Multirings and applications to algebraic theory of quadratic forms and graded rings

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There are many of abstract theories of quadratic forms. The first ones (abstract Witt rings, quaternionic structures and Cordes schemes [28]) have appeared in the late 1970s, by the hands of M. Marshall and C. M. Cordes, with the following central target: analyze the existence (or not) of fields with certain properties relating to quadratic forms. In the decade of 1980s, appears the Marshall's abstract space of orderings (AOS) [30]: they are important because generalize both theory of orderings on fields and the reduced theory of quadratic forms. But only in the early 1990s that arise a (finitary) first-order theory that generalizes the reduced and non-reduced theory of quadratic forms simultaneously. This theory is the special groups (SG) of M. Dickmann and F. Miraglia [18]. At that moment, the focus was to look at generalizations for the theory of quadratic forms with invertibles coefficients (fields, von Neumann rings, semi-local rings..., in general, rings with a significant amount of unities). In the mid 1990s, M. Marshall generalizes the abstract ordering spaces to commutative rings, and called his new theory by "abstract real spectra" (ARS) [30], in a first attempt to develop a theory of quadratic forms over (general) coefficients on rings. The ring-theoretic case is much more difficult to deal than the field one and an algebraic counterpart of the abstract real spectra just appears in the 2000s, with the real semigroups (RS) [20] of M. Dickmann and A. Petrovich. In [27], is provided a general exposition on all these theories.

All those abstract theories constitute categories that are equivalent, or dually equivalent to full subcategories of each other. Also, each one has a particular motivation and advantage. In particular, some of them are categories of first-order theories and the corresponding language homomorphisms, thus allowing the application of model-theoretical notions and methods in this subject of algebra.

In the present notes, we show, *initially*, that every such theory (and category) can be represented by a category of *commutative multirings*. The notion of a multialgebraic structure – an "algebraic like" structure but endowed with some multivalued operations – has been studied since the 1930s. In particular, the concept of hyperring, rougly a ring with a multivalued addition, were introduced by Krasner in the 1950s [26]. The notion of multiring, essentially a hyperring but satisfying an weak version of distributive laws, was joined to the quadratic forms tools by the hands of M. Marshall in 2006 ([29]).

Since the middle of the 2000s decade, the notions of (commutative) hyperring/multiring have obtained more attention. These multialgebraic structures has been studied for applications many areas: in real algebraic geometry and abstract quadratic forms theory ([29], [22], [8]), tropical geometry ([32], [23]), algebraic geometry ([25], [6]), valuation theory ([24]), Hopf algebras ([21]), etc ([5], [1], [2], [7]).

A more detailed account of variants of concept of polynomials over multirings/hyperrings is even more recent, having less than five years ([23], [3], [6], [13]). These polynomials are organized in a multialgebraic structure called "superring". In [13], is started a model-theoretic oriented analysis of multialgebras introducing the class of algebraically closed hyperfields and providing variant proof of quantifier elimination flavor, based on new results on superring of polynomials. In [9], are provided new steps the program of studying the hyperfields/superfields under a natural notion of algebraic extension

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and roots of polynomials - this shares some common features with the recent work in [6] - it is showed that every well behaved superfield has a (unique up to isomorphism) algebraic extension to a superfield that is algebraically closed.

We emphasize that multirings: by on hand, can be described by a convenient first-order theory on a relational language and, by another hand, allows natural and useful generalization of "commutative multialgebra". These subsume the general idea of the present work, whose main references are: [29], [14], [11], [8], [15], [10], [12].

Overview of the work:

The work is organized in six chapters, briefly described below:

The initial chapter of these notes provides the main tools on multiring theory needed to develop the subsequent applications in abstract theories of quadratic forms and in graded rings: a general notion of multialgebraic structure and (multi)equational theory are given; the notions of hyperrings, multirings, hyperfields are presented and illustrated by examples; are described the main notions of morphisms and the main constructions in the (sub)categories of multirings (of hyperbolic multirings, in particular); it is explored some fragment of the order theory of multifields and multirings.

In Chapter 2, it is presented the notion of *superring*, a kind of multialgebraic structure that generalizes the algebraic notion of ring, where the sum *and* the product are multivalued operations. It is provided here a fragment of the theory of superrings (and superfields) needed to present the main constructions and developments envisaging applications in quadratic forms theory summarized in Chapter 5: are considered some constructions of superrings (as superrings of polynomials and Marshall's quotient of a superring by a "coherent" multiplicative subset); are explored some properties of the superring of polynomials with coefficients in a hyperfield; are developed some fragments of the theory of algebraic extension for superfields, such as the addition of a root of an irreducible polynomial in a superfield.

The third chapter is devoted to: (i) present a detailed functorial encoding of main abstract theories of quadratic forms – abstract ordering spaces, (pre)special groups, abstract real spectra and real semigroups – into the theory of multirings (respectively: real reduced hyperfields, special hyperfields, real reduced multirings); (ii) present other notions of multirings R (as quadratic multirings) or pairs (R, T), where R is a multiring and T is a certain multiplicative subset (as DM-multirings, DP-multirings and quadratic pairs), that seem to be relevant to algebraic theory of quadratic forms.

In Chapter 4, the focus is on constructions of multirings associated with real semigroups: (i) are described further properties of the functor Q, the reflection (= left adjoint functor) of the natural inclusion of the category of real reduced multirings into the category of pre-ordered multirings and explored some of these properties; (ii) by the employ of sheaf-theoretic methods, it is presented a characterization of the real reduced hyperrings as certain "geometric" von Neumann regular real hyperring and is described the functor V, "geometric" von Neumann regular hull of a multiring; (iii) it is presented some interesting logical-algebraic interactions between the functors Q and V that are useful to describe the Witt ring of a real semigroup (or real reduced multiring) [16].

In Chapter 5, are explored the significance of the previously described multialgebraic methods in the theories of superrings and superfields to the development of the algebraic theory of quadratic forms: (i) obtaining new relevant constructions in the category of special groups (or its equivalent category special hyperfields) – more specifically, obtaining quadratic extensions of special groups; (ii) extending to *all* special groups the validity of the Arason-Pfister Hauptsatz [4] – a positive answer to a question posed by J. Milnor in a classical paper of 1970 [31] – and established by Dickmann-Miraglia to the realm of *reduced* special groups (or its equivalent category of real reduced hyperfields) in 2000 [18].

The last Chapter of these notes is devoted to expand a fundamental tool in the algebraic theory of quadratic forms to the more general multivalued setting: the K-theory. It is introduced and developed the K-theory of hyperbolic hyperfields that generalize simultaneously Milnor's K-theory [31] and Special Groups K-theory, developed by Dickmann-Miraglia [19]. Are presented some properties of this generalized K-theory, that can be seen as a free construction in the category of inductive graded ring, a concept introduced in order to provide a solution of Marshall's Signature Conjecture by Dickmann-Miraglia [17]. Moreover, it is shown how the extended version of Arason-Pfister Hauptsatz - presented in the previous Chapter 5 - entails some interesting properties concerning K-theory and Marshall's conjecture.

References

[1] R Ameri, M Eyvazi, and S Hoskova-Mayerova. Advanced results in enumeration of hyperfields. *Aims Mathematics*, 5(6):6552–6579, 2020.

[2] R Ameri, A Kordi, and S Hoskova-Mayerova. Multiplicative hyperring of fractions and coprime hyperideals. *Analele Universitatii" Ovidius" Constanta-Seria Matematica*, 25(1):5–23, 2017.

[3] Reza Ameri, Mansour Eyvazi, and Sarka Hoskova-Mayerova. Superring of polynomials over a hyperring. *Mathematics*, 7(10):902, 2019.

[4] J. Arason and A. Pfister. Beweis des Krullschen Durschnittsatzes fur den Wittring. *Inventiones Mathematicae*, 12:173–176, 1971.

[5] Matthew Baker and Tong Jin. On the structure of hyperfields obtained as quotients of fields. *Proceedings of the American Mathematical Society*, 149(1):63–70, 2021.

[6] Matthew Baker and Oliver Lorscheid. Descartes' rule of signs, Newton polygons, and polynomials over hyperfields. *Journal of Algebra*, 569:416–441, 2021.

[7] Nathan Bowler and Ting Su. Classification of doubly distributive skew hyperfields and stringent hypergroups. *Journal of Algebra*, 574:669–698, 2021.

[8] Kaique Matias de Andrade Roberto, Hugo Rafael de Oliveira Ribeiro, and Hugo Luiz Mariano. Quadratic structures associated to (multi) rings. *Categories and General Algebraic Structures*, 16(1):105–141, 2022.

[9] Kaique Matias de Andrade Roberto, Hugo Rafael de Oliveira Ribeiro, and Hugo Luiz Mariano. On algebraic extensions and algebraic closures of superfields. *Preliminary version in https://arxiv.org/pdf/2208.08537. Submitted*, 2023.

[10] Kaique Matias de Andrade Roberto, Hugo Rafael de Oliveira Ribeiro, and Hugo Luiz Mariano. Quadratic extensions of special hyperfields and the general Arason-Pfister Hauptsatz. *Preliminary version in https://arxiv.org/abs/2210.03784. Submitted*, 2024.

[11] Kaique Matias de Andrade Roberto, Hugo Rafael de Oliveira Ribeiro, Hugo Luiz Mariano, and Kaique Ribeiro Prates Santos. Examples and counterexamples in the theory of (super)hyperfield: some perspectives. *Submitted*, 2023.

[12] Kaique Matias de Andrade Roberto and Hugo Luiz Mariano. K-theories and free inductive graded rings in abstract quadratic forms theories. *Categories and General Algebraic Structures*, 17(1):1–46, 2022.

[13] Kaique Matias de Andrade Roberto and Hugo Luiz Mariano. On superrings of polynomials and algebraically closed multifields. *Journal of Pure and Applied Logic*, 9(1):419–444, 2022.

[14] Hugo Rafael de Oliveira Ribeiro, Kaique Matias de Andrade Roberto, and Hugo Luiz Mariano. Functorial relationship between multirings and the various abstract theories of quadratic forms. *São Paulo Journal of Mathematical Sciences*, 16:5–42, 2022.

[15] Hugo Rafael de Oliveira Ribeiro and Hugo Luiz Mariano. von Neumann regular hyperrings and applications to real reduced multirings. *Algebra and Logic*, 62:215–256, 2023.

[16] Hugo Rafael de Oliveira Ribeiro and Hugo Luiz Mariano. Witt rings for Real Semigroups. *in preparation*, 2023.

[17] Maximo Dickmann and Francisco Miraglia. On quadratic forms whose total signature is zero mod 2^n: Solution to a problem of M. Marshall. *Inventiones mathematicae*, 133(2):243–278, 1998.

[18] Maximo Dickmann and Francisco Miraglia. *Special groups: Boolean-theoretic methods in the theory of quadratic forms.* Number 689 in Memoirs AMS. American Mathematical Society, 2000.

[19] Maximo Dickmann and Francisco Miraglia. Algebraic k-theory of special groups. *Journal of Pure and Applied Algebra*, 204(1):195–234, 2006.

[20] Maximo Dickmann and Alejandro Petrovich. Real semigroups and abstract real spectra. i. *Contemporary Mathematics*, 344:99–120, 2004.

[21] Chris Eppolito, Jaiung Jun, and Matt Szczesny. Hopf algebras for matroids over hyperfields. *Journal of Algebra*, 556:806–835, 2020.

[22] Pawel Gladki and Krzysztof Worytkiewicz. Witt rings of quadratically presentable fields. *Categories and General Algebraic Structures*, 12(1):1–23, 2020.

[23] Jaiung Jun. Algebraic geometry over hyperrings. *Advances in Mathematics*, 323:142–192, 2018.

[24] Jaiung Jun. Valuations of semirings. *Journal of Pure and Applied Algebra*, 222(8):2063–2088, 2018.

[25] Jaiung Jun. Geometry of hyperfields. *Journal of Algebra*, 569:220–257, 2021.

[26] Marc Krasner. Approximation des corps valués complets de caractéristique p= 0 par ceux de caractéristique 0. In *Colloque d'algèbre supérieure, tenu à Bruxelles du*, volume 19, pages 129–206, 1956.

[27] Hugo Luiz Mariano, Hugo Rafael de Oliveira Ribeiro, and Kaique Matias de Andrade Roberto. *Uma Jornada pelas Teorias Algébricas de Formas Quadráticas*. Editora da Física, 2021.

[28] Murray Marshall. *Abstract Witt rings*. Kingston, Ont.: Queen's University, 1980.

[29] Murray Marshall. Real reduced multirings and multifields. *Journal of Pure and Applied Algebra*, 205(2):452–468, 2006.

[30] Murray A Marshall. *Spaces of orderings and abstract real spectra*. Lecture Notes in Mathematics 1636, Springer, 1996.

[31] John Milnor. Algebraic k-theory and quadratic forms. *Inventiones mathematicae*, 9(4):318–344, 1970.

[32] Oleg Viro. Hyperfields for Tropical Geometry I. Hyperfields and dequantization. *arXiv preprint arXiv:1006.3034*, 2010.