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MEMORIAL VOLUME FOR SASHA ANAN'IN

Alternative M_2-algebras and Γ-algebras

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Abstract
Recently V. H. López Solís and I. Shestakov [9] solved an old problem by N.Jacobson [2] on describing of unital alternative algebras containing the matrix 2×2 algebra M_2 as a unital subalgebra. Here we give another description of M_2-algebras via the 6-dimensional alternative superalgebra $B(4, 2)$ and an auxiliar Z_2-graded algebra Γ. It occurs that the category of alternative M_2-algebras is isomorphic to the category of Γ -algebras. We describe also the free Γ-algebras and construct their bases.

Keywords Alternative algebra · Coordinatization theorem · Category of M_2 -algebras · Jordan superalgebra · Grassmanian

1 Introduction

The classical Wedderburn Coordinatization Theorem says that if a unital associative algebra A contains a matrix algebra $M_n(F)$ over a field F with the same identity element then it is itself a matrix algebra, $A \cong M_n(D)$, "coordinated" by D. Generalizations and analogues of this theorem were proved for various classes of algebras and superalgebras [2, 4, 7, 8, 10–12, 14, 15]. The common content of all these results is that if an algebra (or superalgebra) contains a certain subalgebra (matrix algebra, octonions, Albert algebra) with the same unit then the algebra itself has the same structure, but not over the basic field rather over a certain algebra that "coordinatizes" it. The Coordinatization Theorems play important role in structure theories, especially in classification theorems, and also in the representation theory, since quite often an algebra A coordinated by D is Morita equivalent to D, though they could belong to different classes (for instance, Jordan algebras are coordinated by associative and alternative algebras).

To the memory of our dear friend Sasha Anan'in.

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I. Kaplansky [4] proved an analogue of Wedderburn's theorem for alternative algebras containing the Cayley algebra. He showed that if A is an alternative algebra with identity element 1 which contains a subalgebra B isomorphic to a Cayley algebra and if 1 is contained in B, then A is isomorphic to the Kronecker product $B \otimes T$, where T is the center of A.

The Wedderburn coordinatization theorem in the case $n \geq 3$ admits a generalization for alternative algebras, since every alternative algebra A which contains a subalgebra $M_n(F)$ ($n \geq 3$) with the same identity element is associative (see [15, Corollary 11, Chapter 2]). The result is not true for $n = 2$, the split Cayley algebra and its 6-dimensional subalgebra are counterexamples. The problem of description of alternative algebras containing $M_2(F)$ or, more generally, a generalized quaternion algebra $\mathbb{H}$ with the same identity element was posed by Jacobson [2].

In [9], this problem was solved for the split case $\mathbb{H} \cong M_2(F)$. The corresponding $M_2(F)$-coordinatization in [9] involves two ingredients: an associative algebra D and a commutative D-bimodule V (that is, V is annihilated by any commutator of elements of D), on which a skew-symmetric mapping is defined with values in the center of D, satisfying Plücker relations. More exactly, $A = M_2(D) \oplus V^2$, with a properly defined multiplication.

Here we give another characterization of $M_2(F)$-algebras, based on the 6-dimensional simple alternative superalgebra $B = B(4,2)$ [17] and an auxiliar Z_2-graded algebra Γ: a unital alternative algebra A is an $M_2(F)$-algebra if and only if A is a Γ *-envelope* of the superalgebra B: $A = \Gamma_0 \otimes B_0 + \Gamma_1 \otimes B_1$. Moreover, the category of alternative M_2-algebras is isomorphic to the category of Γ-algebras. We describe the free Γ-algebras and construct their bases. It occurs that these algebras are closely related to coordinate algebras of grassmannians $Gr(2, n)$.

Throughout this paper the ground field F is of arbitrary characteristic.

2 Definitions, examples, and preliminary results

Let A be a composition algebra (see [3, 6, 15, 18]). Recall that A is a unital alternative algebra, it has an involution $a \mapsto \bar{a}$ such that the *trace* $t(a) = a + \bar{a}$ and *norm* $n(a) = a\bar{a}$ lie in F.

An alternative bimodule V over a composition algebra A is called a *Cayley bimodule* if it satisfies the relation

$$av = v\bar{a}, \tag{1}$$

where $a \in A$, $v \in V$, and $a \to \bar{a}$ is the canonical involution in A.

Typical examples of composition algebras are the algebras of (generalized) quaternions $\mathbb{H}$ and octonions $\mathbb{O}$ (or *a Cayley algebra*) with symplectic involutions. Recall that $\mathbb{O} = \mathbb{H} \oplus v\mathbb{H}$, with the product defined by

$$a \cdot b = ab,\ a \cdot vb = v(\bar{a}b),\ vb \cdot a = v(ab),\ va \cdot vb = (b\bar{a})v^2, \tag{2}$$

where $a, b \in \mathbb{H},\ 0 \neq v^2 \in F,\ a \mapsto \bar{a}$ is the symplectic involution in $\mathbb{H}$.

The subspace $v\mathbb{H} \subset \mathbb{O}$ is invariant under multiplication by elements of $\mathbb{H}$ and it gives an example of a Cayley bimodule over $\mathbb{H}$. If $\mathbb{H}$ is a division algebra then $v\mathbb{H}$ is irreducible, otherwise $\mathbb{H} \cong M_2(F)$ and

$$v\mathbb{H} = \langle ve_{22}, -ve_{12}\rangle \oplus \langle -ve_{21}, ve_{11}\rangle,$$

where $M_2(F)$-bimodules $\langle ve_{22}, -ve_{12}\rangle$ and $\langle -ve_{21}, ve_{11}\rangle$ are both isomorphic to the 2-dimensional Cayley bimodule $\mathrm{Cay} = F\cdot m_1 + F\cdot m_2$, with the action of $M_2(F)$ given by

$$e_{ij}\cdot m_k = \delta_{ik} m_j, \quad m\cdot a = \bar{a}\cdot m, \tag{3}$$

where $a \in M_2(F)$, $m \in \mathrm{Cay}$, $i, j, k \in \{1, 2\}$ and $a \mapsto \bar{a}$ is the symplectic involution in $M_2(F)$. In the last case the algebra $\mathbb{O} = M_2(F) \oplus vM_2(F)$ is called *the split octonion algebra*.

Let us call a unital alternative algebra A an M_2-*algebra* if there exists a homomorphism of unital algebras $\phi : M_2 \to A$, where $M_2 = M_2(F)$.

Examples of M_2 -algebras:

1. A associative, $A \supseteq M_2 \ni 1_A \Rightarrow A = M_2(B) \cong M_2 \otimes B,\ B$ associative.
2. $\mathbb{O}$ split octonion algebra, $\mathbb{O} = M_2 \oplus (M_2)v$.
3. $S = M_2(F) \oplus \mathrm{Cay} \subseteq \mathbb{O},\ \mathrm{Cay}^2 = 0$, the split null extension of M_2 by bimodule Cay .
4. $G(B(4,2)) = G_0 \otimes M_2 + G_1 \otimes \mathrm{Cay},\ char\, F = 3$, the Grassmann envelope of the simple alternative superalgebra $B(4,2) = M_2 \oplus \mathrm{Cay}$ (see [17]), with the following multiplication in Cay:

$$m_1^2 = e_{21},\ m_2^2 = -e_{12},\ m_1 m_2 = -e_{11},\ m_2 m_1 = e_{22}.$$

Remark 1 The odd product in $B(4, 2)$ in [17] has different sign; one can get the old product by the following change of the basis:

$$e_{21} \leftrightarrow -e_{21}.\ e_{12} \leftrightarrow -e_{12},\ m_2 \leftrightarrow -m_2.$$

Any M_2-algebra A may be considered as a unital alternative M_2-bimodule. The structure of such bimodules is given by the following result:

Theorem 1 [3, 17] *Let V be a unital alternative M_2-bimodule. Then V is completely reducible, moreover, $V = V_a \oplus V_c$, where V_a is an associative M_2-bimodule and is a direct some of regular bimodules* Reg M_2, *while V_c is a Cayley M_2-bimodule which is a direct sum of irreducible Cayley bimodules of type* Cay .

In particular, any M_2-algebra A may be written as $A = A_a \oplus A_c$. It was proved in [9] that

$$A_c A_a + A_a A_c \subseteq A_c,\ A_a A_a \subseteq A_a,\ A_c A_c \subseteq A_a.$$

That is, A is a Z_2-graded algebra. We have $A_a = (\text{Reg } M_2)^n,\ A_c = (\text{Cay })^m$. Moreover, the subalgebra A_a is associative [9].

The $M_2(F)$-coordinatization in [9] involves two ingredients: an alternative $M_2(F)$ -algebra A is "coordinated" by an associative algebra D and by a commutative D-bimodule V (that is, V is annihilated by any commutator of elements of D), on which a skew-symmetric form is defined with values in the center of D, satisfying Plücker relations. More exactly,

$$A = M_2(D) \oplus V^2,$$

with a properly defined multiplication.

In more details, let D be an associative unital algebra and V be a left D-module such that $[D, D]$ annihilates V. Clearly, in this case V has a structure of a commutative D-bimodule with $v \cdot a = a \cdot v,\ v \in V,\ a \in D$. Assume that there exists a D-bilinear skew-symmetric mapping $\langle , \rangle : V^2 \to D$ such that $\langle V, V\rangle \subseteq Z(D)$ and for any $u, v, w \in V$

$$\langle u, v\rangle w + \langle v, w\rangle u + \langle w, v\rangle u = 0. \tag{4}$$

Consider $A = M_2(D) \oplus V^2$. Let $X, Y \in A,\ X = X_a + (x, y),\ Y = Y_a + (z, t)$, where $X_a,\ Y_a \in M_2(D)$ and $(x, y), (z, t) \in V^2$. Define a product in A by formula:

$$XY = X_aY_a + \begin{pmatrix} -\langle x, t\rangle & -\langle y, t\rangle \\ \langle x, z\rangle & \langle y, z\rangle \end{pmatrix} + (z, t)X_a + (x, y)(Y_a)^*,$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Theorem 2 [9] *The algebra A with the product defined above is an alternative unital algebra containing $M_2(F)$ with the same identity element. Conversely, every unital alternative algebra that contains the matrix algebra $M_2(F)$ with the same identity element has this form.*

3 The new construction

Consider the vector space direct sum $\Gamma = D \oplus V$, where D and V are taken from theorem 2, and define a multiplication on it as follows

$$(a + u)(b + v) = (ab + \langle u, v\rangle) + (av + bu).$$

Then Γ becomes a Z_2-graded algebra with $\Gamma_0 = D,\ \Gamma_1 = V$, that satisfied the following conditions:

(i) Γ_0 is a unital associative algebra, $[\Gamma_0, \Gamma_1] = (\Gamma_0, \Gamma, \Gamma) = 0$,
(ii) $\Gamma_1^2 \subseteq Z(\Gamma_0)$,
(iii) $xy + yx = 0,\ x, y \in \Gamma_1$,

(iv) $(xy)z + (yz)x + (zx)y = 0,\ x, y, z \in \Gamma_1$.

Theorem 3 *The "Γ-envelope" $\Gamma(B(4,2)) = \Gamma_0 \otimes M_2 + \Gamma_1 \otimes Cay$ is isomorphic to the algebra $M_2(\Gamma_0) \oplus \Gamma_1^2$ from theorem 2. In particular, an algebra A is an alternative M_2-algebra if and only if $A = \Gamma(B(4,2))$ for a certain Z_2-graded algebra Γ satisfying the above conditions.*

Remark 2 Note that the superalgebra $B(4, 2)$ is alternative only in $char\, F = 3$ case, but the theorem holds in any characteristic.

Proof of the theorem. It is clear that $\Gamma_0 \otimes M_2 \cong M_2(\Gamma_0)$. Let us prove that the mapping

$$\varphi : X + (x, y) \mapsto X + x \otimes m_1 + y \otimes m_2,\ X \in M_2(\Gamma_0),\ x, y \in \Gamma_1,$$

is an isomorphism of $M_2(\Gamma_0) \oplus \Gamma_1^2$ and $\Gamma(B(4,2))$. Let $X = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}, d_{ij} \in \Gamma_0$. Consider

$$\begin{aligned}\varphi(X \cdot (x,y)) =& \varphi(xd_{11} + yd_{21}, xd_{12} + yd_{22})\\ =&(xd_{11} + yd_{21}) \otimes m_1 + (xd_{12} + yd_{22}) \otimes m_2.\end{aligned}$$

On the other hand,

$$\begin{aligned}\varphi(X)\varphi(x,y) =& X(x \otimes m_1 + y \otimes m_2) = (\sum_{ij} d_{ij} \otimes e_{ij})(x \otimes m_1 + y \otimes n)\\ =&(d_{11}x + d_{21}y) \otimes m_1 + (d_{12}x + d_{22}y) \otimes m_2.\end{aligned}$$

Since $[\Gamma_0, \Gamma_1] = 0$, we have $\varphi(X \cdot (x, y)) = \varphi(X)\varphi(x, y)$. Similarly, $\varphi((x, y) \cdot X) = \varphi(x, y)\varphi(X)$. Let now $z, t \in \Gamma_1$, consider

$$\varphi((x,y)(z,t)) = \varphi\begin{pmatrix} -\langle x,t\rangle & -\langle y,t\rangle \\ \langle x,z\rangle & \langle y,z\rangle \end{pmatrix} = \begin{pmatrix} -\langle x,t\rangle & -\langle y,t\rangle \\ \langle x,z\rangle & \langle y,z\rangle \end{pmatrix}.$$

On the other hand,

$$\begin{aligned}\varphi(x,y)\varphi(z,t) =& (x \otimes m_1 + y \otimes n)(z \otimes m_1 + t \otimes m_2)\\ =&\langle x,z\rangle \otimes e_{21} - \langle x,t\rangle \otimes e_{11} + \langle y,z\rangle e_{22} - \langle y,t\rangle \otimes e_{12}\\ =&\begin{pmatrix} -\langle x,t\rangle & -\langle y,t\rangle \\ \langle x,z\rangle & \langle y,z\rangle \end{pmatrix}.\end{aligned}$$

This proves the theorem. □

Let us consider the examples of M_2-algebras from section 2 and determine the structure of the corresponding algebra Γ in every case.

1. A associative, $A \supseteq M_2 \ni 1_A \Rightarrow A = M_2(B) \cong M_2 \otimes B$, B associative. In this case $\Gamma = \Gamma_0 = B$.
2. $\mathbb{O}$ split octonion algebra, $\mathbb{O} = M_2 \oplus (M_2)v$. Here $\Gamma = B(1,2)$, the 3-dimensional simple superalgebra from [17] which is alternative in characteristic 3 case. $B(1,2)_0 = F$, $B(1,2)_1 = Fx + Fy$, $x^2 = y^2 = 0, xy = -yx = 1$.
3. $S = M_2(F) \oplus \mathrm{Cay} \subseteq \mathbb{O}$, $\mathrm{Cay}^2 = 0$, the split null extension of M_2 by bimodule Cay. In this case $\Gamma = F + Fx$, $x^2 = 0$, is the 2-dimensional algebra with $\Gamma_0 = F$, $\Gamma_1 = Fx$.
4. $G(B(4,2)) = G_0 \otimes M_2 + G_1 \otimes \mathrm{Cay}$, *char* $F = 3$, the Grassmann envelope of the simple alternative superalgebra $B(4,2)$. Here $\Gamma = G$ since the Grassmann algebra G satisfies the conditions for Γ in the case of characteristic 3.

4 Tensor algebras of bimodules and free Γ-algebras

Recall the definition of a tensor algebra of bimodule (see [5]). Let A be an algebra in a variety $\mathcal{M}$ and let V be an $\mathcal{M}$-bimodule over A. Consider the free $\mathcal{M}$-algebra $F_{\mathcal{M}}[A \oplus V]$ and let I be the ideal of this algebra generated by the set $\{a * b - ab,\ a * v - a \cdot v,\ v * a - v \cdot a \mid a, b \in A,\ v \in V\}$, where $*$ denotes the multiplication in $F_{\mathcal{M}}[A \oplus V]$ and $a \cdot v$, $v \cdot a$ denote the action of A on V. Then the quotient algebra $F_{\mathcal{M}}[A \oplus V]/I$ is called *the tensor algebra of the A-bimodule V*.

By the standard arguments, one can prove the following universal property of tensor algebra.

Proposition 1 *Let $B \in \mathcal{M}$ and let $\varphi : A \to B$ be a homomorphism of algebras. Then B has a natural structure of an A-bimodule. Now, for any homomorphism of A-bimodules $\psi : V \to B$ there exists a unique homomorphism of algebras $\tilde{\psi} : F_{\mathcal{M}}[A \oplus V] \to B$ such that $\tilde{\psi}(a) = \varphi(a)$, $\tilde{\psi}(v) = \psi(v)$ for any $a \in A$, $v \in V$.*

In particular, the tensor algebra $M_2[V]$ of an alternative M_2-bimodule V plays a role of a free object in the category of alternative M_2-algebras: for any M_2-algebra B, any homomorphism of M_2-bimodules $\varphi : V \to B$ is uniquely extended to an algebra homomorphism $\tilde{\varphi} : M_2[V] \to B$.

Let us call a Z_2-graded algebra satisfying conditions (i)–(iv) of section 3 a Γ *-algebra*. We want to prove

Theorem 4 *The category of alternative M_2-algebras is isomorphic to the category of Γ-algebras.*

Proof There is a natural functor from the category of Γ-algebras to the category of M_2-algebras: $F : \Gamma \to \Gamma(B(4,2))$, which sends a morphism of Γ-algebras $\varphi : \Gamma \to \Gamma'$ to the morphism $F(\varphi) : \Gamma(B(4,2)) \to \Gamma'(B(4,2))$ identical on $B(4,2)$.

It is clear from the proof of theorem 3 that this functor is bijective on objects: every M_2-algebra $A = M_2(D) \oplus V^2$ defines uniquely $\Gamma_0 = D$ and $\Gamma_1 = V$. It remains to show that any morphism $\varphi : \Gamma(B(4,2)) \to \Gamma'(B(4,2))$ is induced by a morphism $\psi : \Gamma \to \Gamma'$ such that $\varphi = F(\psi)$. Denote $A = \Gamma(B(4,2))$, $A' = \Gamma'(B(4,2))$. Since φ is identical on M_2, it is a homomorphism of M_2-bimodules; in particular, $\varphi(A_a) = \varphi(M_2(\Gamma_0)) \subset A'_a = M_2(\Gamma'_0)$ and $\varphi(A_c) = \varphi(\Gamma_1 \otimes \mathrm{Cay}) \subset (A')_c = \Gamma'_1 \otimes \mathrm{Cay}$. It is well known that a homomorphism of matrix algebras is induced by a homomorphism of their coordinates, hence there exists a homomorphism $\psi_0 : \Gamma_0 \to \Gamma'_0$ which induces $\varphi|_{A_a}$.

Now, fix $0 \neq \gamma \in \Gamma_1$. Let $\varphi(\gamma \otimes m_1) = \alpha \otimes m_1 + \beta \otimes m_2$, $\varphi(\gamma \otimes m_2) = \lambda \otimes m_1 + \mu \otimes m_2$ for some $\alpha, \beta, \lambda, \mu \in \Gamma'_1$. We have

$$\begin{aligned}\alpha \otimes m_1 + \beta \otimes m_2 =& \varphi(\gamma \otimes m_1) = \varphi((1 \otimes e_{11})(\gamma \otimes m_1)) \\ =& (1 \otimes e_{11})\varphi(\gamma \otimes m_1) = \alpha \otimes m_1,\end{aligned}$$

which implies $\beta = 0$. Similarly, $\lambda = 0$. Futhermore,

$$\begin{aligned}\mu \otimes m_2 =& \varphi(\gamma \otimes m_2) = \varphi((1 \otimes e_{12})(\gamma \otimes m_1)) \\ =& (1 \otimes e_{12})\varphi(\gamma \otimes m_1) = \alpha \otimes m_2,\end{aligned}$$

which implies $\alpha = \mu$. Therefore, $\varphi(\gamma \otimes \mathrm{Cay}) = \alpha \otimes \mathrm{Cay}$, and we put $\psi_1(\gamma) = \alpha$. One can easily check that $\psi = \psi_0 + \psi_1 : \Gamma \to \Gamma'$ is a homomorphism of algebras such that $F(\psi) = \varphi$. ☐

Corollary 1 *Let $\Gamma[X_0, X_1]$ be the free Γ-algebra on sets X_0 and X_1 of even and odd generators. Then $\Gamma[X_0, X_1](B(4,2)) \cong M_2[V]$, where $V = Reg^{\#X_0} \oplus \mathrm{Cay}^{\#X_1}$.*

In view of the Corollary, it seems important to determine the structure of free Γ -algebras.

Let $V = F^n$ be an n-dimensional vector space over F and $F[Gr(2, V)]$ be the coordinate algebra of the Grassmannian $Gr(2,n) = Gr(2, V)$. Recall that $F[Gr(2, V)] \cong F[V^{\wedge 2}]/P$, where P is the ideal generated by the *double Plücker relations*

$$(u \wedge v)(w \wedge z) + (u \wedge w)(z \wedge v) + (u \wedge z)(v \wedge w),\ u, v, w, z \in V.$$

Furthermore, consider the tensor product $F[Gr(2, V)] \otimes V$ which has a natural structure of $F[Gr(2, V)]$-module. Denote by $F[Gr(2, V)]_1$ the quotient module $(F[Gr(2, V)] \otimes V)/I$ where I is the $F[Gr(2, V)]$-submodule generated by the *ordirnary Plücker relations*

$$(u \wedge v) \otimes w + (v \wedge w) \otimes u + (w \wedge u) \otimes v,\ u, v, w \in V.$$

Define multiplication in $F[Gr(2, V)]_1$ with results in $F[Gr(2, V)]$ by setting

$$(a \otimes u)(b \otimes v) = ab(u \wedge v) \otimes 1,$$

where $a, b \in F[Gr(2, V)],\ u, v \in V$. The product is defined correctly, since

$$((u \wedge v) \otimes w + (v \wedge w) \otimes u + (w \wedge u) \otimes v)(a \otimes z) =$$
$$a((u \wedge v)(w \wedge z) + (v \wedge w)(u \wedge z) + (w \wedge u)(v \wedge z)) \otimes 1 = 0.$$

Theorem 5 *The Z_2-graded algebra $\Gamma[\emptyset;V] = F[Gr(2,V)] + F[Gr(2,V)]_1$ is a free Γ-algebra generated by the space of odd generators V.*

Proof In fact, it is easy to check that the unital algebra $\Gamma[\emptyset;V]$ satisfies conditions (i)–(iv) defining Γ-algebras, and is generated by the space V. Let $\Gamma = \Gamma_0 + \Gamma_1$ be a Γ-algebra and $\varphi : V \to \Gamma_1$ a linear mapping. For $v \in V$ denote by $\bar{v}$ its image in Γ_1. The mapping φ is extended to a linear mapping $V^{\wedge 2} \to \Gamma_0$, $u \wedge v \mapsto \bar{u}\bar{v}$ and further to an algebra homomorphism $F[V^{\wedge 2}] \to \Gamma_0$. In view of condition (iv) the ordinary Plücker relation holds in Γ. Moreover, for any $u, v, w, z \in V$ we have in Γ

$$(\bar{u}\bar{v})(\bar{w}\bar{z}) + (\bar{u}\bar{w})(\bar{z}\bar{v}) + (\bar{u}\bar{z})(\bar{v}\bar{w}) = \bar{u}(\bar{v}\bar{w}\bar{z} + \bar{w}\bar{z}\bar{v}) + \bar{z}\bar{v}\bar{w}) = 0.$$

Therefore, the mapping φ can be extended to an algebra homomorphism $\tilde{\varphi} : \Gamma[\emptyset;V] \to \Gamma$. □

Let now U be another vector space, construct the free Γ-algebra $\Gamma[U;V]$ generated by the space U of even generators and the space V of odd generators. Denote by $F\langle U\rangle$ and by $F[U]$ the free associative and polynomial algebras over the space U. Furthermore, by $F[Gr(2,V)]^0$ denote the augmentation ideal of the algebra $F[Gr(2, V)]$, that is, the ideal of elements without scalar terms. Consider the Z_2-graded vector space

$$\Gamma[U;V] = (F\langle U\rangle \oplus (F[U] \otimes F[Gr(2,V)]^0)) \oplus (F[U] \otimes F[Gr(2,V)]_1),$$

with $\Gamma[U;V]_0 = F\langle U\rangle \oplus (F[U] \otimes F[Gr(2,V)]^0)$ and $\Gamma[U;V]_1 = F[U] \otimes F[Gr(2,V)]_1$. Observe that

$$I = (F[U] \otimes F[Gr(2,V)]^0) \oplus (F[U] \otimes F[Gr(2,V)]_1) =$$
$$F[U] \otimes (F[Gr(2,V)]^0 + F[Gr(2,V)]_1) \subseteq F[U] \otimes \Gamma[\emptyset;V]$$

Define multiplication on $\Gamma[U;V]$ in the following way: the space I is an ideal of $\Gamma[U;V]$ with the product defined as in a subalgebra of the algebra $F[U] \otimes \Gamma[\emptyset;V]$; the algebra $F\langle U\rangle$ is a subalgebra of $\Gamma[U;V]$, and the element $f \in F\langle U\rangle$ acts on I by

$$f \cdot (g \otimes a + h \otimes \ b \otimes v) = \bar{f}g \otimes a + \bar{f}h \otimes \ b \otimes v,$$

where $g, h \in F[U]$, $a, b \in F[Gr(2,V)]$, $v \in V$, and $\bar{f} \in F[U]$ is the image of f under the natural epimorphism $F\langle U\rangle \to F[U]$.

Theorem 6 *The algebra $\Gamma[U;V]$ with the multiplication defined above is a free Γ-algebra generated by the spaces U and V of even and odd generators.*

Proof First of all, one can easily check that the algebra $\Gamma[U;V]$ satisfies conditions (i)–(iv). Furthermore, let $\Gamma = \Gamma_0 + \Gamma_1$ be a Γ-algebra and $\varphi : U \to \Gamma_0$, $\psi : V \to \Gamma_1$ be linear mappings. By above, ψ can be extended to an algebra homomorphism $\tilde{\psi} : \Gamma[\emptyset;V] \to \Gamma$. By the property of free algebras, there exists also an algebra homomorphism $\tilde{\varphi} : F\langle U\rangle \to \Gamma_0$ extending φ. Now the mapping

$$(\tilde{\varphi} + \tilde{\psi}) : f + (h \otimes a + g \otimes b \otimes v) \mapsto \tilde{\varphi}(f) + \tilde{\varphi}(h)\tilde{\psi}(a) + \tilde{\varphi}(g)\tilde{\psi}(b)\psi(v)$$

for $f \in F\langle U\rangle$, $g, h \in F[U]$, $a, b \in F[Gr(2, V)]$ and $v \in V$ is an algebra homomorphism of $\Gamma[U;V]$ to Γ extending $\varphi + \psi$. □

5 Γ-algebras and Jordan superalgebras

Observe that if Γ is a Γ-algebra with commutative even part Γ_0 then Γ is a commutative superalgebra. Moreover, in this case it is a Jordan superalgebra.

Proposition 2 *Let Γ is a Γ-algebra with commutative even part Γ_0. Then Γ is a Jordan superalgebra. Moreover, if $char\, F = 3$ then Γ is an alternative superalgebra.*

Proof Recall that a commutative superalgebra is called a Jordan superalgebra if it satisfies the super-identity

$$(xy, z, t) + (-1)^{\bar{y}\bar{z}+\bar{y}\bar{t}+\bar{z}\bar{t}}(xt, z, y) + (-1)^{\bar{x}(\bar{y}+\bar{z}+\bar{t})+\bar{z}\bar{t}}(yt, z, x) = 0, \tag{5}$$

where $\bar{x}$ for $x \in \Gamma_0 \cup \Gamma_1$ denotes the parity of element x: $\bar{x} = i \Leftrightarrow x \in \Gamma_i$. Note that $[\Gamma_0, \Gamma] = 0$, hence Γ_0 is contained in the center of Γ. Therefore, if at least 2 elements of x, y, t lie in Γ_0 or $z \in \Gamma_0$, all the associators in identity (5) vanish. If all the elements x, y, z, t are in Γ_1 then xy, xt, $yt \in \Gamma_0$, and again (5) holds. Therefore, it suffices to consider the case when $x, y, z \in \Gamma_1$, $t = a \in \Gamma_0$. We have

$$\begin{aligned} &(xy, z, a) - (xa, z, y) + (ya, z, x) = -(xa, z, y) + (ya, z, x) \\ =&a(-(x, z, y) + (y, z, x)) = a(-xz \cdot y + x \cdot zy + yz \cdot x - y \cdot zx) \\ =&a(-(xz + zx)y + x(zy + yz)) = 0. \end{aligned}$$

Furthermore, let $char\, F = 3$. Since $\Gamma_0 \subseteq Z(\Gamma)$, in order to check alternativity we have to consider only associators on odd generators. Let $x, y, z \in \Gamma_1$, then we have

$$\begin{aligned} (x, y, z) - (x, z, y) =& xy \cdot z - x \cdot yz - xz \cdot y + x \cdot zy = xy \cdot z - 2yz \cdot x + zx \cdot y \\ =& xy \cdot z + yz \cdot x + zx \cdot y = 0, \end{aligned}$$

and similarly $(x, y, z) - (y, x, z) = 0$, hence the superalgebra Γ is alternative. □

An important example of supercommutative Γ-algebras can be obtained as follows. Let A be a unital commutative associative algebra, consider $\Gamma(A) = A \oplus A^2$ with the grading $\Gamma(A)_0 = A,\ \Gamma(A)_1 = A^2$ and the following multiplication:

$$a \cdot b = ab,\ a \cdot (b,c) = (b,c) \cdot a = (ab, ac),\ (a,b) \cdot (c,d) = ad - bc;\ a,b,c,d \in A.$$

We have only to check condition (iv) in the definition of Γ-algebra. Consider

$$\begin{aligned}&(a,b)(c,d) \cdot (e,f) + (c,d)(e,f) \cdot (a,b) + (e,f)(a,b) \cdot (c,d)\\ &= (ad - bc)(e,f) + (cf - de)(a,b) + (eb - fa)(c,d) = (0,0).\end{aligned}$$

Therefore, $\Gamma(A)$ is a Γ-algebra. Since A is commutative, $\Gamma(A)$ is a Jordan superalgebra.

Proposition 3 *Let A be a domain, then $\Gamma(A)$ is a central order in the simple Jordan superalgebra of type $B(1,2)$ (the superalgebra of a skew-symmetric bilinear form on a 2-dimensional vector space). In particular, in this case $\Gamma(A)$ is prime and special.*

Proof In fact, let K be the quotient field of A, then we have an inclusion $\Gamma(A) \subseteq \Gamma(K)$; moreover, since $K = A^{-1}A$ and $A = Z(\Gamma(A))$, we have $A^{-1}\Gamma(A) = \Gamma(K)$. It is clear that $\dim_K \Gamma(K) = 3$ and $\Gamma(K) = B(1,2)$ as a K-superalgebra.

It is well known that $B(1,2)$ is a special superalgebra, hence so is $\Gamma(A)$. Finally, a central order in a simple (super)algebra is evidently prime. □

6 Bases of free Γ-algebras

In this section we will construct bases of free Γ-algebras defined in terms of free generators.

Let $A_n = F[x_1, \dots, x_n; y_1, \dots, y_n]$, consider the Γ-algebra $\Gamma(A_n)$. We want to prove that the subalgebra of $\Gamma(A_n)$ generated by the odd elements $v_i = (x_i, y_i),\ i = 1, \dots, n$, is a free Γ-algebra on these set of generators.

Denote $\alpha_{ij} = v_i v_j = x_i y_j - x_j y_i\ (1 \le i < j \le n),\ S_n = F[\alpha_{12}, \dots, \alpha_{(n-1)n}] \subset A_n,\ \ V_i = F v_i,\ V = \sum_{i=1}^{n} V_i$. We have the relations

$$\alpha_{ij} v_k + \alpha_{jk} v_i + \alpha_{ki} v_j = 0, \tag{6}$$

$$\alpha_{ij} + \alpha_{ji} = 0, \tag{7}$$

$$\alpha_{ij}\alpha_{kl} + \alpha_{ik}\alpha_{lj} + \alpha_{il}\alpha_{jk} = 0. \tag{8}$$

The following lemma is well known (see, for instance, [13]).

Lemma 1 *The algebra S_n is the free algebra modulo relations (7), (8). Moreover, it has the following base over F:*

$$B_n = \{\alpha_{i_1j_1}\alpha_{i_2j_2}\cdots\alpha_{i_rj_r} \mid i_1 \le i_2 \le \cdots \le i_r,\ j_1 \le j_2 \le \cdots \le j_r; i_s < j_s\}. \tag{9}$$

In fact, the algebra $S_n = F[Gr(2,n)]$ is the coordinate algebra of grassmanian $Gr(2,n)$ (see, for example, [16, vol. 1, p. 42]).

Let $S_{n,m} = F[\alpha_{ij} \mid j \ge m]$, then we have

$$S_{n,n} \subseteq S_{n,n-1} \subseteq \cdots \subseteq S_{n,2} = S_{n,1} = S_n.$$

Denote by I_m the ideal of S_n generated by the set $\{\alpha_{ij} \mid i < j < m\}$, $3 \le m \le n$; then $0 = I_2 \subseteq I_3 \subseteq I_4 \subseteq \cdots \subseteq I_n$. Let also $\bar{S}_{n,m} = (S_{n,m} + I_m)/I_m$.

Lemma 2 *The image in $\bar{S}_{n,m}$ of the following set forms a base of the algebra $\bar{S}_{n,m}$ over F:*

$$B_{n,m} = \{\alpha_{i_1j_1}\alpha_{i_2j_2}\cdots\alpha_{i_rj_r} \in B_n \mid j_1 \ge m\}.$$

Proof It is easy to prove using identities (8) that the set $B_{n,m}$ spans $S_{n,m}$ modulo I_m. Let us prove that it is linearly independent modulo I_m. Consider the algebra $E = E_{n,m} = F[x_m, \dots, x_n; t; y_1, \dots, y_n]$ and the homomorpism

$$\phi : A_n \to E,\ y_i \mapsto y_i,\ i = 1, \dots, n; x_j \mapsto x_j,\ j \ge m; x_k \mapsto ty_k,\ k < m.$$

Let $D_{n,m} = \phi(S_n)$. Note that $\phi(\alpha_{ij}) = 0$ if $j < m$, hence $I_m \subseteq \ker\phi$. Introduce the deg lex order in E by setting

$$x_n > x_{n-1} > \cdots > x_m > y_1 > y_2 > \cdots > y_n > t,$$

and let $\bar{f}$ denotes the leading term of polynomial f. Then for $i < m \le j$ we have $\overline{\phi(\alpha_{ij})} = \overline{ty_iy_j - x_jy_i} = -x_jy_i$, hence

$$\overline{\phi(\alpha_{i_1j_1}\alpha_{i_2j_2}\cdots\alpha_{i_sj_s})} = (-1)^s x_{j_1}\cdots x_{j_s} y_{i_1}\cdots y_{i_s}.$$

Therefore, if u and v are monomials in $\phi(\alpha_{ij})$ then $u = v$ if and only if $\bar{u} = \bar{v}$. This easily implies that the set $\phi(B_{n,m})$ is linearly independent over F, and thus the set $B_{n,m}$ is linearly independent modulo I_m. □

Consider the elements of the base B_n with more details. For any $u \in B_n$ of form (9) there exist uniquely defined numbers $l < p$ such that

$$u = \alpha_{i_1j_1}\cdots\alpha_{i_lj_l}\cdots\alpha_{i_pj_p}\cdots\alpha_{i_rj_r}, \tag{10}$$

where $j_l < m, j_{l+1} \ge m$; $i_p < m, i_{p+1} \ge m$.

Lemma 3 *The intersection $I_m \cap S_{n,m}$ has a base formed by elements* (10) *with $r - p \ge l \ge 1$.*

Proof Let us first prove that every element u of form (10) with $r - p \ge l \ge 1$ belongs to $I_m \cap S_{n,m}$. Since $l \ge 1$, $u \in I_m$. In order to prove that $u \in S_{n,m}$, it suffices to show that

$$(\alpha_{i_1j_1}\cdots\alpha_{i_lj_l})(\alpha_{i_{p+1}j_{p+1}}\cdots\alpha_{i_rj_r})\in S_{n,m}.$$

Since $r-p\geq l$, it suffices to prove that every product $\alpha_{ij}\alpha_{kl}$ with $j<m$, $k\geq m$ belongs to $S_{n,m}$. But this follows easily from relation (8).

In order to prove the inverse inclusion, we associate with any element $u\in B_n$ the set of its indices $\operatorname{ind}(u)=\{i_1,j_1,\dots,i_r,j_r\}$. Note that relation (8) does not change the set of indices, hence the algebras S_n and $S_{n,m}$ are homogeneous with respect to the sets of indices, that is, they may be represented as direct sums of subspaces with the same sets of indices. Moreover, so is the ideal I_m. Since the elements of $S_{n,m}$ are polynomials in α_{ij} with $j\geq m$, it is clear that for any homogeneous element $u\in S_{n,m}$ with $\operatorname{ind}(u)=\{i_1,j_1,\dots,i_r,j_r\}$ we should have at least r indices that are greater or equal to m. Assume now that $\sum\lambda_iu_i\in I_m\cap S_{n,m}$ for some u_i of form (10), then we have $\operatorname{ind}(u_i)=\operatorname{ind}(u_j)=\{i_1,j_1,\dots,i_l,j_l,\dots,i_p,j_p,\dots,i_r,j_r\}$ with $l\geq 1$ for all the summonds u_i,u_j. The set $\operatorname{ind}(u_i)$ has $l+p$ indices which are smaller than m, hence the sum lies in $S_{n,m}$ only if $l+p\leq r$ or $r-p\geq l$. □

Lemma 4 $(S_{n,m}v_m)\cap(\sum_{j<m}S_{n,j}v_j)=(S_{n,m}\cap I_m)v_m$.

Proof Let us first prove that $(S_{n,m}\cap I_m)v_m\subseteq\sum_{j<m}S_{n,j}v_j$. Let u be an element of form (10) with $r-p\geq l\geq 1$, then $u=\alpha_{i_1j_1}u'$, where $i_1<j_1<m$ and u' is an element of form (10) with $r'=r-1$, $p'=p-1$, $l'=l-1$. In particular, we have $r'-p'=r-p\geq l>l'$, therefore as in the proof of Lemma 3 we have $u'\in S_{n,m}$. Now by (6)

$$uv_m=u'\left(\alpha_{i_1j_1}v_m\right)=u'\left(\alpha_{i_1m}v_{j_1}-\alpha_{j_1m}v_{i_1}\right)\in S_{n,m}v_{j_1}+S_{n,m}v_{i_1}\subseteq\sum_{j<m}S_{n,j}v_j.$$

Note that Lemma 2 implies that $S_{n,m}=(S_{n,m}\cap I_m)\oplus F\cdot B_{n,m}$. Let us prove that $B_{n,m}V_m\cap\left(\sum_{j<m}S_{n,j}v_j\right)=0$. Assume that $w_1=w_2\neq 0$, where

$$w_1=\sum_{a_i\in B_{n,m}}\lambda_ia_iv_m,\quad w_2=\sum_{j<m,\,b_j\in S_{n,j}}\mu_jb_jv_j;\;\;\lambda_i,\mu_j\in F.$$

In particular, we have $\sum_i\lambda_ia_ix_m=\sum_j\mu_jb_jx_j\neq 0$. Consider the leading terms of both parts with respect to the deg lex order in A_n when

$$x_n>x_{n-1}>\cdots>x_1>y_1>y_2>\cdots>y_n.$$

We have

$$\overline{\sum_i\lambda_ia_ix_m}=\overline{\alpha_{i_1j_1}\dots\alpha_{i_sj_s}x_m}=f(y)x_mx_{j_1}\cdots x_{j_s},\ m\leq j_1\leq j_2\leq\cdots\leq j_s,$$
$$\overline{\sum_j\mu_jb_jx_j}=\overline{\alpha_{p_1q_1}\dots\alpha_{p_sq_s}x_j}=g(y)x_jx_{q_1}\cdots x_{q_s},\ j<m,j\leq q_1\leq q_2\leq\cdots\leq q_s.$$

Since $\overline{\sum_i\lambda_ia_ix_m}\neq\overline{\sum_j\mu_jb_jx_j}$, we conclude that $w_1=w_2=0$.

Now, since $(S_{n,m} \cap I_m)v_m \subseteq \sum_{j<m} S_{n,j}v_j$, we have

$$\begin{aligned}(S_{n,m}v_m) \cap \left(\sum_{j<m} S_{n,j}v_j\right) &= \big((S_{n,m} \cap I_m) \oplus F \cdot B_{n,m}\big)v_m \cap \left(\sum_{j<m} S_{n,j}v_j\right) \\ &= \big((S_{n,m} \cap I_m)v_m + B_{n,m}V_m\big) \cap \left(\sum_{j<m} S_{n,j}v_j\right) \\ &= (S_{n,m} \cap I_m)v_m + B_{n,m}V_m \cap \left(\sum_{j<m} S_{n,j}v_j\right) \\ &= (S_{n,m} \cap I_m)v_m.\end{aligned}$$

□

Lemma 5 *The subspace $S_nV \subseteq A_n^2$ is decomposed into a vector space direct sum*

$$S_nV = \oplus_{i=1}^n B_{n,i}V_i.$$

Proof First of all, note that due to (6) we have $S_nV = \sum_{i=1}^n S_{n,i}v_i$. Denote $U_m = \sum_{i=1}^m S_{n,i}v_i$, then $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n = S_nV$. Furthermore, let $W_i = U_i/U_{i-1}$, then we have a vector space isomorphism $S_nV \cong \oplus_{i=1}^n W_i$. Finally, for any $m \le n$ we have

$$\begin{aligned}W_m &= \left(\sum_{i=1}^m S_{n,i}v_i\right) / \left(\sum_{i=1}^{m-1} S_{n,i}v_i\right) \cong S_{n,m}v_m / \left(S_{n,m}v_m \cap \left(\sum_{i=1}^{m-1} S_{n,i}v_i\right)\right) \\ &= \text{(by Lemma 4)} \ = S_{n,m}v_m/(S_{n,m} \cap I_m)v_m \cong \big(S_{n,m}/(S_{n,m} \cap I_m)\big)v_m \\ &\cong \text{(by Lemma 2)} \ \cong B_{n,m}V_m.\end{aligned}$$

□

Theorem 7 *The space $S_n + S_nV$ is a subalgebra of algebra $\Gamma(A_n)$ which is isomorphic to the free Γ-algebra $\Gamma[\emptyset;V]$. It has a base $B_n \cup \left(\cup_{j=1}^n B_{n,j}v_j\right)$.*

Proof Consider the epimorphism $\varphi : \Gamma[\emptyset;V] \to S_n + S_nV$ defined by the conditions $v_i \mapsto (x_i, y_i)$. Relations (6)–(8) hold in the algebra $\Gamma[\emptyset;V]$ as well, and using these relations it is easy to see that it is spanned by the set $B_n \cup \left(\cup_{j=1}^n B_{n,j}v_j\right)$. Since its image is linearly independent in $\Gamma(A_n)$, it forms a base of $\Gamma[\emptyset;V]$, and φ is an isomorphism. □

Theorem 8 *The free Γ-algebra $\Gamma[t_1, \dots, t_m;v_1, \dots, v_n]$ on even generators $t_1, \dots, t_m$ and odd generators $v_1, \dots, v_n$ has the following structure:*

$$\begin{aligned} \Gamma_0 &= F\langle t_1, \dots, t_m\rangle + S_n' \otimes F[\bar{t}_1, \dots, \bar{t}_m], \\ \Gamma_1 &= F[\bar{t}_1, \dots, \bar{t}_m] \otimes \left(\oplus_{j=1}^n B_{n,j} V_j\right), \end{aligned}$$

where $F\langle t_1, \dots, t_m\rangle$ *and* $F[\bar{t}_1, \dots, \bar{t}_m]$ *are the free associative and the polynomial algebras on m variables,* S_n' *stands for the augmentation ideal of the algebra* S_n, $v_i \cdot v_j = \alpha_{ij} \in S_n$, $V_i = Fv_i$, *and for any* $f = f(t_1, \dots, t_m) \in F\langle t_1, \dots, t_m\rangle$ *and* $v \in \Gamma_1$, $f \cdot v = f(\bar{t}_1, \dots, \bar{t}_m) \otimes v$.

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