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# Free k-tuples in linear groups

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## 1. Introduction

A k-tuple of members of a group is free if it freely generates a free subgroup.

In [1]. Chang, Jennings and Ree gave sufficient conditions for a pair of parabolic complex linear transformations to be free. Their result imply that the set of free pairs of members of PGL(2,$\mathbb{C}$) contains an open subset of PGL(2,$\mathbb{C}$)$^2$. Tits [6] has provided a criterion that, when satisfied by a k-tuple $A = (A_1,\ldots,A_k)$ in PGL(n,F), F a locally compact field, implies the existence of a integer r such that $(A_1^r,\ldots,A_k^r)$ is free.

We address the question of how large, topologically, is the set of free k-tuples in PGL(n,F), $n > 1$, when F is endowed with a topology. On the light of Tits' methods, we shall work in the following setting:

Let F be a locally compact field, endowed with a nontrivial valuation $|\ \ |$; V is an n-dimensional vector space over F, and P is the projective space of V. We denote by $\pi$ both canonical projections $\pi$: V → P and $\pi$: GL(V) → PGL(P). The space V is endowed with the only topology compatible with that of F, and so is GL(V). The topologies on P and PGL(P) are those induced by the projection $\pi$. Thus, if one chooses a basis on V, and identifies $V = F^n$, $GL(V) = GL(n,F) \subseteq F^{n^2}$, the identifications are homeomorphisms, where $F^n$ and $F^{n^2}$ have the product topology.

A simple observation shows:

Proposition 1: The set of free k-tuples of PGL(P) is dense in $PGL(P)^k$.

We elaborate on Tits' [6] methods in order to get:

Theorem 2: The set of free k-tuples of PGL(P) has a nonvoid interior in $PGL(P)^k$.

2. Proofs:

Proof of Proposition 1: It is enough to show that the free k-tuples of GL(n,F) are dense in $GL(n,F)^k$.

A k-tuple of matrices in GL(n,F) satisfies a given group identify, provided its entries simultaneously satisfy a set of $n^2$ polynomial equations. Thus the set of k-tuples which do not satisfy that identify is open and dense in $GL(n,F)^k$.

It follows that the set of free k-tuples is a countable intersection (indexed by the members of the free group in k-generators) of dense open sets of $GL(n,F)^k$. Thus it is dense by Baire's theorem applied to $GL(n,F)^k$ and the proposition is proved.

In order to prove the next theorem it will be necessary to recall some known facts and make some topological considerations.

As Tits, our main tool for proving a k-tuple to be free is the following criterion, due to Macbeath [4], Lyndon and Ullman [3] and Tits [6]:

<u>Lemma 2.1</u>: Let G be a group acting on a set P on the left and let $g = (g_1,\ldots,g_k)$ be a k-tuple of members of G. Suppose that there exists a k-tuple $(P_1,\ldots,P_k)$ of subsets of P and a $p \in P - P_1 \cup P_2 \cup\ldots\cup P_k$ such that for all i,j, $1 \leq i,\ j \leq k$, $i \neq j$ and all $m \in Z^*$, $g_i^m\ (P_j \cup \{p\}) \subseteq P_i$. Then $(g_1,\ldots,g_k)$ is free.

<u>Proof</u>: Let $F_k$ be the free group generated by $\overline{g}_1,\ldots,\overline{g}_k$ and let $f: F_k \to \langle g_1,\ldots,g_k\rangle$ be the epimorphism determined by $f(\overline{g}_i) = g_i$, $1 \leq i \leq k$. It follows by induction that if $x = x_n\, x_{n-1}\ldots x_1$ is a reduced word of $F_k$, $f(x)\ (p) \in P_i$, where i is such that $x_n = g_i$ or $x_n = \overline{g_i}^{-1}$. Hence, if $n \geq 1$, $f(x)\ (p) \neq p$, so $f(x) \neq 1$, Ker $f = 1$ and the result follows.

Following Tits [6], we associate to each transformation $g \in PGL(P)$ two linear subspaces of P, A(g) and A'(g) as follows. Choose a representative $\overline{g}$ of g and let $f(t) = \prod_{i=1}^{n} (t - \lambda_i)$ be its characteristic polynomial. Set $\Omega = \{\lambda_i / |\lambda_i| = \sup\{|\lambda_j|,\ 1 \leq j \leq n\}\}$, $f_1(t) = \prod_{\lambda_i \in \Omega} (t - \lambda_i)$ and $f_2(t) = \prod_{\lambda \notin \Omega} (t - \lambda_i)$. We define A(g) A'(g) as the subspaces of P which correspond to the Kernels of $f_1(\overline{g})$ and $f_2(\overline{g})$, respectively.

<u>Definition</u>: A k-tuple $(g_1,\ldots,g_k)$ of $PGL(P)^k$ satisfies <u>Tits' Criterion</u> if:

(a) $A(g_i)$ and $A(g_i^{-1})$ are points.

(b) $i \neq j$ implies that $(A(g_i) \cup A(g_i^{-1})) \cap (A'(g_j) \cup A'(g_j^{-1})) = \emptyset$

In [6], Lemma 3.12 is proved.

<u>Lemma 2.2</u>: If $(g_1,\ldots,g_k)$ satisfies Tits Criterion there exist in P compact neighborhoods $U_i$ of $A(g_i)$ and $U_1'$ of $A(g_i^{-1})$, $1 \leq i \leq k$, $p \in P - \bigcup_i (U_i \cup U_i')$ and $r > 0$ such that

$\forall j \neq i$, $\forall m \geq r$

$$g_i^m (U_j \cup U_j' \cup \{p\}) \subseteq \text{int } U_i$$

$$g_i^{-m}(U_j \cup U_j' \cup \{p\}) \subseteq \text{int } U_i'$$

where int designates topological interior.

Now we need a topological lemma.

<u>Lemma 2.3</u>: Let K, $\mathcal{O}$ be subsets of P, K compact, $\mathcal{O}$ open. Then $A(K,\mathcal{O}) = \{T \in PGL(P)/T(K) \subseteq \mathcal{O}\}$ is open in PGL(P).

<u>Proof</u>: Let us consider the diagram:

$$\begin{array}{ccc} GL(V) \times V^* & \xrightarrow{\ \phi\ } & V^* \\ \downarrow{\scriptstyle \pi \times \pi} & & \downarrow{\scriptstyle \pi} \\ PGL(P) \times P & \xrightarrow{\ \bar{\phi}\ } & P \end{array}$$

where $\phi(t,v) = Tv$ and $\overline{\Phi}$ is defined such that the above diagram commutes. As can be seen easily $\overline{\Phi}$ is well defined.

We claim that $\overline{\Phi}$ is continous.

Let A be an open set contained in P. Then, since $\phi$ and $\pi$ are continous and $\pi \times \pi$ is an open map, it follows that $(\pi \times \pi)(\phi^{-1}(\pi^{-1}(A)))$ is open in PGL(P) $\times$ P and the claim is proved.

Now, let $P^P$ be the set of all continous maps of P into itself, endowed with the compact open topology. Define $\hat{\Phi}$: PGL(P) $\to P^P$ by $\hat{\Phi}(T)(x) = \overline{\Phi}(T,x)$. Since $\overline{\Phi}$ is continous, it follows that $\hat{\Phi}$ is continous too (see [2], Theorem XII 3.1). The set $A(K,O)$ is the inverse image by $\hat{\Phi}$ of a basic open set of $P^P$, thus it is open.

Proof of Theorem 2: Let $(f_1,\dots,f_k)$ be a k-tuple satisfying Tits' Conditions and let r, $U_1,\dots,U_k$, $U'_1,\dots,U'_k$, p be the objects associated to this k-tuple in Lemma 2.2; denote $P_i = U_i \cup U'_i$, $1 \le i \le k$.

Following the notations of Lemma 2.3, let, for $1 \le i, j \le k$, $i \ne j$

$$K_{ij} = A(P_j \cup \{p\}, \text{int } U_i)$$

$$K'_{ij} = A(P_j \cup \{p\}, \text{int } U'_i)$$

$$L_i = A(U_i, \text{int } U_i)$$

$$L'_i = A(U'_i, \text{int } U'_i)$$

$$M_i = (\bigcap_{j \ne i} K_{ij}) \cap L_i \; ; \; M'_i = (\bigcap_{j \ne i} K'_{ij}) \cap L'_i \; ;$$

$$W_i = M_i \cap \{f \in PGL(P)/f^{-1} \in M_i'\}.$$

It follows from Lemma 2.3 that $M_i$ and $M_i'$ are open neighborhoods of $f_i^r$ and $f_i^{-r}$, respectively. Since $x \mapsto x^{-1}$ is a homeomorphism of PGL(P), $W_i$ is an open neighborhood of $f_i^r$.

The open set $W_1 \times \ldots \times W_k$ is comprised of free k-tuples only. This follows from Lemma 2.1, as the construction of the $W_i$ ensures that if $(g_1,\ldots,g_k) \in W_1 \times \ldots \times W_k$, then

$$g_i^m(P_j \cup \{p\} \subseteq \begin{cases} U_i & m > 0 \\ U_i' & m < 0 \end{cases}$$

3. Concluding remarks

Actually, we had intended to show that the set of free k-tuples is open. Theorem 2 is an approximation to that, as we have not succeded in our original purpose (nor have we been able to disprove the conjectured result).

Indeed, we conjecture:

(a) The set of free k-tuples is open in PGL(P); and a weakening form of (a);

(b) The set of free k-tuples contains a dense open set of PGL(P).

In a companion paper [5], we shall give a modified proof of Theorem 2 which is quite computational, and allows one to explicitly describe some open sets os free k-tuples.

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