RT-MAE 2010-02

POINTWISE APPROXIMATIONS FOR SUMS OF NON-IDENTICALLY DISTRIBUTED BERNOULLI TRIALS

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Palavras-Chave: Poisson approximation, non-identically distributed Bernoulli variables, Le Cam's type inequalities, total variation distance, pointwise approximation.

Classificação AMS: 60F05, 60G10, 37A50.

Pointwise approximations for sums of non-identically distributed Bernoulli trials *

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06/12/09

Abstract

Binomial and Poisson approximations to the law of the sum of independent non-identically distributed Bernoulli variables are reexamined. In each case, new bounds on both "pointwise distances" and total variation distance are obtained, improving the well-known Le Cam's type inequalities. The rather simple technique introduced also yields a comparison between Binomial and Poisson models and a geometric approximation to the distribution of the waiting time for the first success.

1 Introduction

The Poisson probability distribution, brought in by Simeon Dénis Poisson in his remarkable 1837 book, plays an important role in probabilistic modelling in various fields of knowledge. Since its appearance in applications to lawsuit and criminal trials, the Poisson model has been increasingly adopted as a mathematical device for the common problem of counting the occurrences of rare events.

If empirical evidence hints the suitability of the Poisson model for count processes, it is the existence of approximation theorems that justifies more formally the use of the Poisson distribution and its generalizations in modelling. For instance, a primary result that states the "Poissonity" of the limit number of successes in a series of n independent Bernoulli trials, each having probability of success p_n, p_n decreasing with n such that np_n converges to a positive constant, is usually taken into account in the choice of the Poisson model for the count of occurrences of rare events (successes).

Since the formulation of the aforementioned result, Poisson approximations to more general sequences of Bernoulli trials, including the cases of weak (local) dependence and of non-identical probabilities of success, have been drawing

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Running title. Approximations for Bernoulli trials.

AMS subject classification. 60F05, 60G10, 37A50.

Key words and phrases. Poisson approximation, non-identically distributed Bernoulli variables, Le Cam's type inequalities, total variation distance, pointwise approximation.

a lot of attention in the Probability Community. A landmark in this field of research is Le Cam's inequality (1960): it establishes bounds on the error of a Poisson approximation to the distribution of a (finite) sum of independent non-identically distributed Bernoulli variables, refining upon the then usual approach based on the evaluation of convergence (in law) of sequences of partial sums. As a matter of fact, much effort has been focused on the improvement and generalization of Le Cam's results. One loses count of the works that followed Le Cam's paper (1960): Hodges and Le Cam (1960), Freedman (1974), Chen (1974, 1975), Serfling (1975, 1978), Presman (1985), Wang (1986), Yakshyavicius (1986), Arratia, Goldstein and Gordon (1989, 1990), Barbour, Chen and Loh (1992), Barbour, Holst and Janson (1992), Loh (1992), Steele (1994), Roos (2003), among others. As emphasized by Steele (1994) "Le Cam's inequality also seems to be one of those facts that repeatedly calls to be proved - and improved".

In this work, binomial and Poisson approximations to the distribution of the sum of independent non-identically distributed Bernoulli variables, known as the Binomial-Poisson model, are discussed (recent works on the matter include Ehm (1991) and Roos (2000, 2001, 2008)). More precisely, in each of the above situations, we present bounds on both "pointwise distances" and total variation distance (T. V. D.) between the distributions (the latter resulting from the formers). By "pointwise distance" between the probability measures P and Q at k we mean the distance $|P(\{k\}) - Q(\{k\})|$. As far as the authors know, the approach to Le Cam's-type inequalities based on pointwise distances has not been thoroughly investigated in the literature. In fact, it seems that only with the recent advances in probabilistic modelling of DNA sequences the examination of local bounds has gained considerable ground, advantageously substituting for global bounds in applied calculations such as approximations to tail probabilities in hypotheses testing (Abadi (2004) and Vergne and Abadi (2008)).

We prove that an upper bound for the binomial approximation is sharpened (in the sense of total variation distance) with the choice of a median of the probabilities of success as the Binomial parameter. For the Poisson approximation, the sum should be picked instead in order to attain an improvement of Le Cam's inequality. The theorems, which are derived directly from a basic lemma and some elementary combinatorial analysis, also yield an evaluation of the geometric approximation to the law of the waiting time for the first success and a comparison between binomial and Poisson probabilities.

The paper is organized as follows: in Section 2, the main results are stated, the proofs of which are developed in Section 3; the conclusions and final comments are presented in Section 4.

2 Results

We begin evaluating pontwise and total variation distances between the law of a sum of independent non-identically distributed Bernoulli variables (Binomial-

Poisson model) and a Binomial distribution.

Theorem 1 (Binomial-Poisson law versus Binomial law) *Let $\{X_i\}_{i\geq 1}$ be a sequence of independent Bernoulli random variables, with corresponding parameters $\{p_i\}_{i\geq 1}$, and $\{Y_i\}_{i\geq 1}$ be independent Bernoulli random variables with common parameter p. Suppose also that the sequences $\{X_i\}_{i\geq 1}$ and $\{Y_i\}_{i\geq 1}$ are independent. Let $S_n = \sum_{i=1}^{n} X_i$ and $B(n,p) = \sum_{i=1}^{n} Y_i$, $B(n,p)$ with Binomial distribution the parameters of which are n and p. Define $\epsilon_i(p) = |p_i - p|$, then the following inequality holds for $k = 0, ..., n$:*

$$|I\!P(S_n = k) - I\!P(B(n,p) = k)| \leq \sum_{i=1}^{n} \epsilon_i(p)[I\!P(S_{n-1}^{(i)} = k-1) + I\!P(S_{n-1}^{(i)} = k)], \tag{1}$$

where, for $i = 1, ..., n$,

$$S_{n-1}^{(i)} = \sum_{j=1}^{i-1} X_j + \sum_{j=i+1}^{n} Y_j \ .$$

(We use the convention that the sum over an empty set of indexes is zero)

From Theorem 1, we obtain the following corollary.

Corollary 2 (Le Cam's inequality for Binomial model) *Under the conditions of Theorem 1*

$$\sum_{k=0}^{n} |I\!P(S_n = k) - I\!P(B(n,p) = k)| \leq 2 \sum_{i=1}^{n} \epsilon_i(p) \ .$$

Remark 3 *Among all Binomial distributions of fixed parameter n approximating the law of S_n, the bests, in the sense of minimizing the above upper bound, are those with the second parameter (p) equal to a median of $p_1, ..., p_n$.*

Next, we introduce some definitions in order to state the approximation theorem between the law of a sum of independent non-identically distributed Bernoulli random variables and a Poisson distribution.

Definition 4 *Under the conditions of Theorem 1, we define, for each integer $j \geq 1$, the finite measure μ_j over the power set of $\{0,1\}$ by $\mu_j(\{1\}) = p_j e^{-p_j}$ and $\mu_j(\{0\}) = e^{-p_j}$, in such a way that $\mu_j(\{0,1\}) = (1+p_j)e^{-p_j}$.*

Definition 5 *Under the conditions of Theorem 1 and Definiton 4, fixed integer $n \geq 1$, we define, for each $i \in \{1, ..., n\}$, the finite product measure $\mu_n^{(i)}$ over the power set of $\{0,1\}^{n-1}$ by*

$$\mu_n^{(i)}(\{(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)\}) = \prod_{j=1}^{i-1} I\!P(X_j = a_j) \prod_{j=i+1}^{n} \mu_j(\{a_j\}) \ .$$

Thus $\mu_n^{(i)}(\{0,1\}^{n-1}) = \prod_{j=i+1}^{n}(1+p_j)e^{-p_j}$. (We use the convention that the product over an empty set of indexes is one)

The above measures, which are not probability ones (as they do not sum up one), figure in the following theorem.

Theorem 6 (Binomial-Poisson law versus Poisson law) *Let $\{X_i\}_{i\geq 1}$ be a sequence of independent Bernoulli random variables, with corresponding parameters $\{p_i\}_{i\geq 1}$, $S_n = \sum_{i=1}^n X_i$ and $Z(\lambda)$ a Poisson random variable with parameter $\lambda = \sum_{i=1}^n p_i$. Then, the following inequality holds for all $k \geq 0$:*

$$|I\!P(S_n = k) - I\!P(Z(\lambda) = k)| \leq \sum_{i=1}^n p_i^2 \left[\mu_n^{(i)}(A_{n-1,k-1}) + \frac{1}{2}\mu_n^{(i)}(A_{n-1,k})\right] + \delta(k),$$

where $A_{n,k} = \{(a_1, ..., a_n) \in \{0,1\}^n \mid a_1 + ... + a_n = k\}$, $\delta(0) = \delta(1) = 0$ and

$$\delta(k) = e^{-\lambda}\frac{\lambda^{k-2}}{(k-2)!}\sum_{i=1}^n \frac{p_i^2}{2}, fork \geq 2 .$$

From Theorem 6, we have the corollary below.

Corollary 7 (Le Cam's inequality for Poisson model) *Under the conditions of Theorem 6*

$$\sum_{k=0}^{\infty} |I\!P(S_n = k) - I\!P(Z(\lambda) = k)| \leq \sum_{i=1}^n p_i^2 \ [\frac{3}{2} \prod_{j=i+1}^n (1 + p_j)e^{-p_j} + \frac{1}{2}] \ .$$

It is easy to see that the function $h(p) = (1+p)e^{-p}$ decreases in the interval $[0,1]$ and that the maximum of $\{h(p) : p \in [0,1]\} = 1$. Thus, the upper bound obtained in Corollary 7 is strictly smaller, for $p \in (0,1)$, than the remarkable $2\sum_{i=1}^n p_i^2$ that figures in Le Cam's inequality. The upper bound we get attains its minimum if the random variables $X_1, ..., X_n$ are rearranged in such a way that the new (finite) sequence of parameters is non-decreasing.

The next corollary provides a pointwise approximation between the Binomial and the Poisson distributions.

Corollary 8 (Binomial law versus Poisson law) *Let $B(n,p)$ be a binomial random variable with parameters n and p and $Z(\lambda)$ a Poisson random variable with parameter $\lambda = np$. Then, the following inequality holds for all $k \geq 0$:*

$$|I\!P(B(n,p) = k) - I\!P(Z(\lambda) = k)| \ \leq \ p^2 \sum_{i=1}^n \left[\mu_n^{(i)}(A_{n-1,k-1}) + \frac{1}{2} \ \mu_n^{(i)}(A_{n-1,k})\right] + \delta(k) \ ,$$

where $A_{n,k}$, $\mu_n^{(i)}$, $n \geq 1$, $i = 1, ..., n$, and $\delta(k)$ are as in Definitions 4 and 5 and Theorem 6.

From Corollary 8, we have

Corollary 9 (T.V.D. between Binomial and Poisson laws) *Let $B(n,p)$ be a binomial random variable with parameters n and p and $Z(\lambda)$ a Poisson random variable with parameter $\lambda = np$. Then*

$$\sum_{k=0}^{\infty} |I\!P(B(n,p)=k) - I\!P(Z(\lambda)=k)| \leq \frac{np^2}{2} + \frac{3p^2}{2}\,\frac{1-[(1+p)e^{-p}]^n}{1-(1+p)e^{-p}}.$$

As the second term in the right-hand side of the inequality above is strictly smaller than $3np^2/2$ for all p in $(0,1)$, the upper bound we present is sharper than the well-known Le Cam's result.

We finish this section considering a geometric approximation to the law of the waiting time for the first success in a sequence of independent non-identically distributed Bernoulli variables.

Theorem 10 (First success law versus Geometric law) *Considering the conditions of Theorem 1, let us define the quantities $T = inf\{n \geq 1 | X_n = 1\}$ and $G = inf\{n \geq 1 | Y_n = 1\}$ (namely, G is a geometric random variable with parameter p). Then, the following inequality holds:*

$$|I\!P(T=k) - I\!P(G=k)| \leq \sum_{i=1}^{k-1} \epsilon_i(p) I\!P(T^{(i)} = k-1) \;+\; \epsilon_k(p) I\!P(T^{(k)} \geq k).$$

Here $T^{(1)} = G$ and, for $i \geq 2$, $T^{(i)} = inf\{n \geq 1 | X_n^{(i)} = 1\}$, where $X_n^{(i)} = X_n$ for $n \leq i-1$ and $X_n^{(i)} = Y_n$, for $n \geq i$. That is to say, $T^{(i)}$ is the waiting time for the first success in the sequence of independent Bernoulli trials with parameters $p_1, \ldots, p_{i-1}, p, p, \ldots$.

Corollary 11 (T.V.D. between the first success and Geometric laws) *Under the conditions of Theorem 10, we have:*

$$\sum_{k=1}^{\infty} |I\!P(T=k) - I\!P(G=k)| \leq 2\sum_{k=1}^{\infty} \epsilon_k(p) \prod_{j=1}^{k-1}(1-p_j) = 2\sum_{k=1}^{\infty} \epsilon_k(p) I\!P(T^{(k)} \geq k).$$

3 Proofs

A fundamental tool for our proofs is the following lemma which may be seen as a discrete version of the Mean Value Theorem.

Lemma 12 *Let p_i and θ_i, $i = 1, \ldots, n$, be nonnegative real numbers. Define $\epsilon_i = |p_i - \theta_i|$ for all $i = 1, \ldots, n$, then the following inequality holds:*

$$\left|\prod_{i=1}^{n} p_i - \prod_{i=1}^{n} \theta_i\right| \;\leq\; \sum_{i=1}^{n} \Big(\prod_{j=1}^{i-1} p_j\Big)\, \epsilon_i \,\Big(\prod_{j=i+1}^{n} \theta_j\Big).$$

In the case $\theta_i = 0$ for all $i = 1, \ldots, n$, the above expression reduces to

$$\left|\prod_{i=1}^{n} p_i - \theta^n\right| \leq \sum_{i=1}^{n}(\prod_{j=1}^{i-1} p_j)\ \epsilon_i(\theta)\ \theta^{n-i}\ ,$$

where $\epsilon_i(\theta) = |p_i - \theta|$, $i = 1, ..., n$.

Proof of Lemma 12. We write the following triangle inequality

$$\begin{aligned}
&\left|\prod_{i=1}^{n} p_i - \prod_{i=1}^{n} \theta_i\right| \\
= &\left|\prod_{i=1}^{n} p_i - (\prod_{i=1}^{n-1} p_i)\theta_n + (\prod_{i=1}^{n-1} p_i)\theta_n - (\prod_{i=1}^{n-2} p_i)\theta_{n-1}\theta_n + (\prod_{i=1}^{n-2} p_i)\theta_{n-1}\theta_n - \ldots + p_1 \prod_{i=2}^{n}\theta_i - \prod_{i=1}^{n}\right. \\
\leq &\sum_{i=1}^{n}\left|(\prod_{j=1}^{n-i+1} p_j)(\prod_{j=n-i+2}^{n} \theta_j) - (\prod_{j=1}^{n-i} p_j)(\prod_{j=n-i+1}^{n} \theta_j)\right| \\
= &\sum_{i=1}^{n}(\prod_{j=1}^{n-i} p_j)\ |p_{n-i+1} - \theta_{n-i+1}|\ (\prod_{j=n-i+2}^{n} \theta_j)\ .
\end{aligned}$$

This ends the proof. □

Proof of Theorem 1. In the sequel, we denote by $|A|$ the cardinality of the set A, $I_k = \{I \subseteq \{1, ..., n\} \mid |I| = k\}$ and $I_k^{(i)} = \{I \subseteq \{1, ..., n\} - \{i\} \mid |I| = k\}$, $k = 0, ..., n$. The cases $k = 0$ and $k = n$ are proved by a (single) direct application of Lemma 12 since $|I_0| = |I_n| = 1$. In fact, $|\mathbb{P}(S_n = 0) - \mathbb{P}(B(n,p) = 0)|$ and $|\mathbb{P}(S_n = n) - \mathbb{P}(B(n,p) = n)|$ are bounded by

$$\sum_{i=1}^{n}(\prod_{j=1}^{i-1}(1 - p_i))\ (1-p)^{n-i}\ \epsilon_i(p) = \sum_{i=1}^{n} \epsilon_i(p)\mathbb{P}(S_{n-1}^{(i)} = 0)$$

and

$$\sum_{i=1}^{n}(\prod_{j=1}^{i-1} p_i)\ p^{n-i}\ \epsilon_i(p) = \sum_{i=1}^{n} \epsilon_i(p)\mathbb{P}(S_{n-1}^{(i)} = n-1)\ ,$$

respectively. This ends the proofs of these cases.

For each $k \in \{1, ..., n-1\}$, let us write

$$\{S_n = k\} = \bigcup_{I \in I_k} \{X_i = 1, i \in I;\ X_j = 0, j \in I^c = I_k - I\}\ . \qquad (2)$$

We first notice that $|I_k| = \binom{n}{k}$ and that for each set $I \in I_k$ we have

$$\mathbb{P}(X_i = 1, i \in I\ ; X_j = 0, j \in I^c) = \prod_{i \in I} p_i \prod_{i \in I^c}(1 - p_i)\ ,$$

in such a way that

$$|I\!P(S_n = k) - I\!P(B(n,p) = k)| \le \sum_{I \in I_k} \left| (\prod_{i \in I} p_i) \; (\prod_{i \in I^c} (1 - p_i)) - p^k (1-p)^{n-k} \right| .$$

Applying Lemma 12 to each modulus in the above summation, we bound it by

$$\sum_{I \in I_k} \sum_{i=1}^{n} (\prod_{j=1}^{i-1} f_j) \; \epsilon_i(p) \; (\prod_{j=i+1}^{n} g_j) \, ,$$

where $\epsilon_i(p) = |p_i - p|$,

$$f_j = \begin{cases} p_j & j \in I \\ 1 - p_j & j \in I^c \end{cases} \quad \text{and} \quad g_j = \begin{cases} p & j \in I \\ 1 - p & j \in I^c \end{cases} .$$

Now rearranging the last summation, it yields

$$\begin{aligned} & \sum_{i=1}^{n} \sum_{I \in I_k} (\prod_{j=1}^{i-1} f_j) \; \epsilon_i(p) \; (\prod_{j=i+1}^{n} g_j) \\ = \; & \sum_{i=1}^{n} \epsilon_i(p) [\sum_{I \in I_k - I_k^{(i)}} (\prod_{j=1}^{i-1} f_j)(\prod_{j=i+1}^{n} g_j) \; + \; \sum_{I \in I_k^{(i)}} (\prod_{j=1}^{i-1} f_j)(\prod_{j=i+1}^{n} g_j)] \\ = \; & \sum_{i=1}^{n} \epsilon_i(p) \; [I\!P(S_{n-1}^{(i)} = k-1) + I\!P(S_{n-1}^{(i)} = k)] \, , \end{aligned} \tag{3}$$

as $|I_k^{(i)}| = \binom{n-1}{k}$ for all $i \in \{1, \ldots, n\}$ (and, consequently, $|I_k - I_k^{(i)}| = \binom{n-1}{k-1}$). This ends the proof of the theorem. □

Proof of Corollary 2. Summing (1) over k and rearranging the double series, the result is immediate as $\sum_{k=0}^{n-1} I\!P(S_{n-1}^{(i)} = k) = 1$. □

Proof of Theorem 6. The proof follows the steps of the proof of Theorem 1. For any $k \ge 0$ we have

$$|I\!P(S_n = k) - I\!P(Z = k)| = \left| \sum_{I \in I_k} (\prod_{i \in I} p_i)(\prod_{i \in I^c} (1 - p_i)) - \frac{e^{-\lambda} \lambda^k}{k!} \right| .$$

Considering the multinomial expansion of $\lambda^k = \left(\sum_{i=1}^{n} p_i\right)^k$, we have

$$\frac{e^{-\lambda} \lambda^k}{k!} = \prod_{i=1}^{n} e^{-p_i} \sum_{(a_1, \ldots, a_n) \in S_k} \prod_{i=1}^{n} \frac{p_i^{a_i}}{a_i!}$$

$$= \sum_{(a_1,\ldots,a_n)\in S_k^*} \prod_{i=1}^{n} e^{-p_i} p_i^{a_i} + \sum_{(a_1,\ldots,a_n)\in S_k - S_k^*} \prod_{i=1}^{n} \frac{e^{-p_i} p_i^{a_i}}{a_i!} ,$$

where $S_k = \{(a_1, \ldots, a_n) \in I\!N^n \mid a_1 + \ldots + a_n = k\}$ and $S_k^* = \{(a_1, \ldots, a_n) \in S_k \mid a_i \leq 1, i = 1, \ldots, n\}$. As $|S_k^*| = \binom{n}{k}$, the last equality may be written as

$$\frac{e^{-\lambda}\lambda^k}{k!} = \sum_{I\in I_k} (\prod_{i\in I} p_i e^{-p_i})(\prod_{i\in I^c} e^{-p_i}) + R(k) ,$$

with

$$R(k) = e^{-\lambda} \sum_{(a_1,\ldots,a_n)\in S_k - S_k^*} \prod_{i=1}^{n} \frac{p_i^{a_i}}{a_i!} .$$

By the triangle inequality, we have

$$|I\!P(S_n = k) - I\!P(Z = k)| =$$

$$= \left| \sum_{I\in I_k} (\prod_{i\in I} p_i)(\prod_{i\in I^c}(1-p_i)) - \sum_{I\in I_k} (\prod_{i\in I} p_i e^{-p_i})(\prod_{i\in I^c} e^{-p_i}) - R(k) \right| .$$

$$\leq \sum_{I\in I_k} \left| (\prod_{i\in I} p_i)(\prod_{i\in I^c}(1-p_i)) - (\prod_{i\in I} p_i e^{-p_i})(\prod_{i\in I^c} e^{-p_i}) \right| + R(k)$$

$$\leq \sum_{I\in I_k} \sum_{i=1}^{n} (\prod_{j=1}^{i-1} f_j)\ |f_i - g_i|\ (\prod_{j=i+1}^{n} g_j) + R(k) ,$$

where

$$f_j = \begin{cases} p_j = P(X_j = 1) & j \in I \\ 1 - p_j & j \in I^c \end{cases} \quad \text{and} \quad g_j = \begin{cases} p_j e^{-p_j} = \mu_j(\{1\}) & j \in I \\ e^{-p_j} & j \in I^c \end{cases} .$$

Note that, $\forall p \in [0, 1]$,

$$|p - pe^{-p}| \leq p^2 \quad \text{and} \quad |(1-p) - e^{-p}| \leq \frac{p^2}{2} .$$

Now rearranging the terms of the above summation, we obtain

$$\sum_{i=1}^{n} \sum_{I\in I_k} (\prod_{j=1}^{i-1} f_j)\ |f_i - g_i|\ (\prod_{j=i+1}^{n} g_j) + R(k)$$

$$\leq \sum_{i=1}^{n} p_i^2 \sum_{I\in I_k - I_k^{(i)}} (\prod_{j=1}^{i-1} f_j)(\prod_{j=i+1}^{n} g_j) + \sum_{i=1}^{n} \frac{p_i^2}{2} \sum_{I\in I_k^{(i)}} (\prod_{j=1}^{i-1} f_j)(\prod_{j=i+1}^{n} g_j) + R(k) .$$

But

$$\sum_{I\in I_k^{(i)}} (\prod_{j=1}^{i-1} f_j)(\prod_{j=i+1}^{n} g_j) =$$

$$\sum_{(a_1,\dots,a_{i-1},a_{i+1},\dots,a_n)\in A_{n-1,k}} \left(\prod_{j=1}^{i-1} P(X_j = a_j)\right)\left(\prod_{j=i+1}^{n} \mu_j(\{a_j\})\right) = \mu_n^{(i)}(A_{n-1,k}).$$

Analogously,

$$\sum_{I\in I_k - I_k^{(i)}} \left(\prod_{j=1}^{i-1} f_j\right)\left(\prod_{j=i+1}^{n} g_j\right) = \mu_n^{(i)}(A_{n-1,k-1}).$$

Thus

$$|I\!P(S_n = k) - I\!P(Z = k)| \le$$

$$\sum_{i=1}^{n} p_i^2 \mu_n^{(i)}(A_{n-1,k-1}) + \sum_{i=1}^{n} \frac{p_i^2}{2} \mu_n^{(i)}(A_{n-1,k}) + R(k) .$$

To complete the proof, we need only to set an upper bound for $R(k)$. Clearly, $R(0) = R(1) = 0$ as $S_0 - S_0^* = S_1 - S_1^* = \emptyset$. For $k \ge 2$, we have

$$R(k) = e^{-\lambda} \sum_{(a_1,\dots,a_n)\in S_k - S_k^*} \prod_{i=1}^{n} \frac{p_i^{a_i}}{a_i!} =$$

$$= \frac{e^{-\lambda}\,\lambda^k}{k!} \sum_{(a_1,\dots,a_n)\in S_k - S_k^*} k! \prod_{i=1}^{n} \frac{(\frac{p_i}{\lambda})^{a_i}}{a_i!} = \frac{e^{-\lambda}\,\lambda^k}{k!} P\left(\bigcup_{i=1}^{n} \{W_i \ge 2\}\right) ,$$

where $(W_1, \dots, W_n)$ is a multinomial vector with parameter $(k; (p_1/\lambda, \dots, p_n/\lambda))$. Thus

$$R(k) \le \frac{e^{-\lambda}\,\lambda^k}{k!} \sum_{i=1}^{n} P(W_i \ge 2) .$$

As, for each $i = 1, \dots, n$, W_i and $W_{i1} + \dots + W_{ik}$ are identically distributed, where $W_{i1}, \dots, W_{ik}$ are independent and identically distributed Bernoulli variables with parameter p_i/λ, we may write

$$P(W_i \ge 2) = P\left(\bigcup_{l=1}^{k-1} \bigcup_{m=l+1}^{k} \{W_{il} = W_{im} = 1\}\right)$$

$$\le \sum_{l=1}^{k-1} \sum_{m=l+1}^{k} P(W_{il} = W_{im} = 1) = \sum_{l=1}^{k-1} \sum_{m=l+1}^{k} (\frac{p_i}{\lambda})^2 = \binom{k}{2} (\frac{p_i}{\lambda})^2 .$$

Finally,

$$R(k) \le \frac{e^{-\lambda}\,\lambda^{k-2}}{(k-2)!} \sum_{i=1}^{n} \frac{p_i^2}{2} = \delta(k) ,$$

concluding the proof. □

Proof of Corollary 7. Summing (1) over k, rearranging the double series and noting that $\sum_{k=0}^{n-1} \mu_n^{(i)}(A_{n-1,k}) = \mu_n^{(i)}(\{0,1\}^{n-1}) = \prod_{j=i+1}^{n}(1+p_j)e^{-p_j}$, for $i = 1, ..., n$, and that $\sum_{k=0}^{\infty} \delta(k) = \sum_{i=1}^{n} p_i^2/2$, the result follows. □

Proof of Corollary 8. This is just an application of Theorem 6 with $p_i = p$ for all $i = 1, ..., n$. In this case, S_n and $B(n,p)$ are identically distributed. □

Proof of Corollary 9. Taking $p_i = p$ for all $i = 1, ..., n$ in Corollary 7 and noting that S_n and $B(n,p)$ are both binomial variables with parameters n and P, the result follows. □

Proof of Theorem 10. The theorem results from a single application of Lemma 11 and from the definition of $T^{(i)}$, $i \geq 1$. □

Proof of Corollary 11. The proof is similar to the proof of Corollary 9. □

Acknowledgments. The authors wish to thank Claudia Monteiro Peixoto for fruitful discussions. M. A. also ackowledges partial support from CNPq through grant 308250/2006-0.

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