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On Two Problems Related to Associators of Moufang Loops

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Abstract—A Moufang loop M of order 3^{19} is constructed, together with a pair a, b of elements of M, such that the set of all elements of M associating with a and b is not a subloop. This also gives an example of a nonassociative Moufang loop with a generating set in which every three elements have trivial associator.

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1. INTRODUCTION

A loop in which the identity $(xy)(zx) = (x(yz))x$ holds is said to be a *Moufang loop*. As is well known, the Moufang loops are *diassociative*, i.e., every subloop generated by a pair of elements is a group. In particular, this implies that every left inverse to an element is its right inverse and the usual definition of powers and orders of elements holds. A finite Moufang loop is called a *p-loop* for a prime p is the order of every element of the loop is a power of p. As in the case of groups, this is equivalent to the condition that the order of the loop is a power of p. For the other main properties of Moufang loops, see [1] and [2]. For elements x, y, z of Moufang loops, denote by $[x, y]$ a unique element c, which is called the *commutator* of x and y, such that $xy = (yx)c$ and by (x, y, z) a unique element a, which is called the *associator* of x, y, and z, such that $(xy)z = (x(yz))a$.

The following two interrelated problems arise naturally in the study of Moufang loops.

Problem 1. *Let L be a Moufang loop and let $a, b \in L$. Consider the set*

$$l(a, b) = \{x \in L \mid (x, a, b) = 1\}.$$

Is $l(a, b)$ a subloop of L?

Problem 2. *Let L be a Moufang loop generated by a set X. Suppose that $(x, y, z) = 1$ for all $x, y, z \in X$. Does this imply that L is associative?*

Recall that Moufang's theorem [1, Theorem IV.2.1] claims that, if $(x, y, z) = 1$ for three elements x, y, z of a Moufang loop, then the subloop generated by x, y, and z is a group. This obviously implies a relationship of Problem 2 to Moufang's theorem and the validity of the statement of the problem for $|X| \le 3$. Note the short proof of Moufang's theorem recently obtained in [3].

As far as Problem 1 is concerned, it is a special case of the following general problem. Let F be a free Moufang loop with base $x, x_1, \dots, x_m, y_1, \dots, y_n$. It is of interest to describe the words $w \in F$ for which the set

$$l_w(a_1, \dots, a_m) = \{t \in F \mid w(t, a_1, \dots, a_m, b_1, \dots, b_n) = 1 \text{ for all } b_1, \dots, b_n \in F\}$$

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is a subloop of F for every $a_1, \dots, a_m \in F$. For example, as is well known, the word $w = (x, y_1, y_2)$ has this property, because the corresponding subset $l_w(\varnothing)$ is the *nucleus* of the loop F, and it is an (associative) subloop.

Problems 1 and 2 are solved in the affirmative for all known Moufang loops, which we have tested. However, we found a new example of a 4-generated loop of order 3^{19}, which disproves both assertions. This loop is constructed in the next subsection. In connection with Problem 2, we also note that the existence of 3-torsion in the given example is essential. We formulate the following conjecture.

Conjecture. *The answer to the question in Problem 2 is in the affirmative if L is a finite Moufang p-loop and $p \neq 3$.*

Some evidence in favor of this conjecture is given by the following simple fact of the theory of Mal'tsev algebras.

Proposition. *Let M be a Mal'tsev algebra with a set of generators X over a field whose characteristic differs from 2 and 3. If $\mathrm{J}(x, y, z) = 0$ for all $x, y, z \in X$, then M is a Lie algebra.*

Indeed, every counterexample to this assertion would be a step to a counterexample to the Hypothesis, which one would be able to construct using the Campbell–Hausdorff formula, by analogy with [4].

Our example belongs to the class of "polynomial" Moufang loops, i.e., their underlying set is a vector space $\mathbb{F}_p^n$ over the field of p elements, and the multiplication is defined by polynomials over $\mathbb{F}_p$. We treated a free, in a sense, 4-generated polynomial 3-group G of nilpotency class 4 and then evaluated a general central one-dimensional polynomial extension of G in the variety of Moufang loops. The correctness of the last step was also tested using a computer. The parameters obtained in this way were specialized to obtain the desired counterexample. The loop presented below is the factorization by a maximal normal subgroup which does not contain some nonidentity associator.

2. A LOOP

Let M be the 19-dimensional vector space over the field $\mathbb{F}_3$. We represent elements of M as vectors $x = (x_1, x_2, \dots, x_{19}) \in M$. We introduce a new operation $\circ$ on M defined for $x, y \in M$ by the formula

$$x \circ y = x + y + f, \tag{2.1}$$

where $f = (f_1, \dots, f_{19})$ and f_k are polynomials in x_i and y_j written out explicitly as follows:

$$f_1 = f_2 = f_3 = f_4 = 0, \qquad f_5 = -x_2y_1, \qquad f_6 = -x_3y_1,$$
$$f_7 = -x_4y_1, \qquad f_8 = -x_3y_2, \qquad f_9 = -x_4y_2, \qquad f_{10} = -x_4y_3,$$
$$f_{11} = -x_2x_3y_1 - x_2y_1y_3 + x_5y_3 - x_8y_1, \qquad f_{12} = -x_2x_4y_1 - x_2y_1y_4 + x_5y_4 - x_9y_1,$$
$$f_{13} = -x_3y_1y_2 + x_6y_2 + x_8y_1, \qquad f_{14} = -x_3x_4y_1 - x_3y_1y_4 + x_6y_4 - x_{10}y_1,$$
$$f_{15} = -x_4y_1y_2 + x_7y_2 + x_9y_1, \qquad f_{16} = -x_4y_1y_3 + x_7y_3 + x_{10}y_1,$$
$$f_{17} = -x_3x_4y_2 - x_3y_2y_4 + x_8y_4 - x_{10}y_2, \qquad f_{18} = -x_4y_2y_3 + x_9y_3 + x_{10}y_2,$$
$$\begin{aligned} f_{19} = {} & -x_1x_2x_4y_3 + x_1x_2y_3y_4 + x_1x_3y_2y_4 + x_1x_4y_2y_3 - x_1y_2y_3y_4 \\ & - x_2x_3x_4y_1 + x_2x_3y_1y_4 + x_2x_4y_1y_3 + x_3x_4y_1y_2 - x_3y_1y_2y_4 + x_1x_8y_4 \\ & - x_1x_9y_3 + x_1x_{10}y_2 - x_1y_2y_{10} + x_1y_3y_9 - x_1y_4y_8 - x_2x_6y_4 + x_2x_7y_3 - x_2x_{10}y_1 \\ & + x_2y_1y_{10} - x_2y_3y_7 + x_2y_4y_6 + x_3x_5y_4 - x_3x_7y_2 + x_3x_9y_1 - x_3y_1y_9 + x_3y_2y_7 \\ & - x_3y_4y_5 - x_4x_5y_3 + x_4x_6y_2 - x_4x_8y_1 + x_4y_1y_8 - x_4y_2y_6 + x_4y_3y_5. \end{aligned}$$

One can verify that $(M, \circ)$ is a Moufang loop. The identity element is the zero vector in M, and, for $x \in (M, \circ)$, the relation

$$x^{-1} = -x + h \tag{2.2}$$

holds, where $h = (h_1, \dots, h_{19})$ and the polynomials h_k are

$$h_1 = h_2 = h_3 = h_4 = 0, \qquad h_5 = -x_1x_2, \qquad h_6 = -x_1x_3, \qquad h_7 = -x_1x_4,$$

$$h_8 = -x_2x_3, \qquad h_9 = -x_2x_4, \qquad h_{10} = -x_3x_4, \qquad h_{11} = -x_1x_8 + x_3x_5,$$
$$h_{12} = -x_1x_9 + x_4x_5, \qquad h_{13} = x_1x_2x_3 + x_1x_8 + x_2x_6,$$
$$h_{14} = -x_1x_{10} + x_4x_6, \qquad h_{15} = x_1x_2x_4 + x_1x_9 + x_2x_7,$$
$$h_{16} = x_1x_3x_4 + x_1x_{10} + x_3x_7, \qquad h_{17} = -x_2x_{10} + x_4x_8,$$
$$h_{18} = x_2x_3x_4 + x_2x_{10} + x_3x_9, \qquad h_{19} = -x_1x_2x_3x_4.$$

Let $e_1, \dots, e_{19}$ be the standard basis of the original vector space M, i.e., $e_i = (\dots, 0, 1, 0, \dots)$, where 1 stands at the ith place. Write

$$a = e_1, \qquad b = e_2, \qquad c = e_3, \qquad d = e_4.$$

Then, using the product formula (2.1), one can prove that the following relations hold in $(M, \circ)$:

$$\begin{array}{llll} e_5 = [a,b], & e_6 = [a,c], & e_7 = [a,d], & e_8 = [b,c], \\ e_9 = [b,d], & e_{10} = [c,d], & e_{11} = [[a,b],c], & e_{12} = [[a,b],d], \\ e_{13} = [[a,c],b], & e_{14} = [[a,c],d], & e_{15} = [[a,d],b], & e_{16} = [[a,d],c], \\ e_{17} = [[b,c],d], & e_{18} = [[b,d],c], & e_{19} = ([a,b],c,d). & \end{array} \tag{2.3}$$

Note that the equation

$$(\cdots((e_1^{n_1} \circ e_2^{n_2}) \circ e_3^{n_3}) \circ \cdots \circ e_{19}^{n_{19}}) = (\overline{n}_1, \dots, \overline{n}_{19}) \tag{2.4}$$

holds for all $n_1, \dots, n_{19} \in \mathbb{Z}$, where $[n \mapsto \overline{n}]$ stands for the natural epimorphism $\mathbb{Z} \to \mathbb{F}_3$. Indeed, using (2.1) and (2.2), one can readily show by induction on $n \in \mathbb{Z}$ that the loop element e_i^n coincides with the element $\overline{n}e_i$ of the space M, where $i = 1, \dots, 19$, and then verify the equation (2.4) parametrically, using (2.1).

This, together with equations (2.3), implies, in particular, that $(M, \circ) = \langle a, b, c, d \rangle$. In this loop, the following equations hold:

$$(a,b,c) = (a,b,d) = (a,c,d) = (b,c,d) = 1, \qquad ([a,b],c,d) = e_{19} \neq 1.$$

Hence, the loop is not associative (since the associativity of an arbitrary loop is obviously equivalent to the fact that all associators are equal to the identity), which gives a negative answer to Problem 2. Moreover, we have $a, b \in l(c,d)$ and $[a,b] \notin l(c,d)$, and this means that $l(c,d)$ is not a subloop, and thus the answer to Problem 1 is negative as well.

3. MAL'TSEV ALGEBRAS

For the definitions and main properties of Mal'tsev algebras, see, e.g., [5]. Recall that, for elements a, b, c of a Mal'tsev algebra, one can define their Jacobian $\mathrm{J}(a,b,c) = (ab)c + (bc)a + (ca)b$. Let us now prove the Proposition.

Proof of the Proposition. Let $\mathrm{J}(M)$ be the ideal of the algebra M generated by $\mathrm{J}(a,b,c)$ for all $a, b, c \in M$. It suffices to show that $\mathrm{J}(M) = 0$. Since the Jacobian $\mathrm{J}(a,b,c)$ is linear with respect to every argument, we may assume without loss of generality that a, b, and c are (nonassociative) words in X. Let us use further the induction by $n = |a| + |b| + |c|$, where $|w|$ stands for the length of the word $w \in M$. If $n = 3$, then $\mathrm{J}(a,b,c) = 0$ by assumption. If $n > 3$, then we may assume that $|a| > 1$, $a = a_1a_2$, and $|a| = |a_1| + |a_2|$. Using the identity [5, (2.15)]

$$3\mathrm{J}(wx, y, z) = \mathrm{J}(x,y,z)w - \mathrm{J}(y,z,w)x - 2\mathrm{J}(z,w,x)y + 2\mathrm{J}(w,x,y)z,$$

which holds in the Mal'tsev algebras over the fields whose characteristic differs from 2 and 3, we obtain $\mathrm{J}(a,b,c) = 0$ by induction. This completes the proof of the proposition. ☐

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