

RT-MAT 96-08

On a Conjecture of Zassenhaus
for Metacyclic Groups

S.O. Juriaans
and
S.K. Sehgal

Abril/96

On a Conjecture of Zassenhaus for Metacyclic Groups

S. O. Juriaans and S. K. Sehgal

Abstract

Zassenhaus Conjecture is proven for some metacyclic groups. In particular the conjecture holds if $G = \langle a \rangle \rtimes \langle x \rangle$ with $\langle x \rangle$ a p -group acting faithfully on every Sylow p' -subgroup of G . The case in which $\langle a \rangle$ is of prime power order was proved by Polcino Milies-Sehgal.

1 INTRODUCTION

Let $\mathbf{Z}G$ be the integral group ring of a finite group G . Denote by $\mathcal{U}\mathbf{Z}G$ the group of units and by $\mathcal{U}_1\mathbf{Z}G$ those of augmentation one. A well known conjecture of Zassenhaus states:

(ZC1): If $u \in \mathcal{U}_1\mathbf{Z}G$ is a torsion unit then u is rationally conjugate to a group element. That is there exists elements $g \in G$, $\beta \in \mathcal{U}\mathbf{Q}G$ such that $\beta^{-1}u\beta = g$.

This conjecture has been verified for several families of groups including the following:

Theorem 0 (Weiss [11]) *If G is a finite nilpotent group then (ZC1) holds for G .*

Theorem 0' (Polcino Milies-Ritter-Sehgal [6]) *Let G be a metacyclic group of the form $\langle a \rangle \rtimes \langle b \rangle$ with a, b of coprime order. Then (ZC1) holds for G .*

⁰AMS Subject Classification: Primary 20C05. Secondary 16S34, 16U60.

Key words and phrases: Group Rings, torsion units, metabelian group.

Research supported by Fapesp-Brazil (Proc. 95/2844-1) and NSERC Grant A-5300.

In this paper we extend the last theorem, which was originally proved in a series of three papers, to allow the possibility of some common factors in the orders of a and b . First we fix some notation:

Let $G = A \rtimes X$ where $A = B \times K$. Let π be a set of rational primes, K, X cyclic π -groups and B a cyclic π' -group. We suppose that X acts faithfully on all Sylow subgroups of B . We shall denote by $\mathcal{F}_0$ the family of groups with these properties and by $\mathcal{F}_1$ a family of finite groups, closed with respect to forming subgroups and quotients, satisfying (ZC1). Our main result is:

Theorem 1.1 *Let $G \in \mathcal{F}_0$. Suppose that $K \rtimes X$ belongs to a family $\mathcal{F}_1$, closed with respect to forming subgroups and quotients. If (ZC1) holds for all elements of $\mathcal{F}_1$ then (ZC1) holds for G .*

By applying Weiss' theorem we have:

Corollary 1.2 *If $G \in \mathcal{F}_0$ and K, X are p -groups then (ZC1) holds for G .*

In Section 2 we derive some representations of G . In Section 3 we give some reductions and Section 4 is dedicated to possible counter examples. In Section 5 we give a complete proof of Theorem 0' which is contained in [5], [6] and [7]. We also give a short proof of a result of Luthar and Trama [3]. In the last section we complete the proof of the theorem. We remark that our result includes the following result of [4]: Let G be a split extension of a cyclic group and a group of prime order. Then (ZC1) holds for G .

The first author would like to thank the University of Alberta, especially the department of Mathematics, for its warm hospitality.

2 REPRESENTATIONS OF G

In this section we determine some representations of $G \in \mathcal{F}_0$. Let $\Phi \in \text{Irr}(G)$ be an irreducible representation of G . By [10, pg. 62] Φ is induced from AS_Ψ , where Ψ is an irreducible representation of A and S_Ψ is the stabilizer of Ψ in X namely $S_\Psi = \{x \in X : \Psi(x^{-1}ax) = \Psi(a), \forall a \in A\}$. Also the degree of Φ is equal to $[X : S_\Psi]$. From [8, 8.4.3] we see easily that

$\ker(\Psi) = A \cap \ker(\Phi)$. Hence Φ is faithful on A if and only if Ψ is faithful on A .

Fix a Φ faithful on A and let S_Ψ be as above. If $x \in S_\Psi$ then $\Psi(x^{-1}ax) = \Psi(a)$ for all $a \in A$. Since Ψ is faithful on A it follows that $x^{-1}ax = a$ for every $a \in A$ and thus $x = 1$. Hence we have proved that Φ is induced from A .

Lemma 2.1 *Let $A = B \times K$ be cyclic, $G = A \rtimes X$ with X acting faithfully on B . If $\Phi \in \text{Irr}(G)$ is faithful on A then it is induced from A and has degree $|X|$.*

Corollary 2.2 *Let $G \in \mathcal{F}_0$ and p a prime dividing $|A|$. Suppose that p^2 divides $|A|$ and $\Phi_p \in \text{Irr}(G)$ is such that $A \cap \ker(\Phi_p)$ has order p . Then Φ_p is induced from A and has degree $|X|$.*

Proof: Set $N = A \cap \ker(\Phi_p)$. We shall see later on, Corollary 3.2, that $G/N \in \mathcal{F}_0$. But Φ_p is defined on G/N and is faithful on A/N . The result follows from the last Lemma. $\square$

Let $G \in \mathcal{F}_0$ and let $\Phi_0 \in \text{Irr}(G)$ be faithful on A . Set $X = \langle x \rangle$, $A = \langle a \rangle$ and $\Psi \in \text{Irr}(A)$ such that $\Phi_0 = \Psi^G$. For the action of x on A we write $a^x = a^j$ and let $\Psi(a) = \xi$, $\phi(\xi) = \phi(a)$. Setting $s = |X|$ we see easily that:

$$\Phi_0(a) = \begin{pmatrix} \xi & 0 & 0 & \cdots & 0 \\ 0 & \xi^j & 0 & \cdots & \vdots \\ 0 & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \xi^{j^{s-1}} \end{pmatrix}, \quad \Phi_0(x) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \vdots \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & 1 \\ 1 & \cdots & \cdots & 0 & 0 \end{pmatrix}$$

So $\Phi_0 : G \rightarrow M_s(\mathbb{Q}(\xi))$. Up to rational equivalence this is the only $\Psi \in \text{Irr}(G)$ which is faithful on A i.e. If Φ is an irreducible character of G which is faithful on A then there exists a field automorphism $\sigma \in \text{Aut}(\mathbb{Q}(\xi))$ such that $\Phi = \sigma \circ \Phi_0$.

Note that if $a_1, a_2 \in A$ are such that $\Phi_0(a_1), \Phi_0(a_2)$ are conjugate then they are already conjugate in G and this depends only on the fact that A is cyclic. To see this we only have to compare eigenvalues and remember that ξ and a have the same order. We shall frequently use this observation in the proof of our results.

Let Φ_p be as in Corollary 2.2. It follows from this corollary that, up to rational equivalence, Φ_p is given by:

$$\Phi_p(a) = \begin{pmatrix} \xi^p & 0 & 0 & \cdots & 0 \\ 0 & \xi^{pj} & 0 & \cdots & \vdots \\ 0 & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \xi^{pj^{t-1}} \end{pmatrix}, \quad \Phi_p(x) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \vdots \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & 1 \\ 1 & \cdots & \cdots & 0 & 0 \end{pmatrix}$$

So we see that $\Phi_p = \Phi_0 \circ \Psi_p$ where $\Psi_p : G \rightarrow G$ is the homomorphism defined by $\Psi_p(a^k x^l) = a^{pk} x^l$. Note that $\text{Ker}(\Phi_p) = \text{Ker}(\Psi_p)$. The Corollary is still true if $K = 1$ and X acts faithfully on B . In this case Φ_0 represents a more or less as above but the image of X consists of permutation matrices.

In the next Lemma $A = B \rtimes K$ and B, X are groups of coprime order. We write $A = \langle a \rangle$.

Lemma 2.3 *Let $G = A \rtimes X$ with X acting faithfully on B . Suppose that $u \in 1 + \Delta(G, A)$ and $o(u)$ has exactly the same prime divisors as $|A|$. Then there exists $a_0 \in A$ such that $\Phi_0(u)$ and $\Phi_0(a_0)$ are conjugate.*

Proof: Since X acts faithfully on B it must be abelian. As remarked above Φ_0 maps elements of X to permutation matrices. Denote by $u_1, \dots, u_r$ the generators of the Sylow subgroups of $\langle u \rangle$. By [1, Theorem 2.9] every u_i is rationally conjugate to an element of A . Each matrix $\Phi_0(u_i)$ has an eigenvalue of the form ξ^{a_i} . It follows that $\Phi_0(u)$ has an eigenvalue of the form ξ^t and we may suppose that it is the product of these ξ^{a_i} . Write $X = \{1, x_1, \dots, x_s\}$ and $a^{x_i} = a^{k_i}$. We want to show that ξ^{tk_i} are all eigenvalues of $\Phi_0(u)$. But this is easily seen because $\Phi_0(x_i)$ is a permutation matrix and hence is fixed by the field automorphism $\sigma_i : \xi \rightarrow \xi^{k_i}$. Therefore, applying σ_i to the entries of $\Phi(u)$ is the same as conjugating by $\Phi_0(x_i)$. To see this write $u = \sum_k f_k(a)x_k$ and notice that $\Phi_0(a^{k_i}) = \sigma_i(\Phi_0(a))$. Then $\Phi_0(x_i^{-1})\Phi_0(u)\Phi_0(x_i) = \Phi_0(x_i^{-1}ux_i) = \sum_k f_k(\Phi_0(a^{k_i}))\Phi_0(x_k) = \sigma_i(\Phi_0(u))$.

Since X acts faithfully on B and $o(u)$ has the same prime divisors as $|A|$ it follows that these eigenvalues are all distinct. In fact suppose that $\xi^{tk_i} = \xi^{tk_j}$. Since ξ and a have equal orders it follows that $a^{tk_i} = a^{tk_j}$. Write $a^t = a_1 a_2$ with $a_1 \in B$ and $\langle a_2 \rangle \cap B = 1$. Then we have that $a_1^{k_i} = a_1^{k_j}$ and thus

$x_i(x_j)^{-1}$ centralizes a_1 . Note now that $o(a_1)$ has exactly the same prime divisors as $|B|$ and since the latter and $|X|$ are relatively prime it follows that $x = x_i(x_j)^{-1}$ centralizes B . To see this let $b \in B$ be of prime order p , say, and notice that b is centralized by x . Write $b^x = b^k$ for the action of x on b . It follows that $k = 1 \pmod p$. Since $(o(x), |B|) = 1$ it follows that x centralizes the Sylow p -subgroup containing b and hence the statement is proved. Therefore, $x_i = x_j$ and the eigenvalues ξ^{ik} are all distinct. It follows that $\Phi_0(u), \Phi_0(a^t)$ are conjugate. $\square$

3 SOME REDUCTIONS

In this section we establish some reductions. Notations of former sections will be used freely and $\sim$ stand for conjugation of units or matrices.

Lemma 3.1 *Let p be a prime, $\langle a \rangle$ a cyclic p -group and X a p' -group acting faithfully on $\langle a \rangle$. Then X acts faithfully on every non-trivial homomorphic image of $\langle a \rangle$.*

Proof: Let $x \in X$ be given and set $a^x = a^j$. If $\langle \bar{a} \rangle$ is a non-trivial homomorphic image of $\langle a \rangle$ then $\bar{a}^x = \bar{a}^j$ and hence if x acts trivially on $\bar{a}$ we have that $j = 1 \pmod p$. Since X is a p' -group it follows that $j = 1 \pmod{o(a)}$ and thus by assumption $x = 1$. $\square$

Corollary 3.2 *Let $G \in \mathcal{F}_0$ and $N \subset B$ a proper subgroup. Then $G/N \in \mathcal{F}_0$.*

Proof: By the last Lemma X acts faithfully on every Sylow subgroup of B/N and hence the result is proved. $\square$

Lemma 3.3 *Let $G \in \mathcal{F}_0$, $g \in G$, $g = byx$ with $b \in B$, $y \in K$ and $x \in X$. If $x \neq 1$, then g is a π -element.*

Proof: We use induction on the number of prime divisors of $|B|$. If B is of prime power order then, by [9, 37.16], bx is a π -element. Since K is normal in G it follows that g is a π -element. In the general case write $b = b_1b_2$ with $\langle b_1 \rangle$ a Sylow subgroup of $\langle b \rangle$. By induction b_2x is a π -element and hence, by [8, 9.1.3] it is conjugate to an element of $K \rtimes X$. It follows that a conjugate

h of g is of the form $h = b_3 y_1 x_1$ with $X \ni x_1 \neq 1$, $y_1 \in K$ and $b_3 \in B$ has prime power order. It follows by the first case that h and hence g is a π -element. $\square$

Lemma 3.4 *Let $G \in \mathcal{F}_0$ be a counter example of least order to (ZC1) and let $u \in \mathcal{U}_1 \mathbb{Z}G$ be a torsion unit which is not conjugate, in $\mathbb{Q}G$, to a group element. If $K \rtimes X \in \mathcal{F}_1$ then $u \in 1 + \Delta(G, A)$.*

Proof: Denote by $W = \Delta(G) \Delta(G')$ the Whitcomb ideal. Since G/A and A are abelian we use Whitcomb's Argument and [9, 31.3] to find $g \in G$ such that $u - g \in W$ and $o(u) = o(g)$. Write $g = byx$, as in the last Lemma. If $x \neq 1$ then, by the last Lemma, u is a π -element and thus, by [9, 37.13], it is rationally conjugate to an element of $\mathbb{Z}(K \rtimes X)$. Since (ZC1) holds for the latter group we have a contradiction. So $x = 1$ and since $W \subset \Delta(G, A)$ the result is proved. $\square$

Lemma 3.5 *Let G and u be as in Lemma 3.4 and $K \rtimes X \in \mathcal{F}_1$. Then every prime dividing $|A|$ divides $o(u)$.*

Proof: Suppose that p is a prime dividing $|A|$ and p does not divide $o(u)$. Let N be a subgroup of G of order p . Note that N is normal in G . Now either G/N is isomorphic to $K \rtimes X$ or $G/N \in \mathcal{F}_0$. If the first case happens then we know that (ZC1) holds for this group. In the second case (ZC1) holds by the choice of G . Since N and u have coprime order we apply [1, Lemma 2.1] to obtain that u is conjugate, in $\mathbb{Q}G$, to a group element, a contradiction. $\square$

Lemma 3.6 *Let G be a finite nilpotent-by-nilpotent group and $A < G$ a normal subgroup. If $\Phi \in \text{Irr}(G)$ is faithful on A then it is injective on finite subgroups of $1 + \Delta(G, A)$.*

Proof: Let $u \in 1 + \Delta(G, A)$ be such that $\Phi(u) = 1$. We may suppose that u is of prime power order. Since G is nilpotent-by-nilpotent it follows, by [1, Theorem 2.9], that u is rationally conjugate to a group element $A \ni a$, say. So $\Phi(a) = 1$ and $u = 1$. $\square$

4 COUNTER EXAMPLES

In this section we prove some results about characters and units. Unless explicitly stated groups G in this section do not necessarily belong to $\mathcal{F}_0$. Also $\sim$ will stand for conjugation of matrices or units. We begin by making the following assumptions:

Assumptions:

1. G is a finite nilpotent-by-nilpotent-group belonging to some family $\mathcal{F}$ and it is a counter example to (ZC1).
2. A is a normal abelian subgroup of G .
3. Φ_0 is a fixed irreducible representation of G which is faithful on A . It satisfies the following: If $\alpha \in U_1 \mathbb{Z}G$ is a torsion unit which is rationally conjugate to a group element $g \in A$ and if $\Phi_0(\alpha) \sim \Phi_0(h)$, $h \in G$; then $g \sim h$ in G .
4. $u \in 1 + \Delta(G, A)$ is a torsion unit of least order which is not rationally conjugate to a group element; $|A|$ and $o(u)$ have exactly the same prime divisors.
5. $\Phi_0(u) \sim \Phi_0(a_0)$ for some $a_0 \in A$.
6. Φ_0 is faithful on finite subgroups of $1 + \Delta(G, A)$.

In all what follows G, Φ_0, a_0 and u will be as above.

Lemma 4.1 *Let $\Phi \in \text{Irr}(G)$ be so that there exists a prime p dividing $o(u)$, but p does not divide $o(\Phi(u))$. Then $\Phi(u) \sim \Phi(a_0)$.*

Proof: Since G is nilpotent-by-nilpotent it follows, by [1, Theorem 2.9], that u is not of prime power order. Write $u = u_1 u_2$ where u_1 is a p -element and u_2 is a p' -element. By our hypothesis $\Phi(u)$ is a p' -element and hence $\Phi(u_1) = 1$. Since u_2 is a p' -element we can find infinitely many integers $k \in \mathbb{N}$ such that $u_2^{p^k} = u_2$ and $u_1^{p^k} = 1$. So we have $u^{p^k} = u_2$ and thus $\Phi_0(u_2) \sim \Phi_0(a_0^{p^k})$. By (3) and (4) it follows that $u^{p^k} \sim a_0^{p^k}$ and consequently $\Phi(u_2) \sim \Phi(a_0^{p^k})$.

Now write $a_0 = a_1 a_2$ where a_1 is a p -element and a_2 is a p' -element. By (6) $o(u_i) = o(a_i)$ and thus $a_0^{p^k} = a_2$. It follows that $\Phi(u_2) \sim \Phi(a_2)$.

Let $m = o(u_2)$. By (5) $\Phi_0(u_1^m) = \Phi_0(u^m) \sim \Phi_0(a_0^m) = \Phi_0(a_1^m)$. By (3) and (4) it follows that $u_1^m \sim a_1^m$ and since p does not divide m and $\Phi(u_1) = 1$ we have that $\Phi(a_1) = 1$. Finally it follows that $\Phi(u) = \Phi(u_1 u_2) = \Phi(u_2) \sim \Phi(a_2) = \Phi(a_0)$. $\square$

Corollary 4.2 Suppose that A is cyclic and p is a prime dividing the order of u but p^2 does not divide the order of u . If $\Phi \in \text{Irr}(G)$ is such that p divides $|N = A \cap \text{Ker}(\Phi)|$ then $\Phi(u) \sim \Phi(a_0)$.

Proof: We use the notation of the proof of the last Lemma. Since p^2 does not divide $o(u)$ it follows that u_1, a_1 have order p . Since A is cyclic and p divides $|N|$ we must have that $\Phi(a_1) = 1$ and thus $\Phi(u) = \Phi(u_2)$. It follows that p does not divide the order of $\Phi(u)$. The result follows by Lemma 4.1. $\square$

Let $u = \sum u(g)g \in \mathbb{Z}G$, $g_0 \in G$ and denote by C the conjugacy class of g_0 . Define $\tilde{u}(g_0) = t_C(u) = \sum_{g \in C} u(g)$ and $\Lambda = [\mathbb{Z}G, \mathbb{Z}G]$. It is easy to see that $u \in \Lambda$ if and only if $\tilde{u}(g) = 0$ for every $g \in G$. It is also known that (ZC1) is equivalent to the following: For every $u \in \mathcal{U}_1 \mathbb{Z}G$ of finite order there exists a unique conjugacy class C such that $t_C(u) \neq 0$ (see [9, 41.5]). As in the previous sections we shall denote by W the Whitcomb ideal $\Delta(G)\Delta(G')$.

Suppose that $A = \langle a \rangle$ and $G = A \rtimes X$ with X an abelian group. Define $\Psi_p : G \rightarrow G$ by $\Psi_p(a^k x_l) = a^{pk} x_l$, $x_l \in X$. We shall denote by s the order of X . This notation will be used in what follows.

Theorem 4.3 Let p be a prime dividing $|A|$, π a set of rational primes and $\Psi_p : G \rightarrow G$ the homomorphism defined above. Suppose that $\Phi \in \text{Irr}(G)$ satisfies $\Phi = \Phi_0 \circ \Psi_p$. Suppose that A is not a π -group and that one of the following holds:

1. Elements of the form $a^k x_l$ are π -elements.
2. p does not divide the order of X .

Then $\Phi(u) \sim \Phi(a_0)$.

Proof: Write $u = \sum_{0 \leq j < s} f_j(a)x_j$, $\mathbf{Z}(A) \ni f_j(a) = \sum_i \alpha_{ij}a^i$. Then $\Psi_p(u) = \sum_{0 \leq j < s} f_j(a^p)x_j = f_0(a^p) + \sum_{1 \leq j < s} f_j(a^p)x_j$ and, by [9, 7.1], $u^p = \sum_{0 \leq j < s} (f_j(a)x_j)^p \pmod{\Lambda + p\mathbf{Z}G}$. Further, we have that $u^p = f_0(a^p) + \sum_{i,j \neq 0} \alpha_{ij}(a^i x_j)^p \pmod{\Lambda + p\mathbf{Z}G}$.

Note that, by 1. and 2., elements of the form $a^i x_j$ either do not have the same prime divisors as A or are not in A . So if $C = C_{g_0}$ is the conjugacy class of an element $g_0 \in A$ such that $o(g_0)$ and $|A|$ have precisely the same prime divisors then $t_C((a^i x_j)^p) = 0$ if $x_j \neq 1$.

It is easy to see that $A \cap \text{Ker}(\Phi)$ has order p . Thus if p^2 does not divide the order of u then we are done by Corollary 4.2.

So we may suppose that $o(u)$ is divisible by p^2 . By Whitcomb's Argument and [9, 31.3] we can find $g_u \in G$ such that $u - g_u \in W$ and $o(u) = o(g_u)$. Since Ψ_p is a homomorphism we see that $\Psi_p(u) - \Psi_p(g_u) \in W$ and hence, by [9, 31.3], their orders are equal. But it should be noticed that $W \subset \Delta(G, A)$ and thus $g_u \in A$. It follows that $\Psi_p(g_u) = g_u^p$ and consequently $\Psi_p(u)$ has order $\frac{o(u)}{p}$. So we have proved that u^p and $\Psi_p(u)$ have order equal to $\frac{o(u)}{p}$ and hence, by assumption (4), they are rationally conjugate to group elements. We show that they are in fact conjugate to each other. Note first that since p^2 divides the order of u , the orders of u^p and $\Psi_p(u)$ have precisely the same prime divisors as $|A|$ which is not a π -group. Let $C = C_{g_0}$ be as above. Then, by what was proved above, it follows that $t_C(u^p) = t_C(f_0(a^p)) \pmod{p}$ and $t_C(\Psi_p(u)) = t_C(f_0(a^p))$ and hence $t_C(u^p) = t_C(\Psi_p(u)) \pmod{p}$.

On the other hand there exists unique conjugacy classes $C_1 = C_{g_1}$, $C_2 = C_{g_2}$, with g_1, g_2 having precisely the same prime divisors as $|A|$, such that $t_{C_1}(u^p) \neq 0$ and $t_{C_2}(\Psi_p(u)) \neq 0$. Taking $C = C_1$ we conclude that u^p and $\Psi_p(u)$ are conjugate in $\mathbf{Q}G$.

Finally we have $\Phi(u) = \Phi_0 \circ \Psi_p(u) \sim \Phi_0(u^p) \sim \Phi_0(a_0^p) = \Phi_0 \circ \Psi_p(a_0) = \Phi(a_0)$. Hence the proof is complete. $\square$

Remark: Note that all we needed is that the orders of the elements $(a^i x_j)^p$ which belong to A do not have the same prime divisors as $|A|$.

5 SOME APPLICATIONS

In this section we give a uniform proof of all what is known about (ZC1) for split metacyclic groups (see introduction and references).

Theorem 5.1 (Luthar-Trama [3]) *Let $G = \langle a \rangle \rtimes X$ with X any group such that $(|X|, o(a)) = 1$. Let $u \in 1 + \Delta(G, \langle a \rangle)$. Then u is rationally conjugate to a group element.*

Proof: Note that such groups form a family $\mathcal{F}$ which is closed with respect to forming quotients and subgroups. Let G be a counter example of least order in $\mathcal{F}$. Let $u \in U_1 \mathbb{Z}G$ be an element of least order which is not conjugate in $\mathbb{Q}G$ to a group element and set $N = C_X(a)$. Note that N is normal in G and, by [1, Corollary 2.8], has order prime to that of u . If $N \neq 1$ then, by the choice of G and [1, Lemma 2.1], u would be rationally conjugate to a group element. Hence $N = 1$ and thus X is abelian and acts faithfully on $\langle a \rangle$. Let q be a prime divisor of $o(a)$. If q does not divide $o(u)$ we take N to be a Sylow q -subgroup of G and use [1, Lemma 2.1] and the choice of G to get a contradiction. Hence $o(a)$ and $o(u)$ have exactly the same prime divisors. Since $u \in 1 + \Delta(G, \langle a \rangle)$ and the orders of $\langle a \rangle$ and X are relatively prime we obtain, by Lemma 2.3, that $\Phi_0(u) \sim \Phi_0(a_0)$ for some $a_0 \in \langle a \rangle$.

Take π to be the set of prime divisors of $|X|$ and let Φ_p be as in section 2. It follows by Theorem 4.3 that $\Phi_p(u) \sim \Phi_p(a_0)$ for every prime divisor of n . Let $\Phi \in Irr(G)$ be given. Suppose first that Φ is not faithful on $\langle a \rangle$. Then $N = (A \cap Ker \Phi) \cap Ker \Phi_p \neq 1$ for some prime divisor p of the order of A and, by Corollary 4.2, we may suppose that p^2 divides the order of A . Consider the factor group $\overline{G} = G/N$ and denote by $\bar{u}$ the image of u in $\mathbb{Z}\overline{G}$. Note that Φ_p and Φ are both defined on $\overline{G}$ and Φ_p is faithful on A/N . Also $\Phi_p(\bar{u}) = \Phi_p(u) \sim \Phi_p(a_0) = \Phi_p(\bar{a}_0)$, where $\bar{a}_0$ is the image of a_0 in $\overline{G}$. It follows, by the choice of G and the fact that A/N is cyclic, that $\bar{u} \sim \bar{a}_0$ and hence $\Phi(u) = \Phi(\bar{u}) \sim \Phi(\bar{a}_0) = \Phi(a_0)$.

On the other hand if Φ is faithful on A then $\Phi = \sigma \circ \Phi_0$, where σ is a field automorphism. Since $\Phi_0(u) \sim \Phi_0(a_0)$ it follows that $\Phi(u) \sim \Phi(a_0)$.

So we have seen that $\Phi(u) \sim \Phi(a_0)$ for every $\Phi \in Irr(G)$. By [9, 37.5] $u \sim a_0$ in $\mathbb{Q}G$, a final contradiction. So there are no counter examples in $\mathcal{F}$. $\square$

Theorem 5.2 Let $G = \langle a \rangle \rtimes X$, X abelian, with $n = o(a)$ and $t = |X|$ relatively prime. Then (ZC1) holds for G .

Proof: Denote by $\mathcal{F}$ the family of all such groups and let G be a counter example of least order in this family. Let also $u \in \mathcal{U}_1 \mathbb{Z}G$ be an element of least order which is not rationally conjugate to a group element. By Whitcomb's Argument and [9, 31.3] there exists $G \ni g_u = a^k x_1$ such that $u - g_u \in W$, W being the Whitcomb ideal, and $o(u) = o(g_u)$. By the choice of G and [9, 37.13] we may suppose that $o(u)$ is divisible by every prime divisor of $o(a)$. Hence, by [6, Lemma 2.3], x_1 is central in G and thus, by the choice of u , we may suppose that $o(u)$ has the same prime divisors as n and $u \in 1 + \Delta(G, \langle a \rangle)$. The result follows by Theorem 5.1. $\square$

The case in which $\langle a \rangle$ is a p -group was proved in [2] and A. Valenti proved a stronger version of this theorem (see [9, 42.9]).

6 PROOF OF THE THEOREM

We shall use freely the notations and results of the former sections and G will always be in $\mathcal{F}_0$. In particular, G and u will be as in section 3 and Φ_0 is as in section 2. By Lemma 3.6, Φ_0 is faithful on finite subgroups of $1 + \Delta(G, A)$.

Proof of Theorem 1.1: Let G be a counter example of least order and let $u \in \mathcal{U}_1 \mathbb{Z}G$ be an element of least order which is not rationally conjugate to a group element. We want to apply the results of section 4. Note that we just have to check that conditions (3) and (5) are satisfied: If $\alpha \sim g \in A$ and $\Phi_0(\alpha) \sim \Phi_0(h)$, $h \in A$ then $\Phi_0(g) \sim \Phi_0(h)$ and, since A is cyclic, $g \sim h$. By Lemma 2.3 $\Phi_0(u) \sim \Phi_0(a_0)$ for some $a_0 \in A$.

By Theorem 4.3 we have that $\Phi_p(u) \sim \Phi_p(a_0)$ for every prime divisor of $|A|$. Let $\Phi \in \text{Irr}(G)$ be given. We proceed as in the proof of Theorem 5.1. Suppose first that Φ is not faithful on $\langle a \rangle$. Then $N = (A \cap \text{Ker} \Phi) \cap \text{Ker} \Phi_p \neq 1$ for some prime divisor p of the order of A and, by Corollary 4.2, we may suppose that p^2 divides the order of A . Consider the factor group $\overline{G} = G/N$ and denote by $\bar{u}$ the image of u in $\mathbb{Z}\overline{G}$. Note that Φ_p and Φ are both defined on $\overline{G}$ and Φ_p is faithful on A/N . By the choice of G and the fact that A/N is cyclic it follows that $\bar{u} \sim \bar{a}_0$ and hence $\Phi(u) \sim \Phi(a_0)$.

If Φ happens to be faithful on A then $\Phi = \sigma \circ \Phi_0$, where σ is a field automorphism. The fact that $\Phi_0(u) \sim \Phi_0(a_0)$ implies that $\Phi(u) \sim \Phi(a_0)$.

So we concluded that $\Phi(u) \sim \Phi(a_0)$ for every $\Phi \in \text{Irr}(G)$ and hence, by [9, 37.5], $u \sim a_0$ in $\mathbb{Q}G$, a final contradiction. So there are no counter examples and the theorem is proved. $\square$

REFERENCES

- [1] Dokuchaev, M. A., Juriaans, S. O., Finite Subgroups in Integral Group Rings, *Canadian Journal of Math.* (to appear).
- [2] Juriaans, S. O., Torsion Units in Integral Group Rings, *Commun. Algebra* 22 (12) (1994), 4905-4913.
- [3] Luthar, I. S., Trama, P., Zassenhaus Conjecture for Certain Integral Group Rings, *Journal of Indian Math. Soc.* 55 (1990), 199-212.
- [4] Luthar, I. S., Trama, P., Torsion Units in Integral Group Rings of Some Metacyclic Groups (preprint).
- [5] Polcino Milies, C., Sehgal, S. K., Torsion Units in integral group rings of metacyclic groups, *J. Number Theory* 19 (1984), 103-114.
- [6] Polcino Milies, C., Ritter, J., Sehgal, S. K., On a conjecture of Zassenhaus on torsion units in integral group rings II, *Proc. Amer. Math. Soc.* 97 (2) (1986), 206-210.
- [7] Ritter, J., Sehgal, S. K., On a conjecture of Zassenhaus on torsion units in integral group rings, *Math. Ann.* 264 (1983), 257-270.
- [8] Robinson, D. J. S., *A Course in the Theory of Groups*, Springer-Verlag, New York, Heidelberg, Berlin, 1980.
- [9] Sehgal, S. K., *Units of Integral Group Rings*, Longman's, Essex, 1993.
- [10] Serre J. -P., *Linear Representations of Finite Groups.*, Springer-Verlag N.Y. Heidelberg Berlin 1971
- [11] Weiss, A., Torsion units in integral group rings, *J. Reine Angew. Math.* 415 (1991), 175-187.

Instituto de Matemática e Estatística,
Universidade de São Paulo,
Caixa Postal 66281,
05389-970 - São Paulo S. P.- Brazil.
ostanley@ime.usp.br.

Department of Mathematical Sciences,
University of Alberta,
Edmonton, AB,
T6G 2G1,
Canada.
s.sehgal@ualberta.ca

TRABALHOS DO DEPARTAMENTO DE MATEMÁTICA

TÍTULOS PUBLICADOS

- 95-01 BARROS, L.G.X. de and Juriaans, S.O. Loops whose Loop Algebras are Flexible. 25p.
- 95-02 GUIDORIZZI, H.L. - Jordan canonical form: an elementary proof, 12 p.
- 95-03 CATALAN A., and COSTA R. E-Ideals in Baric Algebras. 14p.
- 95-04 MARTIN, P.A. On the generating function of the decimal expansion of an irrational real numbers. 7p.
- 95-05 COELHO F.U., MARCOS E.N., MERKLEN H.A. and PLATZECK.M.I. Modules of Infinite Projective Dimension over Algebras whose Idempotent Ideals are Projective. 13p.
- 95-06 GUIDORIZZI, H. L. The family of functions S_k and the Liénard Equation. 22p.
- 95-07 GUIDORIZZI, H. L. On the Existence of Periodic Solution for the Equation $\ddot{x} + \alpha^{2n+1}\dot{x} + x^{4n+3} = 0$. 5p.
- 95-08 CORTIZO, S.F. Extensões Virtuais. 27p.
- 95-09 CORTIZO, S.F. Cálculo Virtual. 31p.
- 95-10 GUIDORIZZI, H. L. On Periodic Solutions of Systems of the Type $\ddot{x} = H(y)$, $\dot{y} = -\sum_{i=1}^n f_i(x)H_i(y) - g(x)$. 16p.
- 95-11 OLIVA, S. M., PEREIRA, A. L. Attractors for Parabolic Problems with Non linear Boundary Conditions in Fractional Power Spaces. 28p.
- 95-12 CORDARO, P. D. Global hypoellipticity for $\bar{\partial}_b$ on certain compact three dimensional CR manifolds. 11p.
- 95-13 COELHO, F.U. and SKOWRONSKI, A. On Auslander-Reiten Components for Quasitilted Algebras. 16p.
- 95-14 COELHO, F.U. and HAPPEL, D. Quasitilted algebras admit a preprojective component. 12p.
- 95-15 GOODAIRE, E.G. and POLCINO MILIES, C. The torsion product property in alternative algebras. 10p.
- 95-16 GOODAIRE, E. G. and POLCINO MILIES, C. Central idempotents in alternative loop algebras. 7p.
- 95-17 GOODAIRE, E. G. and POLCINO MILIES, C. Finite conjugacy in alternative loop algebras. 7 p.
- 95-18 EXEL, R. Unconditional integrability for dual actions. 22p.
- 95-19 OLIVA, W.M. and SALLUM, E.M. The dynamic of malaria at a rice irrigation system. 11p.
- 95-20 FIGUEIREDO, L.M.V., GONÇALVES, J.Z. and SHIRVANI, M. Free Group Algebras in Certain Division Rings. 28p.
- 95-21 SHIRVANI, M. and GONÇALVES, J. Z. Algebraically Independent Orbits and Free Algebras. 17p.
- 95-22 ARAGONA, J. Generalized functions on quasi-regular sets. 17p.
- 95-23 GUZZO JR., H. On normal and composition series for baric algebras. 18p.

- 95-24 DRUCK, I. de F. Um pouco da história de potências, exponenciais e logaritmos. 25p.
- 95-25 BARROS, L.G.X. de and JURIAANS, S.O. Integral Loop Rings of Code Loops. 7p.
- 95-26 GARCIA D., LOURENÇO M.L., MORAES L.A., and PAQUES O.W. The spectra of some algebras of analytic mappings. 15p.
- 95-27 BRASIL, A. JR Complete hypersurfaces of S^{n+1} with constant mean curvature and constant scalar curvature. 11p.
- 95-28 GUZZO JR., H. The bar-radical of baric algebras. 19p.
- 95-29 MARTINS, M.I.R. Composition factors of indecomposable modules. 26p.
- 95-30 CORTIZO, S.F. Cálculo Virtual - Parte II. 19p.
- 95-31 CORTIZO, S.F. Sobre o Cálculo Delta de Dirac. 18p.
- 95-32 COSTA, R. and SUAZO, A. The Multiplication Algebra of a Bernstein Algebra: Basic Results. 13p.
- 95-33 FABEL, E., GORODSKI, C. and RUMIN, M. Holonomy of Sub-Riemannian Manifolds. 34p.
- 95-34 MELO, S.T. Characterizations of Pseudodifferential Operators on the Circle. 9p.
- 96-01 GUZZO JR., H. On commutative train algebras of rank 3. 15p.
- 96-02 GOODAIRE, E. G. and POLCINO MILIES, C. Nilpotent Moufang Unit Loops. 9p.
- 96-03 COSTA, R. and SUAZO, A. The multiplication algebra of a train algebra of rank 3. 12p.
- 96-04 COELHO, S.P., JESPERS, E. and POLCINO MILIES, C. Automorphisms of Groups Algebras of Some Metacyclic Groups. 12p.
- 96-05 GIANNONI, F., MASIELLO, A. and PICCIONE, P. Sur une Théorie Variationnelle pour Rayons de Lumière sur Variétés Lorentziennes Stablement Causales. 7p.
- 96-06 MASIELLO, A. and PICCIONE, P. Shortening Null Geodesics in Lorentzian Manifolds. Applications to Closed Light Rays. 17p.
- 96-07 JURIAANS, S.O. Trace Properties of Torsion Units in Group Rings II. 22p.
- 96-08 JURIAANS, S.O. and SEHGAL, S.K. On a conjecture of Zassenhaus for Metacyclic Groups. 13p.

NOTA: Os títulos publicados nos Relatórios Técnicos dos anos de 1980 a 1994 estão à disposição no Departamento de Matemática do IME-USP.

Cidade Universitária "Armando de Salles Oliveira"
 Rua do Matão, 1010 - Butantã
 Caixa Postal - 66281 (Ag. Cidade de São Paulo)
 CEP: 05389-970 - São Paulo - Brasil