q}(p_k), \widehat{Q}(p_j)] = 0, \tag{7}$$
$$[\widehat{Q}(q_k), \widehat{Q}(p_j)] = -2i\delta_{kj} Id, k, j = 1, \ldots, \dim M. \tag{8}$$

Furthermore, if $\phi : \mathbb{R} \to \mathbb{R}$ is an analytic function then

$$\Re\widehat{Q}(\phi(f)) = \phi(\Re\widehat{Q}(f)). \tag{9}$$

Proof Property (5) follows from definition of Q and R, commutation relation (6) is a consequence of Lemma 21, property (8) immediately follows from (6) and Poisson brackets properties. At last, analogue of von Neumann rule is enough to show when $\phi(x) = x^n$ is a monomial. Now the result follows by induction w.r.t. n (applying properties (3) and (4)). □

Remark 24 Mapping Q itself satisfies property (6), but we have that $Q(1) = 0$. Nevertheless, working separately with Q and P allows us to get representation of canonical commutation relationsas following example shows. Let $M = \mathbb{R}^{2n}$ with the standard Poisson brackets, $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis in $L^2(\mathbb{R}^{2n}, d\mu)$, $f_i = e_i, i \in \mathbb{N}$, μ–standard Gaussian measure

$$d\mu = e^{-\sum_{i=1}^{n} p_i^2 + q_i^2} \prod_i dp_i dq_i,$$

and as the duality, we take scalar product in $L^2(\mathbb{R}^{2n}, d\mu)$. Then, as in the previous example,

$$\begin{aligned} Q_{2i} &:= Q(q_i), Q_{2i+1} := Q(p_i), \\ P_{2i} &:= R(p_i), P_{2i+1} := R(q_i), \\ [Q_i, Q_j] = [P_i, P_j] = 0, [Q_i, P_j] &= (-1)^i \delta_{ij} 1, i, j = 1, \ldots, 2n. \end{aligned}$$

Furthermore, by integration by parts, we can deduce that

$$P_i = (-1)^i (Q_i + Q_i^*), i = 1, \ldots, 2n.$$

Therefore, we can conclude that

$$[Q_i, Q_j] = [Q_i^*, Q_j^*] = 0, [Q_i, Q_j^*] = \delta_{ij} 1, i, j = 1, \ldots, 2n,$$

and formulas (3) and (4) allow us to calculate $Q(f)$, $R(f)$ for arbitrary polynomial $f = f(q, p)$ as a polynomial of operators $Q_i, Q_i^*, i = 1, \ldots, 2n$.

Remark 25 Notice that operators $P_k = T_k T_k^*, k \in \mathbb{N}$ are mutually orthogonal projections. Consequently, we have representation of H as a direct sum

$$H = \oplus_{k=1}^{\infty} H_k, H_k := P_k(H).$$

Let us show that operators $Q(h)$, $R(h)$ are bounded on each $H_k, k \in \mathbb{N}$ under some natural assumptions about h. We will consider only the operator $Q(h)$. The case of $R(h)$ is similar. First, let us notice that

$$Q(h) T_k T_k^* \psi = \sum_{i=1}^{\infty} \langle \{h, e_k\}, f_i \rangle T_i T_k^* \psi.$$

Consequently, by mutual orthogonality of isometries $\{T_l\}_{l=1}^{\infty}$ we can deduce that

$$||Q(h)T_kT_k^*\psi||_H^2 = ||T_k^*\psi||_H^2 \sum_{l=1}^{\infty}\langle\{h, e_k\}, f_l\rangle^2$$
$$= ||T_kT_k^*\psi||_H^2 \sum_{l=1}^{\infty}\langle\{h, e_k\}, f_l\rangle^2,$$

and, therefore,

$$||Q(h)||_{H_k}^2 \leq \sum_{l=1}^{\infty}\langle\{h, e_k\}, f_l\rangle^2.$$

Thus, if we assume that for any $k \in \mathbb{N}$

$$\sum_{l=1}^{\infty}\langle\{h, e_k\}, f_l\rangle^2 < \infty$$

we have that $Q(h)$ has dense in H domain of definition which corresponds to the finite linear combinations of elements of the subspaces H_k, for $k \in \mathbb{N}$.

3 An infinite-dimensional extension via white noise calculus

Starting point of the white noise distribution theory is the Gel'fand triple

$$S \subset L^2(\mathbb{R}, dt) \subset S^*,$$

where S is the space of Schwartz test functions over $\mathbb{R}$ densely embedded in the Hilbert space of square integrable functions with respect to the Lebesgue measure $L^2(\mathbb{R}, dt)$ and S^* the space of tempered distributions, see., for example, [15] for a construction. Via the Bochner–Minlos–Sazonov theorem, see, for example, [16], we obtain the white noise measure μ on S^* by its Fourier transform

$$\int_{S^*} \exp(i\langle x, \xi\rangle)\, d\mu(x) = \exp(-\frac{1}{2}|\xi|_0^2), \quad \xi \in S,$$

where $|.|_0$ denotes the Hilbertian norm on $L^2(\mathbb{R}, dt)$. The topology on S is induced by a positive self-adjoint operator A on the space of real-valued functions $H := L^2(\mathbb{R}, dt)$ with $\inf \sigma(A) > 1$ and Hilbert–Schmidt inverse A^{-1}. Note that the complexification $S_{\mathbb{C}}$ are equipped with the norms $|\xi|_p := |A^p\xi|_0$ for $p \in \mathbb{R}$. We denote $H_{\mathbb{C}} := L^2(\mathbb{R}, \mathbb{C}, dt)$; furthermore,

$$S_{\mathbb{C},p} := \left\{\xi \in S_{\mathbb{C}} |\ |\xi|_p < \infty\right\}$$

and

$$S_p^* := \left\{ \xi \in S^* | \ |\xi|_p < \infty \right\},$$

for $p \in \mathbb{R}$ resp.

Now we consider the following Gel'fand triple of Hida test functions and Hida distributions.

$$(S) \subset (L^2) := L^2(S^*, \mu) \subset (S)^*.$$

By the Wiener–Ito chaos decomposition theorem, see, for example, [10], [17] or [18], we have the following unitary isomorphism between (L^2) and the Boson Fock space $\Gamma(H_{\mathbb{C}})$:

$$(L^2) \ni \Phi(x) = \sum_{n=0}^{\infty} \left\langle : x^{\otimes n} :, f_n \right\rangle \leftrightarrow (f_n) \sim \Phi \in \Gamma(H_{\mathbb{C}}),$$
$$f_n \in L^2(\mathbb{R}, dt)_{\mathbb{C}}^{\hat{\otimes} n},$$

where $: x^{\otimes n} :$ denotes the Wick ordering of $x^{\otimes n}$ and $\hat{\otimes} n$ denotes the symmetric tensor product of order n. Moreover, the (L^2) norm of $\Phi \in (L^2)$ is given by

$$\|\Phi\|_0^2 = \sum_{n=0}^{\infty} n! \, |f_n|_0^2 .$$

We denote by $\langle\langle ., . \rangle\rangle$ the canonical $\mathbb{C}$ bilinear form on $(S)^* \times (S)$. For each $\Phi \in (S)^*$, there exists a unique sequence $(F_n)_{n=0}^{\infty}$, $F_n \in (S_{\mathbb{C}}^{\hat{\otimes} n})^*$ such that

$$\langle\langle \Phi, \varphi \rangle\rangle = \sum_{n=0}^{\infty} n! \, \langle F_n, f_n \rangle , \quad (f_n) \sim \varphi \in (S). \tag{10}$$

Thus, we have, see, for example, [10], [17] or [18]: $(S) \ni \Phi \sim (f_n)$, if and only if for all $p \in \mathbb{R}$ we have

$$\|\Phi\|_p := \left(\sum_{n=0}^{\infty} (n!) \, |f_n|_p^2 \right)^{\frac{1}{2}} < \infty.$$

Moreover, for its dual space we obtain $(S)^* \ni \Phi \sim (F_n)$, if and only if there exists a $p \in \mathbb{R}$ such that

$$\|\Phi\|_p := \left(\sum_{n=0}^{\infty} (n!) \, |F_n|_p^2 \right)^{\frac{1}{2}} < \infty.$$

For $p \in \mathbb{R}$, we define

$$(S)_p := \left\{ \varphi \in (L^2) : \ \|\varphi\|_{p,\beta} < \infty \right\}$$

and

$$(S)_{-p} := \left\{ \varphi \in (S)^*_\beta : \|\varphi\|_{-p,} < \infty \right\}.$$

We then obtain

$$(S) := \operatorname{proj}\lim_{p \to \infty} (S)_p$$

and

$$(S)^* = \operatorname{ind}\lim_{p \to -\infty} (S)_p.$$

Moreover, (S) is a nuclear (F)-space.

The exponential vector or Wick ordered exponential is defined by

$$\Phi_\xi(x) := \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle : x^{\otimes n} :, \xi^{\otimes n} \right\rangle, \tag{11}$$

where $\xi \in S_{\mathbb{C}}$ and $x \in S^*$.

For $y \in S^*_{\mathbb{C}}$, we use the same notation and define $\Phi_y \in (S)^*$ by:

$$(S) \ni \psi \sim (f_n)_{n \in \mathbb{N}} : \quad \langle\!\langle \psi, \Phi_y \rangle\!\rangle := \sum_{n=0}^{\infty} \left\langle y^{\otimes n}, f_n \right\rangle.$$

Since $\Phi_\xi \in (S)$, for $\xi \in S_{\mathbb{C}}$, we can define the so called S transform of $\Psi \in (S)^*$ by

$$S(\Psi)(\xi) = \langle\!\langle \Psi, \Phi_\xi \rangle\!\rangle.$$

The S transform can be used to characterize the Hida distributions via a space of ray analytic functions, which is due to the well-known characterization theorem, see, for example, [10, 17–19].

We call $S(\Psi)(0) = \langle\!\langle \Psi, \mathbb{1} \rangle\!\rangle$ the generalized expectation of $\Psi \in (S)^*$.
The Wick product of $\Psi_1 \in (S)^*$ and $\Psi_2 \in (S)^*$ is defined by

$$\Psi_1 \diamond \Psi_2 := S^{-1}(S(\Psi_1) \cdot S(\Psi_2)) \in (S)^*,$$

see, for example, [10, 17, 18].

Naturally we can define a directional derivative on (S) by

$$\partial_u \Phi(x) = \sum_{n=1}^{\infty} n \langle :x^{\otimes n-1}:, \langle f_n, u \rangle \rangle,$$

where $\langle f_n, u \rangle$ denotes the contraction of $f_n \in S(\mathbb{R})_{\mathbb{C}}^{\hat{\otimes} n}$ with respect to $u \in S^*$.

It is known that $\partial_u \in L((S), (S))$, see, for example, [10].

It is shown, see, for example, [10], that ∂_u is indeed a derivation on the space of Hida test functions (S).

In Physics applications, it plays the role of the annihilation operator in the Fock space, while its dual operator is the creation operator, also known as Skorokhod integral, see, for example,[10, 17, 18].

There are several studies on Poisson algebraic structures on the Hida Test function space see, for example, [20–23] and their q-deformation. We will follow this streamline here, but exploit the derivation structure of the derivative.

For this, we work on the triple

$$(S) \subset L^2(S^*(\mathbb{R}, \mathbb{R}^2)) \subset (S)^*.$$

Theorem 31 *Let $\Phi, \Psi \in (S)$ and $K \in \mathcal{L}(L^2(\mathbb{R}, \mathbb{R}))$ a symmetric trace-class operator with eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ and corresponding eigenvectors $e_n \in S(\mathbb{R})$. We define the Poisson bracket of Φ and Ψ by*

$$\{\Phi, \Psi\}_K = \sum_{n=0}^{\infty} \lambda_n (\partial_{q_n} \Phi \partial_{p_n} \Psi - \partial_{p_n} \Phi \partial_{q_n} \Psi),$$

where $q_n = (e_n, 0)$ and $p_n = (0, e_n)$. With this definition $((S), \{,\})$ is an infinite-dimensional Poisson algebra.

Proof For $\Phi, \Psi \in (S)$, we have also the derivative is in (S). However, it is a priori unclear whether the infinite series is still a Hida test function. For this, we show that indeed the Poisson bracket is in all (H^p) spaces. It is enough to show this for the first part. We have for all $p \geq 0$ and $q > 0$:

$$\begin{aligned}
\| \sum_{n=0}^{\infty} \lambda_n (\partial_{q_n} \Phi \partial_{p_n} \Psi \|_p &\leq \sum_{n=0}^{\infty} \| \lambda_n (\partial_{q_n} \Phi \partial_{p_n} \Psi \|_p \\
&\leq C \max_n |e_n|_{-q} \sum_{n=0}^{\infty} |\lambda_n| \|\Phi\|_{p+q} \|\Psi\|_{p+q} \\
&= C \max_n |e_n|_{-q} \|\Phi\|_{p+q} \|\Psi\|_{p+q} \sum_{n=0}^{\infty} |\lambda_n| < \infty.
\end{aligned}$$

Leibniz rule, bilinearity and Jacobi identity follow directly from the gradient structure and the product rule of the derivative. □

Define operators $Q, R \in \mathcal{L}((S), \mathcal{O}_\infty)$ for $\Phi \in (S)$ as follows

$$Q_K(\Phi) := \sum_{i,j=1}^{\infty} \langle\langle \{\Phi, b_i\}_K, b_j \rangle\rangle T_i T_j^*,$$

$$R(\Phi) := \sum_{i,j=1}^{\infty} \langle\langle \Phi \cdot b_j, b_i \rangle\rangle T_i T_j^*, \quad \Phi \in (S),$$

where $\{b_i\}_{i=1}^{\infty} \subset (S)$ is an orthogonal system in (S) extending to $(S)^*$. Moreover, R is well defined since (S) is a Banach algebra, see, for example,[17]. Then we have

Lemma 32 *Let $f, g \in (S)$.*

$$[Q(f), Q(g)] = Q(\{f, g\}_K), \tag{12}$$

$$[Q(f), R(g)] = R(\{f, g\}_K), \tag{13}$$

$$Q(g)R(f) + Q(f)R(g) = Q(fg), \tag{14}$$

$$R(f)R(g) = R(fg). \tag{15}$$

Proof Follows immediately as before. □

Definition 33 Define the quantization $\widehat{Q} \in \mathcal{L}((S), \mathcal{O}_\infty)$ as $\widehat{Q} := R - 2iQ$.

Theorem 34 *$\widehat{Q} \in \mathcal{L}((S), \mathcal{O}_\infty)$ satisfies for $j, k \in \mathbb{N}$*

$$\widehat{Q}(1) = Id, \tag{16}$$

$$[\widehat{Q}(f), \widehat{Q}(g)] = -2i\widehat{Q}(\{f, g\}_K), \tag{17}$$

$$[\widehat{Q}(q_k), \widehat{Q}(q_j)] = [\widehat{Q}(p_k), \widehat{Q}(p_j)] = 0, \tag{18}$$

$$[\widehat{Q}(q_k), \widehat{Q}(p_j)] = -2i\delta_{kj} Id, k, j \in \mathbb{N}. \tag{19}$$

Furthermore, if $\phi : \mathbb{R} \to \mathbb{R}$ is an analytic function then

$$\Re\widehat{Q}(\phi(f)) = \phi(\Re\widehat{Q}(f)) \tag{20}$$

Proof Similar to the proof of Theorem 23. □

Remark 35 As in the finite-dimensional case, we can get a representation of canonical commutation relations also in the infinite-dimensional setting.

Let $M = (S)$ with the Poisson bracket defined as before of a symmetric trace-class operator K. Let μ be the two-dimensional white noise measure on $S^*(\mathbb{R}, \mathbb{R}^2)$ given via:

$$\int_{S*(\mathbb{R},\mathbb{R}^2)} \exp(i\langle(x_q, x_p), (\xi_q, \xi_q)\rangle)\, d\mu((x_q, x_p))$$
$$= \exp\left(-\frac{1}{2}(|\xi_q|_0^2 + |\xi_p|_0^2\right), \quad \xi_q, \xi_p \in S$$

Let $q_n = (e_n, 0)$ and $p_n = (0, e_n)$, where $(e_n)_n \subset S(\mathbb{R})$ are the eigenvectors of K. Then,

$$\begin{aligned} Q_{2i} &:= Q(\langle \cdot q_i \rangle),\, Q_{2i+1} := Q(\langle \cdot p_i \rangle), \\ P_{2i} &:= R(\langle \cdot p_i \rangle),\, P_{2i+1} := R(\langle \cdot q_i \rangle), \\ [Q_i, Q_j] &= [P_i, P_j] = 0, [Q_i, P_j] = (-1)^i \delta_{ij} Id. \end{aligned}$$

Furthermore, by integration by parts, we can deduce that

$$P_i = (-1)^i (Q_i + Q_i^*), i = 1, \ldots .$$

Therefore, we can conclude that

$$[Q_i, Q_j] = [Q_i^*, Q_j^*] = 0, [Q_i, Q_j^*] = \delta_{ij} Id,$$

and formulas (14) and (15) allow us to calculate $Q(f)$, $R(f)$ for arbitrary polynomial $f = f(q, p)$ as a polynomial of operators Q_i, Q_i^*, for $i \in \mathbb{N}$.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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