DEPARTAMENTO DE MATEMÁTICA APLICADA

Relatório Técnico

RT-MAP- 9905

The Method of Stationary Phase with Minimal Smoothness

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Agosto de 1999

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The method of stationary phase treats the behavior of oscillatory integrals when the frequency of oscillations tends to infinity. If $f, g : [a, b] \to \mathbb{R}$ are sufficiently smooth and the derivative $f'(x) \neq 0$ in $a \leq x \leq b$ with the exception of one point x_0 in $a < x_0 < b$ where $f'(x_0) = 0$, $f''(x_0) \neq 0$, then as $\omega \to +\infty$

$$\int_a^b e^{i\omega f} g \approx \int_{-\infty}^{\infty} g(x_0) exp\left(i\omega\left(f(x_0) + \frac{1}{2}f''(x_0)y^2\right)\right) dy$$

or, more explicitly: as $\omega \to +\infty$,

$$\begin{aligned} \sqrt{\omega} e^{-i\omega f(x_0)} \int_a^b exp\left(i\omega f(x)\right) g(x) dx \; &\to \int_{-\infty}^{\infty} g(x_0) e^{i f''(x_0) \frac{t^2}{2}} dt \\ &= g(x_0) \sqrt{\frac{2\pi}{|f''(x_0)|}} e^{\pm \frac{i\pi}{4}} \end{aligned} \tag{1}$$

where $\pm = \text{sgn } f''(x_0)$. In the first approximation, only the behavior near x_0 (where the phase f is "stationary") is important, provided f, g are smooth.

How much smoothness is sufficient? Or necessary?

Erdélyi [1] pag. 52, referring to the one-dimensional method of stationary phase, says "Perhaps the best available theorem is one given by Watson (1920)". The author has found no reason in later literature to modify that judgement.

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G.N. Watson [3] treat a more general case - as we will also, in Theorems 2 and 3 - but for the case above:

If f is $\mathbf{C}^2$ and

$$x \mapsto G(x) = g(x)\frac{f''(x_0)(x-x_0)}{f'(x)}\left(\frac{f(x)-f(x_0)}{\frac{1}{2}f''(x_0)(x-x_0)^2}\right)^{\frac{1}{2}}$$

is of bounded variation (B.V.) in $[a,b]$, the limit (1) above holds with $(g(x_0^+)+g(x_0^-)/2)$ in the place of $g(x_0)$. (In fact, it is enough that f is $\mathbf{C}^1$ and f' need be differentiable only at x_0.) Note, if f is a nontrivial quadratic polynomial, then $G = g$.

We will prove (special case of Theorem 2) the limit (1) also holds if G is Hölder continuous with exponent $\alpha > \frac{1}{2}$ (of class $\mathbf{C}^\alpha$) - for example, if g is $\mathbf{C}^\alpha$ and f is $\mathbf{C}^{2,\alpha}$ or $x \mapsto \dfrac{f'(x)}{(x-x_0)}$ is $\mathbf{C}^\alpha$ - or if G is of class $\mathbf{C}^\alpha$ + B.V., the sum of one of each type. We also show the limit (1) may fail for G of class $\mathbf{C}^\alpha$ with $\alpha \le \frac{1}{2}$. Specifically (Example 1, following Theorem 2) for every α in $0 < \alpha < 1$, there exists a real-valued G_α, locally of class $\mathbf{C}^\alpha$, with $G_\alpha(0) = 0$ and such that, for any $a < b$,

$$\omega^\alpha \int_a^b e^{i\omega x^2} G_\alpha(x)dx$$

is bounded but has no limit when $\omega \to +\infty$. (Neither the real nor the imaginary part has a limit). The method of stationary phase would say, if $a < 0 < b$, that $\sqrt{\omega}\displaystyle\int_a^b e^{i\omega x^2} G_\alpha(x)dx$ tends to zero as $\omega \to +\infty$. In fact it does if $\alpha > \frac{1}{2}$, but it has no limit if $\alpha \le \frac{1}{2}$ and is unbounded if $\alpha < \frac{1}{2}$. (This is true for both the real and imaginary parts.)

We will also treat "Watson's lemma" in a half-space $\{Rez \ge 0\}$, Theorem 3, rather than the usual restriction $\{|argz| \le constant < \frac{\pi}{2}\}$ (for example, [2]). This reduces to the method of stationary phase on the imaginary axis and the argument near the imaginary axis is similar to that case. The corollary to Theorem 3 gives sharp results for higher-order smoothness in the method of stationary phase. (Example 2, 3 and (2+3) prove necessity.)

We begin with a simpler case: no stationary point.

Theorem 1 *Let* $f : [a,b] \to \mathbb{R}$ *be* C^1 *with* $f'(x) \neq 0$ *in* $a \leq x \leq b$ *and suppose* $x \mapsto \dfrac{g(x)}{f'(x)} : [a,b] \to \mathbb{C}$ *is of class* C^α $(0 < \alpha \leq 1)$ *[or B.V.]. Then as* $\omega \to +\infty$,

$$\int_a^b e^{i\omega f} g = O(\omega^{-\alpha}) \quad \left[\textit{or } O(\omega^{-1}), \textit{ respectively}\right].$$

For $0 < \alpha < 1$, *the function* g_α *defined by*

$$g_\alpha(x) = \sum_{k=1}^{\infty} 2^{-k\alpha} \cos(2^k x)$$

is of class C^α *and for any* $a < b$,

$$\omega^\alpha \int_a^b e^{i\omega x} g_\alpha(x) dx$$

is bounded but has no limit as $\omega \to +\infty$. *If* $a + b \neq 0$, *neither the real or the imaginary part has a limit. Thus the estimate of the first part cannot generally be improved.*

Remark 1 *As an immediate consequence,* $g_\alpha|_{[a,b]}$ *is not of bounded variation; in fact, it is not of class* C^β + *B.V. for any* $\beta > \alpha$. *(Otherwise* $\omega^\alpha \int_a^b e^{i\omega x} g_\alpha(x) dx = O(\omega^{\alpha-\beta} + \omega^{\alpha-1}) \to 0$ *as* $\omega \to +\infty$, *which is false.) This holds for any* $a < b$; *the singularity of* g_α *is not localized.*

If g *is Lipschitzian, Hölder continuous with exponent 1, it is locally B.V. If* g *is a nontrivial step function (such as* $g \equiv 1$*), it is easy to see* $\omega \int_a^b e^{i\omega x} g(x) dx$ *is bounded but has no limit as* $\omega \to +\infty$.

Proof: $\displaystyle\int_a^b e^{i\omega f} g = \int_{f(a)}^{f(b)} e^{i\omega x} \tilde{g}(x) dx$, where $\tilde{g}(f(x)) = \dfrac{g(x)}{f'(x)}$, so $\tilde{g}$ is equally of class C^α or B.V.; it suffices to treat the case $f(x) = x$.

If g is of bounded variation,

$$i\omega \int_a^b e^{i\omega x} g(x) dx = \left[e^{i\omega x} g(x)\right]_{a+}^{b-} - \int_{a+}^{b-} e^{i\omega x} dg(x)$$

is bounded as $\omega \to +\infty$.

If g is C^α and $h = \frac{\pi}{\omega}$ so $e^{i\omega h} = -1$,

$$2\int_a^b e^{i\omega x} g(x)dx = \left(\int_a^{a+h} + \int_{b-h}^b\right) e^{i\omega x} g(x) + \int_a^{b-h} e^{i\omega x}\left(g(x) - g(x+h)\right)dx$$

which is $O(h + h^\alpha) = O(\omega^{-\alpha})$ as $\omega \to +\infty$.

If $0 < |y| \le \pi$, define the integer $p \ge 0$ by $2^p|y| \le \pi < 2^{p+1}|y|$; Then

$$|g_\alpha(x+y) - g_\alpha(x-y)| = \left|-2 \sum_{n=2^k \ge 2} n^{-\alpha} \sin nx \sin ny\right| \le$$

$$\le 2\left(\sum_1^p 2^{k(1-\alpha)}|y| + \sum_{p+1}^\infty 2^{-k\alpha}\right) \le L_\alpha |y|^\alpha$$

for a constant L_α depending only on α, $0 < \alpha < 1$. Since g_α is 2π–periodic, the inequality holds for all x, y.

On the sequence $\omega = 2^k$ or $2^k + m$ (fixed $m \ne 0$) or $\omega = 3 \cdot 2^k$, we have for large k ($2^k > 4|m|$)

$$|2^j - \omega| \ge \frac{1}{4}\max\{2^j, \omega\}$$

for every integer $j \ne k$ (and also $j = k$, if $\omega = 3 \cdot 2^k$). This implies, as $\omega \to +\infty$ on these sequences,

$$\omega^\alpha \int_a^b e^{i\omega x} g_\alpha(x)dx - \left\{\begin{array}{ll} \dfrac{(b-a)}{2} & \text{, if } \omega = 2^k \\ exp\left(\dfrac{im}{2}(a+b)\right)\dfrac{1}{m}\sin\dfrac{m}{2}(b-a) & \text{, if } \omega = 2^k + m \\ 0 & \text{, if } \omega = 3 \cdot 2^k \end{array}\right\}$$

$$= O(\omega^{\alpha-1})$$

and completes the proof.

■

Watson [3] treated general stationary points where

$$f(x) = f(x_0) + L(x - x_0)^p + o(x - x_0)^p, \text{ as } x \to x_0^+ \ (L \ne 0, p > 1)$$

with analogous behavior as $x \to x_0$- (but perhaps different L, p), also allowing an algebraic singularity in g at the point x_0. Clearly it is enough to consider only one side ($x > x_0$) and to suppose $x_0 = 0$, $f(x_0) = 0$.

Theorem 2 *Suppose $f : [0,a] \to \mathbf{R}$ is continuous, $f(0) = 0$, f is continuously differentiable with $f'(x) \neq 0$ in $0 < x \leq a$ and for some $L \neq 0$, $p > 0$, as $x \to 0^+$*

$$\frac{f(x)}{x^p} \to L \text{ and } \frac{f'(x)}{x^{p-1}} \to pL.$$

(The second limit implies the first.)

Further assume $q \in \mathbf{C}$ with $p > \mathrm{Re}\ q + 1 > 0$, $g : (0,a] \to \mathbf{C}$ has a right-hand limit $g(0^+)$ at 0, and let $r = \frac{(q+1)}{p}$, so $0 < \mathrm{Re}\ r < 1$. Let

$$G(x) = g(x)\frac{pLx^{p-1}}{f'(x)}\left(\frac{f(x)}{Lx^p}\right)^{1-r}, \ 0 < x \leq a,$$

(note $G(0^+) = g(0^+)$) and assume G is of class B.V. + $\mathbf{C}^\alpha$ with $\alpha > \mathrm{Re}\ r$. (A sufficient condition: both g and $x \mapsto \frac{f'(x)}{x^{p-1}}$ are $\mathbf{C}^\alpha$ or both are B.V.). Then as $\omega \to +\infty$,

$$\omega^r \int_0^a x^q g(x) e^{i\omega f(x)} dx \to \quad g(0^+) \int_0^\infty t^q e^{iLt^p} dt$$

$$= e^{\pm ir\frac{\pi}{2}} \frac{g(0^+)\Gamma(r)}{p|L|^r}$$

where $\pm = \mathrm{sgn}\ L$. (For $\int_0^a x^q g(x) \sin(\omega f(x)) dx$, we may allow $-1 < \mathrm{Re}\ r < 1$ by the same argument.)

Remark 2 *Away from 0, $G(x) = \left(\frac{g(x)}{f'(x)}\right) \times$ (nonzero $\mathbf{C}^1$ function), so this part of the integral is $O(\omega^{-\alpha})$ comparatively small, by Theorem 1.*

If $f(x) = Lx^p$, $g(x) = 1$, $\omega^r \int_0^a e^{i\omega Lx^p} x^q dx = \frac{1}{p}\int_0^{\omega a^p} t^{r-1} e^{iLt} dt$ has a finite limit as $\omega \to +\infty$ only when $0 < \mathrm{Re}\ r < 1$.

Example 1 bellow shows the conclusion may fail for real r and Hölder exponent $\alpha = r$ when $p \geq r$ (for example, $p \geq 1$).

Watson's theorem [3] assumes G is B.V. and assumes more smoothness for f.

The usual case of stationary phase has $p = 2$, $q = 0$, $r = \frac{1}{2}$. *Suppose* $f(0) = 0$, $f'(0) = 0$, $f''(0) = 2L \neq 0$ *and* $f'(x) \neq 0$ *in* $0 < x \leq a$. *Then if*

$$x \mapsto g(x)\frac{f''(0)x}{f'(x)}\left(\frac{2f(x)}{f''(0)x^2}\right)^{\frac{1}{2}}$$

is of class $C^\alpha + B.V.$ *with* $\alpha > \frac{1}{2}$, *as* $\omega \to +\infty$

$$\omega^{\frac{1}{2}}\int_0^a ge^{i\omega f} \to g(0^+)\int_0^\infty e^{if''(0)\frac{t^2}{2}}dt.$$

(If the stationary point 0 is an interior point, there is a similar contribution from the other side, $x < 0$. *This is the case mentioned in the introduction.)*

Proof: Suppose $\left|\frac{f'(x)}{x^{p-1}} - pL\right| \leq \varepsilon$ when $0 < x \leq \delta$; then in $0 < x \leq \delta$, $|f(x) - Lx^p| \leq \int_0^x \varepsilon t^{p-1}dt = \varepsilon\frac{x^p}{p}$. Thus $\frac{f'(x)}{x^{p-1}} \to pL$ as $x \to 0^+$ (with $f(0) = 0$) implies $\frac{f(x)}{x^p} \to L$ as $x \to 0^+$. If $\frac{f'(x)}{x^{p-1}} = \varphi(x)$, we have $\frac{f(x)}{x^p} = \int_0^1 \varphi(\theta x)\theta^{p-1}d\theta$, which is C^α or B.V. when φ is C^α or B.V., respectively.

Define $h(x) \geq 0$ for $0 \leq x \leq a$ by $f(x) = Lh(x)^p$; $h(0) = 0$ and $\frac{h(x)}{x} \to 1$ as $x \to 0^+$. For $x > 0$, $h(x) > 0$ and h is differentiable with

$$h'(x) = \left(\frac{f'(x)}{pLx^{p-1}}\right)\left(\frac{f(x)}{Lx^p}\right)^{\frac{1}{p}-1}$$

so $h'(x) \to 1 = h'(0)$ as $x \to 0^+$, $h'(x) \geq$ *constant* > 0 in $0 \leq x \leq a$.

We have

$$\int_0^a x^q g(x)e^{i\omega f(x)}dx = \int_0^{h(a)} y^q\tilde{g}(y)e^{i\omega Ly^p}dy$$

where $\tilde{g}(h(x)) = G(x) = \frac{g(x)x^q}{h'(x)h(x)^q}$, and $\tilde{g}$ is equally C^α + B.V.. Thus it is sufficient to treat the case where $f(x) = Lx^p$, $h(x) = x$, $g = G = \tilde{g}$, and in fact we may suppose $g(0^+) = 0$:

$$\omega^r\int_0^a x^q g(x)e^{i\omega Lx^p}dx - g(0^+)\int_0^\infty y^q e^{iLy^p}dy =$$

$$= \omega^r\int_0^a x^q(g(x) - g(0^+))e^{i\omega Lx^p}dx + O(\omega^{r-1}).$$

We show: if g is C^α + B.V., $\alpha > \rho = Re\ r$ $(0 < \rho < 1)$, with $g(0^+) = 0$, then as $\omega \to +\infty$,

$$\int_0^a x^q g(x) e^{i\omega L x^p} dx = \frac{1}{p} \int_0^{a^p} g(x^{\frac{1}{p}}) x^{r-1} e^{i\omega L x} dx = o(\omega^{-\rho}).$$

Let $F_\omega(x) = \int_x^\infty t^{r-1} e^{i\omega L t} dt = \omega^{-r} F_1(\omega x)$; it is easy to see $|F_1(y)| \le C \min\{y^{\rho-1}, 1\}$, $\rho = Re\ r$, for $y > 0$ and a constant C. Thus if g is B.V. with $g(0^+) = 0$, $\omega a^p > 1$,

$$\left| \omega^r \int_0^{a^p} g(x^{\frac{1}{p}}) x^{r-1} e^{i\omega L x} dx \right| = \left| -F_1(\omega a^p) g(a) + \int_{0^+}^{a^p} F_1(\omega x) d_x g(x^{\frac{1}{p}}) \right|$$

$$\le C|g(a)|(\omega a^p)^{\rho-1} + C \text{Var}\ g(0,\delta] + \text{Var}\ g(\delta, a](\omega \delta^p)^{\rho-1},$$

for any δ in $0 < \delta < a$. Since Var $g(0,\delta] \to 0$ as $\delta \to 0^+$, choose δ small and then ω large to show the limit is zero.

Now suppose g is C^α, $\alpha > \rho = Re\ r$, $g(0) = 0$; say $|g(x) - g(y)| \le B|x-y|^\alpha$ for $0 \le x, y \le a$. Let $h = \frac{\pi}{\omega |L|}$ so $e^{i\omega L h} = -1$; then if $0 < h < a^{\frac{p}{2}}$,

$$2\int_0^{a^p} g(x^{\frac{1}{p}}) x^{r-1} e^{i\omega L x} dx = \left(\int_0^{2h} + \int_0^h + \int_{a^p - h}^{a^p} \right) g(x^{\frac{1}{p}}) x^{r-1} e^{i\omega L x}$$

$$+ \int_h^{a^p - h} e^{i\omega L x} \left(g(x^{\frac{1}{p}}) x^{r-1} - g((x+h)^{\frac{1}{p}})(x+h)^{r-1} \right)$$

so

$$2\left| \int_0^{a^p} g(x^{\frac{1}{p}}) x^{r-1} e^{i\omega L x} dx \right| \le \left(\int_0^{2h} + \int_0^h + \int_{a^p - h}^{a^p} \right) B x^{\rho + \frac{\alpha}{p} - 1} dx$$

$$+ \int_h^{a^p - h} B \left(x^{\rho-1} \left(\frac{1}{p} h \left(x^{\frac{1}{p}-1} + (x+h)^{\frac{1}{p}-1} \right) \right)^\alpha + x^{\frac{\alpha}{p}} h (1-p) x^{\rho-2} \right)$$

$$= O(h + h^\alpha + h^{\rho + \frac{\alpha}{p}})$$

plus $O(h^\alpha |\log h|)$ if $\alpha = \rho + \frac{\alpha}{p}$, plus $O(h|\log h|)$ if $\rho + \frac{\alpha}{p} = 1$. In any case, if $\alpha > \rho = Re\ r$, $\omega^r \int_0^{a^p} g(x^{\frac{1}{p}}) x^{r-1} e^{i\omega L x} dx \to 0$ as $\omega \to +\infty$, and the proof of Theorem 2 is complete.

■

Example 1 *Suppose* $0 < \alpha < 1$, $0 < r < 1$, $p > 0$, g_α *is the function defined in Theorem 1, and let*

$$G_\alpha(x) = px^{p(1-r)}(g_\alpha(x^p) - g_\alpha(0)), \quad \text{for } x \geq 0.$$

Then G_α *is locally* C^α *if* $p \geq 1$ *or generally* $C^{\alpha \wedge p(\alpha+1-r)}$ $(p > 0)$ *with* $G_\alpha(0) = 0$ *and, if* $q = pr - 1$, $a > 0$,

$$\omega^\alpha \int_0^a x^q G_\alpha(x) e^{i\omega x^p} dx = \omega^\alpha \int_0^{a^p} (g_\alpha(x) - g_\alpha(0)) e^{i\omega x} dx$$

$$= \omega^\alpha \int_0^{a^p} g_\alpha(x) e^{i\omega x} dx + O(\omega^{\alpha-1})$$

so neither the real nor the imaginary part has a limit as $\omega \to +\infty$. *In particular, if* $p \geq r$, G_r *is locally* C^r *but* $\omega^r \int_0^a x^q G_r(x) e^{i\omega x^p} dx$ *has no limit as* $\omega \to +\infty$, *so the Hölder condition of Theorem 2 is sharp, at least for* $p \geq r$.

In the usual method of stationary phase, $p = 2$, $q = 0$, $r = \frac{1}{2}$. *If* $G_\alpha(x) = 2|x|(g_\alpha(x^2) - g_\alpha(0))$, G_α *is locally* C^α *with* $G_\alpha(0) = 0$ *and in fact* $|G_\alpha(x)| = O(|x|^{1+2\alpha})$ *as* $x \to 0$, *and for* $a < 0 < b$

$$\omega^\alpha \int_a^b G_\alpha(x) e^{i\omega x^2} dx = \omega^\alpha \left(\int_0^{a^2} + \int_0^{b^2} \right) g_\alpha(x) e^{i\omega x} dx + O(\omega^{\alpha-1})$$

has limit points $\frac{a^2+b^2}{2}$ *(if* $\omega = 2^k \to \infty$*) and 0 (if* $\omega = 3 \cdot 2^k \to \infty$*) and the imaginary part has limit* $\frac{1}{m}\left(\left(\sin\frac{ma^2}{2}\right)^2 + \left(\sin\frac{mb^2}{2}\right)^2\right)$ *on the sequence* $\omega = 2^k + m$ *(fixed* $m \neq 0$*). If* $a < b$ *but* $ab \geq 0$, *we still conclude neither the real nor the imaginary part has a limit as* $\omega \to +\infty$.

Now we will study $\int_0^\infty e^{-zt^p} t^q g(t) dt = \frac{1}{p} \int_0^\infty e^{-zx} x^{r-1} g(x^{\frac{1}{p}}) dx$ as $|z| \to \infty$ in $Re\ z \geq 0$, where $p > 0$, $r = \frac{q+1}{p}$, $0 < Re\ r < 1$. There are possible difficulties near 0, for positive finite x, and near infinity; we concentrate on the first two, assuming g has compact support except in the final remark.

Lemma 1 *Let* $\varphi : \mathbf{R} \to \mathbf{R}$ *be a* C^∞ *"cutoff", having compact support and* $\varphi(t) = 1$ *for all* t *near 0. Then for any* $N > 0$ *and* $r \in \mathbf{C}$ *with* $Re\ r > 0$,

$$\int_0^\infty e^{-tz} t^{r-1} \varphi(t) dt = \Gamma(r) z^{-r} + O(|z|^{-N})$$

as $|z| \to \infty$ with $Re\ z \geq 0$. (We use the principal branch of z^{-r} since $|arg\ z| \leq \frac{\pi}{2}$.) If r is restricted to a compact set in $\{Re\ r > 0\}$, the estimate is uniform in r.

Proof: Choose a positive integer n sufficiently large; integration by parts (for $z > 0$, $r > 0$) shows

$$\int_0^\infty e^{-tz}t^{r-1}\varphi(t)dt - \Gamma(r)z^{-r} = z^{-n}\int_0^\infty e^{-tz}\left(\frac{d}{dt}\right)^n t^{n-1}(\varphi(t)-1).$$

By analytic continuation, this holds for $Re\ z > 0$, $0 < Re\ r < n$, and by continuity, also for $Re\ z \geq 0$, $z \neq 0$, which gives the result, provided $n \geq N$, $n > Re\ z$.

■

Lemma 2 *Let $g : [0, \infty) \to \mathbf{C}$ have compact support and for certain constants $\alpha > 0$, β and γ with $0 \leq \gamma < \beta \leq 1$, suppose*

$$|g(t)| = O(t^\alpha), \quad |g(t+h) - g(t)| = O(h^\beta t^{-\gamma}),$$

for $0 < h \leq t$. (The second condition with $g(0^+) = 0$ implies the first with $\alpha = \beta - \gamma$, so we might suppose $\alpha \geq \beta - \gamma$.) Further assume $s > 0$, $r \in \mathbf{C}$ with $Re\ r + s\alpha > 0$ and $Re\ r + s(\beta - \gamma) > 0$.

Then as $|z| \to \infty$, with $Re\ z \geq 0$,

$$\int_0^\infty e^{-tz}t^{r-1}g(t^s)dt = O(|z|^{-\mu})$$

where $\mu = \min\{Re\ r + \alpha s, 1, \beta, Re\ r + (\beta - \gamma)s\}$, unless $Re\ r + \alpha s = 1 = \mu$ or $Re\ r + (\beta - \gamma)s = \beta = \mu$, when the estimate is $O(|z|^{-\mu}\log|z|)$.

For example, if g is $\mathbf{C}^\alpha$ $(0 < \alpha \leq 1)$ we may take $\beta = \alpha$, $\gamma = 0$, $\mu = \min\{Re\ r + \alpha s, \alpha\}$. If g is $\mathbf{C}^1$ on $\mathbf{R}^+$ with $|\dot{g}(t)| = O(t^{\alpha-1})$ and $g(0^+) = 0$, take $\beta = 1$, $\gamma = 1 - \alpha$ and $\mu = \min\{Re\ r + \alpha s, 1\}$. (In either case the "log" appears in case of equality.)

Proof: Say $g(t) = 0$ for $t \geq C$, $|g(t) \leq Bt^\alpha$, $|g(t+h) - g(t)| \leq Bh^\beta t^{-\gamma}$ for $0 < h \leq t$, and let $\mu_1 = Re\ r + \alpha s$.

For $Re\ z > 0$

$$\left|\int_0^\infty e^{-tz}t^{r-1}g(t^s)dt\right| \leq B\int_0^\infty e^{-tRe\ z}t^{\mu_1 - 1}dt = O((Re\ z)^{-\mu_1}).$$

We use this estimate when $Re\ z \geq |Im\ z|$, so $1 \leq \frac{|z|}{Re\ z} \leq 2$.

If $0 \leq Re\ z < |Im\ z|$, let $\delta = \frac{\pi}{|Im\ z|}$ so $e^{-\delta z} = -e^{-\delta Re\ z}$ and $0 \leq \delta Re\ z \leq \pi$, $\pi \leq \delta|z| \leq 2\pi$. Then

$$(1 + e^{\delta Re\ z}) \int_0^\infty e^{-tz} t^{r-1} g(t^s) dt = \left(e^{\delta Re\ z} \int_0^{2\delta} + \int_0^\delta \right) e^{-tz} t^{r-1} g(t^s) dt$$

$$+ \int_\delta^{c^{\frac{1}{s}}} e^{-tz} (t^{r-1} g(t^s) - (t+\delta)^{r-1} g(t+\delta)^s)$$

so

$$\left| \int_0^\infty e^{-tz} t^{r-1} g(t^s) dt \right| = O(\delta^{\mu_1}) + O(\int_\delta^{c^{\frac{1}{s}}} (\delta t^{\mu_1 - 2} + \delta^\beta t^{\mu_2 - \beta - 1})$$

where $\mu_2 = Re\ r + (\beta - \gamma)s$. It is easy to see for any real λ and $\beta > 0$, as $\delta \to 0^+$,

$$\delta^\beta \int_\delta^* t^{\lambda - \beta - 1} dt = O(\delta^{\min\{\lambda, \beta\}}),$$

unless $\lambda = \beta$ when it is $O(\delta^\beta \log \delta^{-1})$. This gives the result.

■

Theorem 3 (Watson's lemma in a half-space) *Suppose $0 < \alpha \leq 1$, integer $m \geq 0$, $s > 0$, $Re\ r > 0$ and $g : [0, \infty) \to \mathbf{C}$ is $\mathbf{C}^{m,\alpha}$ (i.e. m^{th}–derivative of class $\mathbf{C}^\alpha$) with compact support. Then as $|z| \to \infty$ with $Re\ z \geq 0$,*

$$\int_0^\infty e^{-tz} t^{r-1} g(t^s) dt = \sum_{j=0}^m \frac{g^{(j)}(0)}{j!} \frac{\Gamma(r + js)}{z^{r+js}} + O(|z|^{-\mu})$$

where $\mu = \min\{m + \alpha, Re\ r + (m + \alpha)s\}$, except that the final term is $O(|z|^{-\mu} \log |z|)$ in the case $m + \alpha = Re\ r + (m + \alpha)s$ or $Re\ r + (m + \alpha)s$ is an integer $\leq m$.

Remark 3 *Examples 2,3 bellow show μ is sharp for $0 < \alpha < 1$; it is not known if "$\log |z|$" is necessary.*

Proof: Let φ be any $\mathbf{C}^\infty$ cutoff, i.e. $\varphi \equiv 1$ near 0 but φ has compact support, and let $G(t) = g(t) - \varphi(t)\sum_0^m g^{(j)}(0)\frac{t^j}{j!}$. G is $\mathbf{C}^{m,\alpha}$ with compact support and $G^{(j)}(t) = O(t^{m+\alpha-j})$ as $t \to 0^+$ for $0 \le j \le m$. By lemma 1

$$\int_0^\infty e^{-tz}t^{r-1}\varphi(t^s)\sum_0^m g^{(j)}(0)\frac{t^{js}}{j!} = \sum_0^m g^{(j)}(0)\frac{\Gamma(r+js)}{j!z^{j+rs}} + O(|z|^{-N})$$

for any N, so it suffices to prove the result when all $g^{(j)}(0) = 0$, $g = G$, as we now assume.

Let $Re\ r + s(m+\alpha) = k^* + \alpha^*$ (integer $k^* \ge 0$, $0 < \alpha^* \le 1$).

First suppose $k^* \ge m$; for appropriate constants C_j^m

$$\left(\frac{d}{dt}\right)^m (t^{r-1}g(t^s))) = \sum_{j=0}^m C_j^m t^{r-1-m+js}g^{(j)}(t^s).$$

If $\tilde{g}(x) = \sum_{j=0}^m C_j^m \frac{g^{(j)}(x)}{x^{m-j}} = C_m^m g^{(m)}(x) + \int_0^1 d\theta \sum_0^{m-1} \frac{(1-\theta)^{m-1-j}}{(m-1-j)!}C_j^m g^{(m)}(x\theta)$ for $x > 0$, $\tilde{g}(0) = 0$, then $\tilde{g}$ is $\mathbf{C}^\alpha$ with compact support and

$$\int_0^\infty e^{-tz}t^{r-1}g(t^s)dt = z^{-m}\int_0^\infty e^{-tz}t^{r''-1}\tilde{g}(t^s)dt$$

where $r'' = r + ms - m$, $Re\ r'' + s\alpha > 0$. Then we may apply lemma 2 to see this is $O(|z|^{-\mu})$ or $O(|z|^{-\mu}\log|z|)$.

Now suppose $k^* < m$. We have $g(t) = t^{m-k}g_k(t)$ where $g_m = g$ and $g_k(t) = \int_0^1 \frac{(1-\theta)^{m-k-1}}{(m-k-1)!}g^{(m-k)}(\theta t)d\theta$, for any k in $0 \le k < m$; g_k is $\mathbf{C}^{k,\alpha}$ with compact support and $g_k^{(j)}(t) = O(t^{k-j+\alpha})$ as $t \to 0^+$, $0 \le j \le k$. Note $tg_k(t) = g_{k+1}(t)$ is at least $\mathbf{C}^{k+1,\alpha}$, for $k < m$ and $tg_k^{(k+1)}(t) + (k+1)g_k^{(k)}(t) = g_{k+1}^{(k+1)}(t) = O(t^\alpha)$, $g_k^{(k+1)}(t) = O(t^{\alpha-1})$ as $t \to 0^+$. If $r' = r + s(m-k)$

$$\int_0^\infty e^{-tz}t^{r-1}g(t^s)dt = z^{-k}\int_0^\infty e^{-tz}\left(\frac{d}{dt}\right)^k\left(t^{r'-1}g_k(t^s)\right)dt$$

and

$$\left(\frac{d}{dt}\right)^k\left(t^{r'-1}g_k(t^s)\right) = \sum_{j=0}^k C_j^k t^{r'-1-k+js}g_k^{(j)}(t^s) = t^{r''-1}\tilde{g}(t^s)$$

with $r'' = r + ms - k$ and $\tilde{g}(x) = \sum_{j=0}^{k} C_j^k \frac{g_k^{(j)}(x)}{x^{k-j}}$; $\tilde{g}$ is C^α and (when $k < m$) $|\tilde{g}'(t)| = O(t^{\alpha-1})$. Then $\int_0^\infty e^{-tz} t^{r-1} g(t^s)dt = z^{-k} \int_0^\infty e^{-tz} t^{r''-1} \tilde{g}(t^s)dt = O(|z|^{-k-\nu})$ where $\nu = \min\{Re\ r'' + \alpha s, 1\}$, for any $k < m$, $k < k^* + \alpha^*$, by lemma 2. We choose $k = k^*$; then $k + \nu = \min\{Re\ r + (m+\alpha)s, k^* + 1\} = \min\{k^* + \alpha^*, k^* + 1\} = Re\ r + (m+\alpha)s\ (\leq m < m + \alpha)$. Thus in every case the estimate is $O(|z|^{-\mu})$ - or perhaps $O(|z|^{-\mu} \log |z|)$ - with $\mu = \min\{m + \alpha, Re\ r + s(m+\alpha)\}$.

■

Corollary 1 *Suppose g is $C^{m,\alpha}$ with compact support, $L \neq 0$ and $p > 0$ real, $q \in \mathbb{C}$ with $0 < Re\ q + 1 < p$. Then as $\omega \to +\infty$*

$$\int_0^\infty e^{i\omega L t^p} t^q g(t)dt = \frac{1}{p} \sum_{j=0}^{\infty} \frac{g^{(j)}(0)}{j!} \Gamma\left(\tfrac{q+j+1}{p}\right)(\omega|L|)^{-\frac{q+j+1}{p}}.$$

$$\cdot \exp\left(\pm i \frac{\pi}{2} \frac{q+j+1}{p}\right) + O(\omega^{-\mu})$$

where $\pm = sgn L$, $\mu = \min\{m + \alpha, \frac{Re\ q+1+m+\alpha}{p}\}$, plus $O(\omega^{-\mu} \log \omega)$ when these are equal (or if $\frac{Re\ q+1+m+\alpha}{p} =$ interger $\leq m$).

Proof: If $z = -i\omega L$, $t^p = x$, $r = \frac{q+1}{p}$,

$$\int_0^\infty e^{i\omega L t^p} t^p g(t)dt = \frac{1}{p} \int_0^\infty e^{-zx} x^{r-1} g(x^{\frac{1}{p}})dx$$

and Theorem 3 applies with $arg\ z = -\frac{\pi}{2} sgn\ L$.

■

Remark 4 *The usual method of stationary phase has $p = 2$, $q = 0$ or $r = s = \frac{1}{2}$.*

Example 2 *Let $g(t) = t^{m+\alpha}\varphi(t)$, with φ a C^∞ cutoff, so g is $C^{m,\alpha}$ with compact support. By lemma 1, if $Re\ r > 0$, $s > 0$,*

$$\int_0^\infty e^{-tz} t^{r-1} g(t^s)dt = \frac{\Gamma(r + s(m+\alpha))}{z^{r+s(m+\alpha)}} + O(|z|^{-N})$$

for any N as $|z| \to \infty$, $Re\ z \geq 0$.

Example 3 *Let $0 < \alpha < 1$, integer $m \geq 0$; suppose g_α is the function defined in Theorem 1, $0 < a < b$. For an appropriate polynomial P (of degree $2m+1$) define $G_\alpha(x) = P(x) + g_\alpha(x) - g_\alpha(a)$ in $[a,b]$ if $m = 0$, or for $m \geq 1$ $G_\alpha(x) = P(x) + \int_a^x \frac{(x-y)^{m-1}}{(m-1)!}(g_\alpha(y) - g_\alpha(a))dy$ in $a \leq x \leq b$, with $G_\alpha(x) = 0$ outside $[a,b]$; then G_α is $C^{m,\alpha}$ with compact support. Let $Re\ r > 0, s > 0$ and $g(t) = t^{\frac{1-r}{s}} G_\alpha(t^{\frac{1}{s}})$, so g is $C^{m,\alpha}$ with compact support $[a^s, b^s]$. Then as $|z| \to \infty$ with $Re\ z \geq 0$ - in particular, on the imaginary axis -*

$$\int_0^\infty e^{-tz} t^{r-1} g(t^s)dt = \int_a^b e^{-tz} G_\alpha(t)dt = z^{-m} \int_a^b e^{-tz} g_\alpha(t) + O(|z|^{-m})$$

is of exact order $O(|z|^{-m-\alpha})$, i.e. $|z|^{m+\alpha} \int_0^\infty e^{-tz} t^{r-1} g(t^s)dt$ is bounded but does not tend to zero.

Example 4 (Combining the previous examples) *The sum of the above "g" gives an example of class $C^{m,\alpha}$ $(0 < \alpha < 1)$ with compact support and with an integral of exact order $O(|z|^{-\mu})$, $\mu = \min\{m + \alpha, Re\ r + s(m + \alpha)\}$, as $|z| \to \infty$ with $Re\ z \geq 0$.*

Remark 5 *With appropriate hypotheses (see Theorem 1 and 2) we may treat $\int_0^\infty e^{-zf(x)} x^q g(x)dx$ as $|z| \to \infty$ in $Re\ z \geq 0$. (Here $f(x) \geq 0$ for all $x \geq 0$.)*

If g does not have compact support, suppose g is C^{k+1} outside a neighborhood of zero, $\gamma(t) \equiv t^{r-1} g(t^s) \to 0$ as $t \to +\infty$ and $\int_*^\infty |\gamma^{(k+1)}(t)|dt < \infty$. Then $\gamma^{(k)}(t) \to 0$ and the interpolation inequalities show $\gamma^{(j)}(t) \to 0$ as $t \to +\infty$ for $0 \leq j \leq k$. Choose a C^∞ cutoff function φ so that $g_2 = (1-\varphi)g$ is C^{k+1} and $g_2 \equiv 0$ near 0 while $g_1 = \varphi \cdot g$ has compact support (treated as in Theorem 3), $g = g_1 + g_2$. Then integration by parts shows $\left|\int_0^\infty e^{-tz} t^{r-1} g_2(t^s)dt\right| = O(|z|^{-k-1})$ as $|z| \to \infty$ with $Re\ z \geq 0$.

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