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A SIMPLE NONPARAMETRIC BAYES SOLUTION TO THE ESTIMATION OF THE SIZE OF A CLOSED ANIMAL POPULATION

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1. Summary

This paper presents a simple nonparametric Bayes estimator of the animal population size under a realistic model which allows capture probabilities to vary by animals, trap response and time. This is a conceptually useful model and the population size is not estimable from the classical viewpoint. The main results are based heavily on Smouse's paper (1984) which requires only the first and second moments of the distributions involved. The dynamic feature of the animal population size is studied via Kalman Filter model.

Finally, some numerical results and comparisons with Jackknife estimators are considered.

2. Introduction

It has long been recognized that a model which allows capture probabilities to vary by animals, trap response and time is a complicated and useful model. However, from the classical approach, there is no estimation procedure associated with this model. For a comprehensive material we suggest Ottis et al (1978) and Burnham and Overton (1978).

The objective of this paper is to present a simple nonparametric Bayes procedure for estimating the animal population size when the three sources of variation operates. The nonparametric approach requires only the first and second moments of the distributions involved. Hartigan (1969) introduced this approach and Bolfarine and Rodrigues (1987) extended it for nonlinear model. In the context of finite populations we have Smouse (1982), O'Haggan (1986) and Rodrigues (1987).

3. General Formulation

Let p_{ij} be the probability of capturing the jth animal in the ith occasion, and define for $j = 1,\ldots,N$; $i = 1,\ldots,k$ the indicator variables

$$(1) \qquad X_{ij} = \begin{cases} 1, & \text{if the jth untagged animal is captured on the ith occasion} \\ 0, & \text{otherwise} \end{cases}$$

In order to gain information about the animal population size N, we introduce the following assumptions:

Assumption 1: The animal population is closed and is of size N.

<u>Assumption 2</u>: The random variables X_{ij} are mutually independent for given p_{ij} and M_i, $j = 1, N-M_i$, $i = 1,\ldots,k$ where $M_i = \sum_{j=1}^{i-1} X_j$ and $X_j = \sum_{\ell=1}^{N-M_j} X_{j\ell}$. Observe from (1) that X_j is the number of untagged animals captured on j-th occasion and M_i is the number of animals seen at least once during the trapping until time i. The basic data is $(X_1,\ldots,X_k)$

<u>Assumption 3</u>: (Prior assumptions)

(i) $p_{ij} \sim F$ (F: unknown distribution) where

(2) $$E_F(p_{ij}) = p \text{ (known)}, \; i = 1,\ldots,k, \; j = 1,\ldots,N-M_i.$$

(ii) $N \sim G$ (G: unknown distribution) where

$$E_G(N) = N_o \text{ (known) and } Var_G(N) = \sigma_N^2 \text{ (known)}.$$

From assumption 2, it is not difficult to see that the $COV[X_i,X_j/M_i,M_j] = 0$, $j \neq i$. As in Hartigan, (1969), let us consider the following nonparametric structure w.r.t. N:

(3)

	<u>Expectation</u>	<u>Weight</u>
<u>Prior</u>	$E_G(N) = N_o$	$Var_G^{-1}(N) = \sigma_N^{-2}$
<u>Data</u>	$E[X_i/M_i,N,p_{ij}] = \sum_{j=1}^{N-M_i} p_{ij}$	$Var^{-1}[X_i/M_i,p_{ij}] = [\sum_{j=1}^{N-M_i} p_{ij}(1-p_{ij})]^{-1}$

Because the above model is very complicated for our purposes, we are going to simplify it by using our prior assumption 3(i). So, after some algebraic manipulations we get the following simple model:

		Expectation	Weight
(4)	Prior	$E_F(N) = N_o$	$Var_F^{-1} = \sigma_N^{-2}$
	Data	$E[X_i^*/M_i, N] = N$	$Var^{-1}[X_i^*/M_i, N] = [(N-M_i)\frac{(1-p)}{p}]^{-1}$,

where $X_i^* = \frac{X_i}{p} + M_i$, $i = 1,\ldots,k$. This model will be denoted by ξ_L.

Definition: Under the model ξ_L, $\hat{N}_L$ is the Bayes linear estimator of N if it minimizes the posterior squared error loss among linear functions of $X' = (X_1,\ldots,X_k)$.

The following result gives the nonparametric estimator of N:

Theorem 1; Under the model ξ_L, the linear Bayes estimator of N is given by

$$(5)\quad \hat{N}_L = \frac{\sum_{i=1}^{k} \frac{p}{(N-M_i)(1-p)}}{\frac{1}{\sigma_N^2} + \sum_{i=1}^{k} \frac{p}{(N_o-M_i)(1-p)}}\bar{N} + \frac{\frac{1}{\sigma_N^2}}{\frac{1}{\sigma_N^2} + \sum_{i=1}^{k} \frac{p}{(N_o-M_i)(1-p)}} N_o,$$

$$\text{where } \bar{N} = \frac{1}{p}\;\frac{\sum_{i=1}^{k} \frac{X_i}{(N_o-M_i)}}{\sum_{i=1}^{k} \frac{1}{(N_o-M_i)}} + \frac{\sum_{i=1}^{k} \frac{M_i}{(N_o-M_i)}}{\sum_{i=1}^{k} \frac{1}{(N_o-M_i)}}, \text{ for } \bar{N} \geq M_{k+1},$$

$$V[\hat{N}_L] = [\frac{1}{\sigma_N^2} + \sum_{i=1}^{k} \frac{p}{(N_o-M_i)(1-p)}]^{-1}.$$

Proof: The result can be easily obtained from Smouse's results (1984) (see also, Brunk, 1984) by taking

$$E[X^*] = 1_k N_o, \quad X^{*\prime} = (X_1^*, \ldots, X_k^*)$$

$$(6) \qquad Cov[X^*] = \frac{1-p}{p} \operatorname{diag}[(N_o-M_1), \ldots, (N_o-M_k)] + \sigma_N^2 J,$$

$$Cov[N, X^*] = \sigma_N^2 1_k', \quad \text{where}$$

$1_k' = (1, \ldots, 1)$ and J is the $k \times k$-matrix of ones.

When we have a noninformative prior for N ($\sigma_N^2 = \infty$) and we do not known p and N_o, it is reasonable to take $\hat{p} = \frac{1}{k}$ and $\hat{N}_{0,i} = M_i + K$, $i = 1, \ldots, k$ for some known constant K. In this particular situation we have from (5) a simple and intuitive estimator for N:

$$(7) \qquad \hat{N}_s = M_{k+1} + \sum_{i=1}^{k} \frac{(K-i)}{k} f_i,$$ where f_i is number of animals captured exactly i times, $i = 1, \ldots, k$.

If p is unknown and we add to the model (3) the prior moments $E(p) = p_o$ (known) and $V(p) = \sigma_p^2$ (known), it is not hard to see from Smouse's results (1984) that the Bayes linear estimator of (N, p) is given by

$$\begin{pmatrix} \hat{N} \\ \hat{p} \end{pmatrix} = \begin{pmatrix} N_o \\ p_o \end{pmatrix} - \begin{bmatrix} \sigma_N^2 p_o & \ldots\ldots & \sigma_N^2 p_o \\ \sigma_p^2 (N_o-M_1) & \ldots & \sigma_p^2 (N_o-M_k) \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ a_{k1} & \cdots & a_{kk} \end{bmatrix}^{-1} \begin{bmatrix} X_1-(N_o-M_1)p_o \\ \vdots \\ X_k-(N_o-M_k)p_o \end{bmatrix}$$

where $a_{ij} = [\sigma_N^2 + N_o^2 - (M_i+M_j)N_o + M_iM_j]\sigma_p^2 + \sigma_N^2 p_o^2$,

for $i,j = 1,\ldots,k$.

4. Numerical Illustration

In this section we apply the proposed estimation procedures to the data of Edwards & Eberhart (1967) on a penned population of 135 wild cottontail rabbits (k = 18) and the data for a population sunfish (k = 14) (Seber, 1973, p.143). Recorded capture frequencies for Edwards & Eberhart's data were:

i	1	2	3	4	5	6	7	8	....	18
f_i	43	16	8	6	0	2	1	0	....	0

The results for the Jackknife estimator $\hat{N}_{JK}$, the interpolated estimator $\hat{N}_J$ (Burnham and Overton, 1978) and the simple Bayes Linear estimation are shown in Table 1:

Table 1: Application of $\hat{N}_{JK}$, $\hat{N}_S$ and $\hat{N}_J$ to the data of Edwards & Eberhardt (1976).

Interpolated Estimator		Jackknife Estimator			Simple Linear Bayes Estimator	
$\hat{N}_J$	Est.st.error	K	$\hat{N}_{JK}$	Est st error	$\hat{N}_S$	$(\text{Post.variance})^{\frac{1}{2}}$
142	15.2	0	76	-	144.2	8.0
		1	116.6	8.9		
		2	141.5	14.9		
		3	158.6	21.9		
		4	170.3	31.1		
		5	176.5	43.5		

Burnham and Overton (1978) using a sequential procedure found that the best estimator is $\hat{N}_J$. As we can see from Table 1 the estimator $\hat{N}_S$ is much better than $\hat{N}_J$ in terms of mean squared error and posterior variance.

Table 2 and 3 show the performance of the estimator $\hat{N}_L$ applied to the sunfish data (Seber, 1973, p.143).

<u>Table 2</u>. $\sigma_N^2 = 10$, $M_{15} = 139$, $k = 14$.

No\p	0.01	0.02	0.03	0.06	0.07	0.10	0.30	0.50	0.70	0.90
140 –	166	164	162	157	156	152	139	139	139	139
150 –	170	168	167	162	161	157	139	139	139	139
160 –	176	175	173	169	168	163	141	139	139	139
170 –	184	182	181	177	175	171	148	139	139	139
180 –	192	191	189	185	184	180	155	139	139	139
190 –	201	199	198	194	192	188	163	140	139	139
200 –	210	208	207	203	201	197	171	146	139	139
210 –	219	217	216	212	210	206	179	152	139	139
220 –	228	227	225	221	220	215	187	159	139	139
230 –	237	236	235	230	229	225	196	166	139	139
240 –	247	245	244	240	238	234	205	174	141	139
250 –	256	255	254	249	248	244	214	181	146	139
260 –	266	265	263	259	258	253	223	189	152	139
270 –	276	274	273	269	267	263	232	197	157	139
280 –	285	284	282	278	277	272	241	205	163	139
290 –	295	294	292	288	286	282	250	213	169	139
300 –	305	303	302	298	296	292	259	221	175	139
310 –	314	313	312	307	306	301	269	230	182	139
320 –	324	323	321	317	316	311	278	238	188	139
330 –	334	333	331	327	325	321	288	247	195	139
340 –	344	342	341	337	335	331	297	255	201	139
350 –	354	352	351	346	345	340	307	264	208	139
360 –	363	362	361	356	355	350	316	273	215	139
370 –	373	372	370	366	365	360	326	281	222	140
380 –	383	382	380	376	374	370	335	290	229	143
390 –	393	392	390	386	384	380	345	299	237	146
400 –	403	401	400	396	394	390	354	308	244	150

Table 3. $\sigma_N^2 = 100$, $M_{15} = 139$, $k = 14$

No\p	0.01	0.02	0.03	0.06	0.07	0.10	0.30	0.50	0.70	0.90
140 -	346	300	268	212	201	176	139	139	139	139
150 -	315	280	254	207	196	173	139	139	139	139
160 -	299	271	249	205	196	174	139	139	139	139
170 -	291	266	247	206	197	175	139	139	139	139
180 -	288	265	247	209	200	178	139	139	139	139
190 -	287	266	249	212	203	182	139	139	139	139
200 -	288	269	252	217	208	186	139	139	139	139
210 -	291	272	257	222	213	191	139	139	139	139
220 -	294	277	262	227	218	196	139	139	139	139
230 -	299	282	267	233	224	202	139	139	139	139
240 -	304	287	273	239	230	208	139	139	139	139
250 -	310	294	280	246	237	214	141	139	139	139
260 -	316	300	286	253	243	220	145	139	139	139
270 -	322	307	293	260	250	227	148	139	139	139
280 -	329	314	301	267	258	234	152	139	139	139
290 -	337	322	308	274	265	241	156	139	139	139
300 -	344	329	316	282	272	248	160	139	139	139
310 -	352	337	324	290	280	255	165	139	139	139
320 -	359	345	332	298	288	263	169	139	139	139
330 -	367	353	340	306	296	270	174	139	139	139
340 -	376	361	348	314	304	278	179	139	139	139
350 -	384	370	357	322	312	286	184	139	139	139
360 -	392	378	365	331	320	294	189	139	139	139
370 -	401	387	374	339	329	302	194	142	139	139
380 -	409	395	382	347	337	310	199	145	139	139
390 -	418	404	391	356	346	318	205	149	139	139
400 -	427	413	400	365	354	326	210	152	139	139

5. The Recursive Estimation Procedure

In this section the animal population is not assumed to be closed, as in Section 3, but may change with time according to the following Kalman Filter model:

Observation equation: $E[\underline{X}_t^*/N_t] = 1_k N_t$,

(8)

$$V[X_t^*/N_t] = b_t V(N_t).$$

System equation: $E[N_t/N_{t-1}] = N_{t-1}$,

$$V[N_t/N_{t-1}] = \sigma_t^2 \quad \text{(known)}$$

$t = 1,\dots,T$, where

$$X_t^{*'} = (X_{t1}^*,\dots,X_{tk}), \; X_{ti}^* = \frac{1}{p_t} X_{ti} + M_{ti},$$

$$i = 1,\dots,k, \quad b_t = \frac{(1-p_t)}{p_t} \quad \text{and } V(N_t) = \text{diag}((N_t-M_{t1}),\dots,(N_t-M_{tk})).$$

It is helpful to remark here that the recursive procedure is started off at time 1 by choosing N_o and σ_1^2 to be our prior known moments about N_1. Let $\underset{\sim}{X}_t^* = (X_t^*, X_{t-1}^*,\dots,X_1^*,N_o)$ the observed data up to time t. Our aim is to make inference about N_t given $\underset{\sim}{X}_t^*$. In order to obtain our recursive procedure we assume at time t-1 that the knowledge about N_{t-1} is given by the linear expectation of N_{t-1} given $\underset{\sim}{X}_{t-1}^*$, N_o and is written $LE(N_{t-1}/\underset{\sim}{X}_{t-1}^*, N_o) = \hat{N}_{t-1}$ (see Brunk , 1980; Smouse, 1984). The linear variance of N_{t-1} is $S_{t-1} = LV(N_{t-1}/X_{t-1}^*, N_o)$. We look forward to time t, but in two stages:

<u>Stage 1</u>: Assuming that X_{t-1}^* is independent of N_t (given N_{t-1}), we have that

(10) $$LE[N_t/\underset{\sim}{X}_{t-1}^*, N_o] = \hat{N}_{t-1} \quad \text{and} \quad LV[N_t/\underset{\sim}{X}_{t-1}^*, N_o] = S_{t/t-1} = S_{t-1} + \sigma_t^2, \quad \text{where}$$

$S_{1/0} = \sigma_1^2$ and $LE[N_1/N_o] = N_o$. So, prior to observing X_t^*, our best choice for N_t is given by $(N_{t-1}, S_{t/t-1})$.

<u>Stage 2</u>: Upon observing X_t^*, our goal is to compute $\hat{N}_t = LE(N_t/N_o, X_t^*)$

and $LV(N_t/N_t^*, N_o) = S_t$ such that $\hat{N}_t$ has the smallest posterior mean squared error loss among linear functions of X_t^*. Again, applying Smouse's results (1984) we find that

$$(11)\ \hat{N}_t = \frac{1_k' b_t^{-1} V^{-1}(\hat{N}_{t-1}) X_t^*}{\frac{1}{S_{t/t-1}} + b_t^{-1} 1_k' V^{-1}(\hat{N}_{t-1}) 1_k} + \left(1 - \frac{b_t^{-1} 1_k' V^{-1}(\hat{N}_{t-1}) 1_k}{\frac{1}{S_{t/t-1}} + b_t^{-1} 1_k' V^{-1}(\hat{N}_{t-1}) 1_k}\right) \hat{N}_{t-1}$$

$$S_t = S_{t/t-1} \left(1 - \frac{b_t^{-1} 1_k' V^{-1}(\hat{N}_{t-1}) 1_k}{\frac{1}{S_{t/t-1}} + b_t^{-1} 1_k' V^{-1}(\hat{N}_{t-1}) 1_k}\right).$$

It is interesting to see that $\hat{N}_t$ is similar to $\hat{N}_L(5)$ at time t.

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