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PATTERN'S SIGNATURE PROCESSES

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Abstract

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1.Introduction. As in Barlow and Proschan (1981) a complex engineering system is completely characterized by its structure function Φ which relate its lifetime T and its components lifetimes T_i, $1 \leq i \leq n$, defined in a complete probability space $(\Omega, \Im, P)$

$$T = \Phi(\mathbf{T}), \mathbf{T} = (T_1, ..., T_n).$$

A system is said to be coherent if its structure function Φ is increasing and each component is relevant, that is, there exist a time t and a configuration of $\mathbf{T}$ in t such that the system works if, and only if, the component works.

The performance of a coherent system can be measured from this structural relationship and the distribution function of its components lifetimes, however such representations make the distribution function of the system lifetime analytically very complicated (mainly in the dependent case). An alternative representation for the coherent system distribution function is through the system signatures, that, while narrower in scope than the structure function, is substantially more useful.

To define system signature, Samaniego (1985), consider the component lifetimes $T_i's$ independent and identically distributed with continuous distributions. Under this assumption, the signature of a coherent system of order n is the n-dimensional probability vector whose i-th coordinate is $P(T = T_{(i)})$, where $T_{(i)}$ is the i-th order statistics of the n i.i.d component lifetimes. Follows that the signature vector does not depend on the common continuous lifetimes component distributions and therefore, any property in system lifetime characteristics must be attributed to its structure function. Certainly, in some application the lifetimes of the components actually employed might not reasonably be assumed to be i.i.d., however the relative performance of systems under an i.i.d. assumption can still provide us information about system quality.

In the above context the system lifetime distribution can be set as

$$P(T \leq t) = \sum_{i=1}^{n} P(T = T_{(i)})P(T_{(i)} \leq t).$$

A detailed treatment of the theory and applications of system signatures may be found in Samaniego (2007).

Samaniego (1985), Kochar, et al. (1999) and Shaked and Suarez-Llorens (2003) extended the signature concept to the case where the systems components lifetimes $T_1, ..., T_n$, are exchangeable (i.e. the joint distribution function, $F(t_1, ..., t_n)$, of $(T_1, ..., T_n)$ is the same for any permutation of $t_1, ..., t_n$), an interesting and practical situation in reliability theory.

Navarro et al. (2008) and Samaniego et al. (2009) consider dynamic (conditioned) signatures and their use in comparing the reliability of new and used systems. Their procedures consider the system lifetime conditioned in an event on time. Navarro et al. (2008) consider either the event $\{T > t\}$ and $\{T_{(i)} \leq t\} \cap \{T > t\}$ with system signature $P(T = T_{(i)}|T > t)$ and $P(T = T_{(i)}|\{T_{(i)} \leq t\} \cap \{T > t\})$ respectively. A systems signature has proven to be quite a useful proxy for a systems design, as it is a distribution-free measure (i.e., not depending on F) that efficiently captures the precise features of a systems design which influence it performance. Unhappiness, in both Navarros above situations, the system signatures does depend on $F(t)$. Samaniego et all. (2009) consider the event in time $\{T_{(i)} \leq t < T_{(i+1)}\} \cap \{T > t\}$ and in this case the system signature $P(T = T_{(i)}|\{T_{(i)} \leq t < T_{(i+1)}\} \cap \{T > t\})$ does not depend on t and on $F(t)$ and have the usual signatures properties.

Navarro, et al. (2008) consider the mixture representation of residual lifetimes of used systems. In its conclusion asked about the general case of dependent components which remains an interesting open question. In this work we intend to analyses such a situation and for that, we are going to use a martingale point process approach. In Section 2 we solve the problem observing the system at component level. In Section 3 we inspect the system at a fixed time t.

2. Pattern's signature process in a complete information level

We consider a collection of n components, $C_1, C_2, ..., C_n$. These are often assumed to form a large system ϕ. Each component C_i has a positive lifetime T_i after 0, where 0 can be thought of as the time at which Φ is installed. We let T_i, $1 \leq i \leq n$, be random variables in a complete probability space $(\Omega, \Im, P)$. The system lifetime T can be represented by its series-parallel decomposition

$$T = \Phi(\mathbf{T}) = \min_{1\leq j\leq k} \max_{i\in K_j} T_i,$$

where $\mathbf{T} = (T_1, ..., T_n)$, $K_j, 1 \leq j \leq k$ are minimal cut sets, that is, a minimal set of components whose joint failure causes the system's failure. Φ is the system structure function.

In the following, to simplify the notation, we assume that relations such as $\subset, =, \leq, <, \neq$ between measurable sets and random variables, respectively, always hold with probability one, which means that the term $P - a.s.$ is suppressed. For a mathematical basis of stochastic processes applied to reliabilit theory see the book of Aven and Jensen (1999).

We describe the failures of $C_1, ..., C_n$ as they appear in advancing time, as a stochastic process. For any outcome $T_1(w), ..., T_n(w)$ of the lifetimes of $C_1, ..., C_n$ let $q(w)$ be the number of distinct values in the set $\{T_i(w); 1 \leq i \leq n\}$. We denote the strictly increasing order statistics of this set by $T_{(k)}$, having then

$$T_{(1)} < T_{(2)} < ... < T_{(q(w))}.$$

Also let

$$J_{(k)}(w) = \{i : T_i(w) = T_{(k)}(w), 1 \leq i \leq n, \}$$

be the index set of the components failing at the kth smallest failure time $T_{(k)}$. If there are no multiples failures, the value of $J_{(k)}$ is one of the singletons $\{i\}, 1 \leq i \leq n$. In general, however, $J_{(k)}$ is a Δ-valued random variable, where Δ is the power set of $\{1, 2, ..., n\}$. We call $T_{(k)}$ the kth failure time and $J_{(k)}$ the kth failure pattern.

The mathematical formulation of our observations is given through a family of sub σ-algebras of $\Im_t$, denoted $(\Im_t)_{t\geq 0}$, where

$$\Im_t = \sigma\{1_{\{T_{(k)}>s\}}, J_{(k)}) \in \Delta, 1 \leq k \leq n, 0 \leq s \leq t\}.$$

satisfies the Dellacherie's condition of right continuity and completeness.

Intuitively at at each time t the observer knows if the events $\{T_{(k)} \leq t, J_{(k)}\}$, $1 \leq k \leq n$, either occurred or not and if it does, he knows exactly the value $T_{(k)}$ and the mark $J_{(k)}$.

The random sequence $(T_{(k)}, J_{(k)})_{1\leq k\leq q}$ (of random length q)describes completely how the components $C_1, ..., C_n$ fail. We let

$$T_{(q+1)} = T_{(q+2)} = ... = \infty,$$

$$J_{(q+1)} = J_{(q+2)} = ... = \emptyset$$

and call the multivariate point process $(T_{(k)}, J_{(k)})_{k\geq 1}$ the failure process of $C_1, ..., C_n$.

We consider the lifetimes $T_{(k),J_{(k)}}$ defined by the failure event $\{T_{(k)} \leq t, J_{(k)}\}$ with their sub-distribution function $F_{(i),j}(t) = P(T_{(k),J_{(k)}} \leq t) = P(T_{(k)} \leq t, J_{(k)})$ suitable standardized.

The simple marked point process $N_{(k),J_{(k)}}(t) = 1_{\{T_{(k)}\leq t, J_{(k)}\}}$ is an $\Im_t$-submartingale and from the Doob-Meyer decomposition we know that there exists a unique $\Im_t$-predictable process $(A_{(k),J_{(k)}}(t))_{t\geq 0}$, the $\Im_t$-compensator process of $N_{(k),J_{(k)}}(t)$, with $A_{(k),J_{(k)}}(0) = 0$, such that $N_{(k),J_{(k)}}(t) - A_{(k),J_{(k)}}(t)$ is an $\Im_t$-martingale. We assume the $T_i, 1 \leq i \leq n$ are totally inaccessible stopping time. $A_{(k),J_{(k)}}(t)$ is absolutely continuous from the totally inaccessibility of $T_i, 1 \leq i \leq n$.

The compensator when understood as a measure in the real line, is well known to have the interpretation

$$A_{(k),J_{(k)}}(dt) = P(T_{(k)} \in dt, J_{(k)}|\Im_{t-}).$$

Intuitively, this corresponds to predicting if $T_{(k),J_{(k)}}$ is going to occur "now", based on all observations available up to the present, but not including it. Motivated by this we call $(A_J(t))_{t\geq 0}$ the hazard process of failure pattern J and $(A_J(t); J \in \Lambda)_{t\geq 0}$ the multivariate hazard process.

As $N_{(k),J_{(k)}}(t)$ can only count on the time interval $(T_{(k-1)}, T_{(k)}]$, the corresponding compensator differential $dA_{(k),J_{(k)}}(t)$ must vanish outside that interval. We denote

$$N_{(k)}(t) = 1_{\{T_{(k)}\leq t\}} = \Sigma_{J_{(k)}\in\Delta} N_{(k),J_{(k)}}(t).$$

Another equivalent way to describe the failure is the following: Fixed $J \in \Delta$ let T_J and $N_J(w,t)$ be defined by

$$T_J = \inf\{T_{(k)} : J_{(k)} = J\}$$

where $\inf \emptyset = 0$ and $N_J(w,t) = 1_{\{T_J \le t\}}$.

and

$$N_J(t) = 1_{\{T_J \le t\}} = \Sigma_{k=1}^{n} N_{(k),J}(t).$$

The stopping times $T_{(k),J_{(k)}}$ are rarely of direct concern in reliability theory. One is more interested in system failures times, which depend on the cumulative pattern of failed components. In more detail, let Φ a monotone (or coherent) system with lifetime T. We let

$$D(t) = \begin{cases} J_{(1)} \cup ... \cup J_{(k)}, & \text{if } T_{(k)} \le t < T_{(k+1)}, \\ 0 & \text{if } t < T_{(1)}. \end{cases}$$

be the cumulative pattern of failed components up to time t. The sample paths $t \to D(w,t)$ are then right continuous and increasing in the natural partial order of Δ. We let $D(t-) = \lim_{s \uparrow t} D(s)$. If

$$\Lambda_\Phi = \{K_1, ..., K_{k_0}\}, k_0 \ge k,$$

is the collection of all the cut sets of Φ, we clearly have

$$T = \inf\{t \ge 0 : D(t) \in \Lambda_\Phi\} = \min\{T_{(k),J_{(k)}} : J_{(1)} \cup ... \cup J_{(k)} \in \Lambda_\Phi\}.$$

We can therefore think that the point process with its only point at T, or equivalently the counting process

$$N_\Phi(t) = 1_{\{T \le t\}}, t \ge 0,$$

has been derived from the multivariate point process $(T_{(k)}, J_{(k)})_{k \ge 1}$.

The behavior of the stochastic process $P(T \le t | \Im_t)$, as the information flows continuously in time :

Theorem 2.1 Let $T_1, T_2, ..., T_n$ be the component lifetimes of a coherent system with lifetime T. Then,

$$P(T \le t | \Im_t) = \sum_{k=1}^{q} \sum_{J_{(k)} \in \Delta} 1_{\{T = T_{(k),J_{(k)}}\}} 1_{\{T_{(k),J_{(k)}} \le t\}}.$$

Proof From the total probability rule we have

$$P(T \le t|\Im_t) = \sum_{k=1}^{q} \sum_{J_{(k)} \in \Delta} P(\{T \le t\} \cap \{T = T_{(k),J_{(k)}}|\Im_t) =$$

$$\sum_{k=1}^{q} \sum_{J_{(k)} \in \Delta} E[1_{\{S=T_{(k),J_{(k)}}\}} 1_{\{T_{(k),J_{(k)}} \le t\}} |\Im_t].$$

As T and $T_{(k),J_{(k)}}$ are $\Im_t$-stopping time and it is well known that the event $\{T = T_{(k),J_{(k)}}\} \in \Im_{T_{(k),J_{(k)}}}$ where

$$\Im_{T_{(k),J_{(k)}}} = \{A \in \Im_\infty : A \cap \{T_{(k),J_{(k)}} \le t\} \in \Im_t, \forall t \ge 0\},$$

we conclude that $\{T = T_{(k),J_{(k)}}\} \cap \{T_{(k),J_{(k)}} \le t\}$ is $\Im_t$-measurable.

Therefore

$$P(T \le t|\Im_t) = \sum_{k=1}^{q} \sum_{J_{(k)} \in \Delta} E[1_{\{T=T_{(k),J_{(k)}}\}} 1_{\{T_{(k),J_{(k)}} \le t\}} |\Im_t] =$$

$$\sum_{k=1}^{q} \sum_{J_{(k)} \in \Delta} 1_{\{T=T_{(k),J_{(k)}}\}} 1_{\{T_{(k),J_{(k)}} \le t\}}.$$

The above decomposition allows us to define the signature point process at component level.

Definition 2.2 The vector $(1_{\{T=T_{(k),J_{(k)}}\}}, 1 \le k \le n, J_{(k)} \in \Delta)$ is defined as the pattern's signature point process of the system ϕ with lifetime S.

Remark 2.3

We can calculate the system reliability as

$$P(T > t) = E[P(T > t|\Im_t)] = E[\sum_{k=1}^{q} \sum_{J_{(k)} \in \Delta} 1_{\{T=T_{(k),J_{(k)}}\}} 1_{\{T_{(k),J_{(k)}} > t\}}] =$$

$$\sum_{k=1}^{q} \sum_{J_{(k)} \in \Delta} P(\{T = T_{(k),J_{(k)}}\} \cap \{T_{(k),J_{(k)}} > t\}).$$

If the component lifetimes are continuous, independent and identically distributed we have,

$$P(T > t) = \sum_{k=1}^{n} P(T = T_{(k)})P(T_{(k)} > t)$$

recovering the classical result as in Samaniego (1985).

Example 2.4

We analize a system of three dependent components. To this purpose the Marshall three-dimensional distribution with positive parameters given by $\lambda_1, \lambda_2, \lambda_3, \lambda_{12}, \lambda_{13}, \lambda_{23}$ and λ_{123}, is used with three variate reliability function given by $P(T_1 > t_1, T_2 > t_2, T_3 > t_3) =$

$$e^{-(\lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_3 + \lambda_{12} \max\{t_1, t_2\} + \lambda_{13} \max\{t_1, t_3\} + \lambda_{23} \max\{t_2, t_3\} + \lambda_{123} \max\{t_1, t_2, t_3\})}.$$

where T_1, T_2 and T_3 are the component's lifetimes.

An interpretation of this distribution is as follows: Seven independent exponential random variables $Z_1, Z_2, Z_3, Z_{12}, Z_{13}, Z_{23}, Z_{123}$, with correspondent parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_{12}, \lambda_{13}, \lambda_{23}, \lambda_{123}$ describe the time point when a shock causes failure of component 1 or 2 or 3 or the components 1 and 2 simultaneously, the components 1 and 3 simultaneously, the components 2 and 3 simultaneously or, even, the three components simultaneously, respectively.

The components lifetimes are given by $T_1 = \min\{Z_1, Z_{12}, Z_{13}, Z_{123}\}$, $T_2 = \min\{Z_2, Z_{12}, Z_{23}, Z_{123}\}$ and $T_3 = \min\{Z_3, Z_{13}, Z_{23}, Z_{123}\}$. Note that $T_1 \neq T_{\{1\}}$, as we have on $Z_{123} < \min\{Z_1, Z_2, Z_3, Z_{12}, Z_{13}, Z_{23}\}$.

We consider the system with lifetime $T = \min\{T_1, \max\{T_2, T_3\}\}$ and calculate, for all $I \in \Delta$, the probabilities in the first failure:

$$P(T = T_{(1),I}) = P(Z_I < \min\{Z_J, J \neq I\}) =$$

$$\int_0^\infty e^{-\sum_{J \neq I} \lambda_J x} \lambda_I e^{-\lambda_I x} dx =$$

$$\lambda_I \int_0^\infty e^{-\lambda x} dx = \frac{\lambda_I}{\lambda}.$$

Also, using the same argument we have

$$P(T = T_{(1),I}, T_{(1),I} > t) = \frac{\lambda_I}{\lambda} e^{-\lambda t}.$$

The probabilities of the second failures are: if $I \in \Delta$ and $I \neq \{2,3\}, I \neq \{2\}, I \neq \{3\}$ we have

$$P(T = T_{(2),I}) = P(Z_2 < Z_I < \min\{Z_J, J \in \Delta, J \neq I, J \neq \{2\}\})+$$

$$P(Z_3 < Z_I < \min\{Z_J, J \in \Delta, J \neq I, J \neq \{3\}\}) =$$

$$\int_0^\infty (1 - e^{-\lambda_2 x}) e^{-\sum_{J\in\Delta, J\neq I, J\neq\{2\}} \lambda_J x} \lambda_I e^{-\lambda_I x} dx +$$

$$\int_0^\infty (1 - e^{-\lambda_3 x}) e^{-\sum_{J\in\Delta, J\neq I, J\neq\{3\}} \lambda_J x} \lambda_I e^{-\lambda_I x} dx =$$

$$\frac{\lambda_I}{\lambda - \lambda_2} + \frac{\lambda_I}{\lambda - \lambda_3} - \frac{2\lambda_I}{\lambda},$$

$$P(T = T_{(2),\{2\}}) = \frac{\lambda_2}{\lambda - \lambda_3} - \frac{\lambda_2}{\lambda}$$

and

$$P(T = T_{(2),\{3\}}) = \frac{\lambda_3}{\lambda - \lambda_2} - \frac{\lambda_3}{\lambda}.$$

Also

$$P(T = T_{(2),I}, T_{(2),I} > t) = \frac{\lambda_I}{\lambda - \lambda_2} e^{-(\lambda-\lambda_2)t} + \frac{\lambda_I}{\lambda - \lambda_3} e^{-(\lambda-\lambda_3)t} - 2\frac{\lambda_I}{\lambda} e^{-\lambda t},$$

in the case where $I \in \Delta$ and $I \neq \{2,3\}, I \neq \{2\}, I \neq \{3\}$.

$$P(T = T_{(2),\{2\}}, T_{(2),\{2\}} > t) = \frac{\lambda_2}{\lambda - \lambda_3} e^{-(\lambda-\lambda_3)t} - \frac{\lambda_2}{\lambda} e^{-\lambda t},$$

and

$$P(T = T_{(2),\{3\}}, T_{(2),\{3\}} > t) = \frac{\lambda_3}{\lambda - \lambda_2} e^{-(\lambda-\lambda_2)t} - \frac{\lambda_3}{\lambda} e^{-\lambda t}.$$

Therefore we can conclude

$$P(T > t) = E\{P(T > t | \Im_t)\} =$$

$$E\{ \sum_{1\leq k\leq n, J\in\Delta} 1_{\{T = T_{(k),J}\}} 1_{\{T_{(k),J} > t\}} \} =$$

$$\sum_{I\in\Delta} P(T=T_{(1),I},T_{(1),I}>t)+\sum_{I\in\Delta} P(T=T_{(2),I},T_{(1),I}>t)=$$

$$\sum_{I\in\Delta}\frac{\lambda_I}{\lambda}e^{-\lambda t}+$$

$$\sum_{I\in\Delta,I\neq\{2,3\},\{2\},\{3\}}\frac{\lambda_I}{\lambda-\lambda_2}e^{-(\lambda-\lambda_2)t}+\frac{\lambda_I}{\lambda-\lambda_3}e^{-(\lambda-\lambda_3)t}-2\frac{\lambda_I}{\lambda}e^{-\lambda t}+$$

$$\frac{\lambda_2}{\lambda-\lambda_3}e^{-(\lambda-\lambda_3)t}-\frac{\lambda_2}{\lambda}e^{-\lambda t}+$$

$$\frac{\lambda_3}{\lambda-\lambda_2}e^{-(\lambda-\lambda_2)t}-\frac{\lambda_3}{\lambda}e^{-\lambda t}.$$

We now go on by studying the $\Im_t$-compensator of the counting process $(N_\Phi(t))_{t\geq 0}$ of system failure, denoting it by $(A_\Phi(t))_{t\geq 0}$. It is natural to ask what is the contribution of the failure's component propensity for predicting the system 's failure propensity. Following, Corollary 2.4 characterizes the relationship between the component's $\Im_t$- compensator and the system's $\Im_t$-compensator processes.

Corollary 2.5 Let $T_1, T_2, ..., T_n$, be the components lifetimes of a coherent system with lifetime T. Then, the $\Im_t$-submartingale $P(T\leq t|\Im_t)$, has the $\Im_t$-compensator

$$\sum_{k,j=1}^{n}\int_0^t 1_{\{T=T_{(k),J_{(k)}}\}}dA_{(k),J_{(k)}}(s).$$

Proof

We consider the process

$$1_{\{T=T_{(k),J_{(k)}}\}}(w,s)=1_{\{T=T_{(k),J_{(k)}}\}}(w).$$

As $T_{(k),J_{(k)}}\leq T$, for all $1\leq k\leq q, J_{(k)}\in\Delta$, it is left continuous and $\Im_t$-predictable. Therefore

$$\int_0^t 1_{\{T=T_{(k),J_{(k)}}\}}(s)dM_s((k),j)$$

is an $\Im_t$-martingale.

As a finite sum of $\Im_t$-martingales is an $\Im_t$-martingale, we have

$$\sum_{k=1}^{q}\sum_{J_{(k)}\in\Delta}\int_0^t 1_{\{T=T_{(k),J_{(k)}}\}}d1_{\{T_{(k),J_{(k)}}\leq s\}}-$$

$$\sum_{k=1}^{q}\sum_{J_{(k)}\in\Delta}\int_0^t 1_{\{T=T_{(k),J_{(k)}}\}}dA_{(k),J_{(k)}}(s)=$$

$$\sum_{k=1}^{q}\sum_{J_{(k)}\in\Delta}\int_0^t 1_{\{T=T_{(k),J_{(k)}}\}}dM_s((k),j)$$

is an $\Im_t$-martingale. As the compensator is unique we finish the proof.

3.Inspecting the system at a fixed time t.

As in Samaniego et al.[15], in time dynamics we can observe the event $\{T>t\}\cap\{T_{(i)}<t\leq T_{(i+1)}\}$. However it is well know that

$$\Im_t\cap\{T_{(i)}<t\leq T_{(i+1)}\}=\Im_{T_{(i)}}\cap\{T_{(i)}<t\leq T_{(i+1)}\},$$

that is, the information up to t is the same information up to $T_{(i)}$. It means that, after the i-th failure we continue to observe $(\Im_{T_{(i)}+t})_{t\geq 0}$, where

$$\Im_{T_{(i)}+t}=\{A\in\Im_\infty : A\cap\{T_{(i)}\leq s-t\}\in\Im_s,\forall s>0\}.$$

Theorem 3.1 Let $T_1,T_2,...,T_n$ be the component lifetimes of a coherent system with lifetime T. Then,

$$P((T-T_{(i)})^+\leq t|\Im_{T_{(i)}})=\sum_{k=1}^{q}\sum_{J_{(k)}\in\Delta}\frac{P(T=T_{(k),J_{(k)}})}{P(T>T_{(i)})}1_{\{T>T_{(i)}\}}$$

$$P((T_{(k),J_{(k)}}-T_{(i)})^+\leq t|\Im_{T_{(i)}}\cap\{T=T_{(k),J_{(k)}}\}).$$

Proof

In this situation we count

$$M_\phi(t) = E[N_\phi(T_{(i)} + t) - N_\phi(T_{(i)}|\Im_{T_{(i)}}] =$$

$$E\{E[N_\phi(T_{(i)} + t) - N_\phi(T_{(i)}|\Im_t]|\Im_{T_{(i)}}\} =$$

$$E\{\sum_{k=1}^{q} \sum_{J_{(k)} \in \Delta} 1_{\{T=T_{(k),J_{(k)}}\}} 1_{\{T_{(k),J_{(k)}} \leq T_{(i)}+t\}} -$$

$$\sum_{k=1}^{q} \sum_{J_{(k)} \in \Delta} 1_{\{T=T_{(k),J_{(k)}}\}} 1_{\{T_{(k),J_{(k)}} \leq T_{(i)}\}} |\Im_{T_{(i)}}\} =$$

$$E[\sum_{k=1}^{q} \sum_{J_{(k)} \in \Delta} 1_{\{T=T_{(k),J_{(k)}}\}} 1_{\{(T_{(k),J_{(k)}} - T_{(i)})^+ \leq t\}} |\Im_{T_{(i)}}\} =$$

$$\sum_{k=1}^{q} \sum_{J_{(k)} \in \Delta} P(T = T_{(k),J_{(k)}}, (T_{(k),J_{(k)}} - T_{(i)})^+ \leq t|\Im_{T_{(i)}}) =$$

$$\sum_{k=1}^{q} \sum_{J_{(k)} \in \Delta} P(T = T_{(k),J_{(k)}}|\Im_{T_{(i)}}).$$

$$P((T_{(k),J_{(k)}} - T_{(i)})^+ \leq t|\Im_{T_{(i)}} \cap \{T = T_{(k),J_{(k)}}\}).$$

We can see that a version for $P(T = T_{(k),J_{(k)}}|\Im_{T_{(i)}})$ is

$$P(T = T_{(k),J_{(k)}}|\Im_{T_{(i)}}) = \begin{cases} \frac{P(T=T_{(k),J_{(k)}})}{P(T>T_{(i)})} 1_{\{T>T_{(i)}\}}, & \text{if } P(T > T_{(i)}) > 0; \\ 0, & \text{otherwise.} \end{cases}$$

as, for any $\Delta \in \Im_{T_{(i)}}$ we have

$$\int_\Delta \frac{P(T = T_{(k),J_{(k)}})}{P(T > T_{(i)})} 1_{\{T>T_{(i)}\}} dP =$$

$$\frac{P(T = T_{(k),J_{(k)}})}{P(T > T_{(i)})} P(\Delta, T = T_{(k),J_{(k)}}) =$$

$$\frac{P(T = T_{(k),J_{(k)}})P(T > T_{(i)})}{P(T > T_{(i)})} =$$

$$P(T = T_{(k),J_{(k)}}) = P(T = T_{(k),J_{(k)}}, \Delta).$$

The second and four equalities follows from $\{T > T_{(i)}\} = \cup_{k>i}\{T = T_{(k),J_{(k)}}\}$ and $\Delta \subseteq \{T = T_{(k),J_{(k)}}\}$. Note that, in $\Delta \in \Im_{T_{(i)}}$ we have $\Delta \cap \{T_{(i)} \leq x\} \in \Im_x$. However, if $k > i$ a set in $\Im_x$ is of the form $\{T_{(k),J_{(k)}} > x\}$. We conclude that if $w \in \Delta$, $w \in \{T_{(i)} \leq x\} \cap \{T_{(k),J_{(k)}} > x\}$, which implies $T_{(k),J_{(k)}} - T_{(i)} > 0$ and $\Delta \subseteq \{T_{(i)} > T\}$.

Follows that

$$P((T - T_{(i)})^+ \leq t | \Im_{T_{(i)}}) = \sum_{k=1}^{q} \sum_{J_{(k)} \in \Delta} \frac{P(T = T_{(k),J_{(k)}})}{P(T > T_{(i)})} 1_{\{T > T_{(i)}\}}.$$

$$P((T_{(k),J_{(k)}} - T_{(i)})^+ \leq t | \Im_{T_{(i)}} \cap \{T = T_{(k),J_{(k)}}\}).$$

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